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An approximate method for solving fractional system differential equations

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Abstract

In this research work, we have shown that it is possible to use fuzzy transform method (FTM) for the estimate solution of fractional system differential equations (FSDEs). In numerical methods, in order to estimate a function on a particular interval, only a restricted number of points are employed. However, what makes the F-transform preferable to other methods is that it makes use of all points in this interval. A number of clear and specific examples have been enumerated for the purpose of illustrating the simplicity and the efficiency of the suggested method.

1 Introduction

Fractional arithmetic and fractional differential equations appear in many disciplines, including medicine [1], economics [2], dynamical problems [3, 4], chemistry [5], mathematical physics [6], traffic models [7] and fluid flow [8] and so on. Scholars and researchers are invited to study books that have been written in order to better understand the concept of fractional arithmetic [9, 10]. This study has been conducted for the purpose of finding the estimate solution for the following system differential equations with fractional derivative:

$$D^{\alpha}u_q(t) + \mathfrak{N}_q\left(u_1(t), u_2(t), \ldots, u_p(t)\right) = h_q(t),\ q = 1, 2, \ldots, p, \tag{1.1}$$

with the initial conditions:

$$u_q^{(i)}(t_0) = \eta_q,\ \ q = 1, 2, \ldots, p,\ \ i = 0, 1, \cdots, m-1, \tag{1.2}$$

where p is the number of unknown variables, $\mathfrak{N}_q$ is nonlinear part, h_q are inhomogeneous terms and D^{α} denotes the Caputo derivative of order α in [10]

$$D^{\alpha}u(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1}u^{(m)}(s)ds,\ m-1 < \alpha \le m,\ m \in \mathbb{Z}^+. \tag{1.3}$$

A number of articles can be found to express modeling, deploying and extent of system differential equation (SDEs), system partial differential equation (SPDEs) and fractional system partial differential equations (FSPDEs), which are cited in [10, 11, 12, 13]. There are no accurate analytical solutions for most SDEs, SPDEs and FSPDEs; thus, a relatively large number of estimate solution expressed by scholars are not possible if they find the accurate analytical solutions with the existing procedures for the SDEs, SPDPs and FSPDEs. Accordingly, for such differential equations, we have to employ some direct and iterative methods. Some of these techniques which

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have been used by scholars include new homotopic perturbation method [13], Adomian's decomposition method (ADM) [14, 15], variational iteration method (VIM) [15], homotopy perturbation method (HPM) [16], homotopy analysis method (HAM) [17] and so on [18, 19]. The FTM has recently been utilized by authors in [20, 21, 22] to find the estimate solution of the first order fuzzy differential equations and two-point boundary value problems. Along the same line of research, Chen and his associates in [23] have established an algorithm to gain the numerical solutions of the second order primary amount problems. This research work is organized as follows: in Section 2, fuzzy partition and fuzzy transform are presented. In Section 3, we have expressed the new approach with Fuzzy transform. In Section 4, the applications of Fuzzy transform method to the system differential equations of real order are illustrated, and some numerical examples are presented. And conclusions are drawn in Section 5.

2 Discretization of the fractional derivative

Assume that $u(t)$ is the solution to equations (1.1). To calculate the approximation of $u(t)$, we use the discretization of the Caputo derivative with the assumption $\tau = t_{j+1} - t_j$ and $t_j = a + j\,\tau$, $j = 0, 1, 2, \cdots$.

Utilizing the approximation for the Caputo derivative [25] of Eq. (1.3) we have:

$$D^{\alpha}u(t_{k+1}) \approx \frac{1}{\tau^{\alpha}\Gamma(2-\alpha)} \sum_{j=0}^{k}(u(t_{j+1}) - u(t_j))\left((k-j+1)^{1-\alpha} - (k-j)^{1-\alpha}\right), \tag{2.1}$$

in which $0 < \alpha \leq 1$, $u(t_0)$ is known and

$$D^{\alpha}u(t_{k+1}) \approx \frac{1}{\tau^{\alpha}\Gamma(3-\alpha)} \sum_{j=0}^{k}(u(t_{j+1}) - 2u(t_j) + u(t_{j-1}))\left((k-j+1)^{2-\alpha} - (k-j)^{2-\alpha}\right), \tag{2.2}$$

in which $1 < \alpha \leq 2$, $u(t_0)$ and $u'(t_0)$ are known and $u(t_{-1}) = u(t_0) - \tau\, u'(t_0)$.

3 Fuzzy partition and Fuzzy transform

In this section, only the main definitions of F-transform to be utilized in the subsequent sections of numerical implementations will be presented.

Definition 3.1. *[24] Presuming that for $n \geq 2$, $a = t_1 < t_2 < \cdots < t_{n-1} < t_n = b$ to be specified nodes, we express that fuzzy sets $B_1, \cdots, B_n$ defined on $[a,\, b]$ with their membership functions $B_1(t), \cdots, B_n(t)$, form a fuzzy partition of $[a,\, b]$ if they meet the following properties:*

(1) B_k of $[a, b]$ to $[0, 1]$ is continuous, $\sum\limits_{k=1}^{n} B_k(t) = 1$ for all $t \in [a, b]$ and $B_k(t_k) = 1$, $k = 1, 2, \cdots, n$.

(2) $B_k(t) = 0$ if $t \notin (t_{k-1}, t_{k+1})$, with $t_0 = a$ and $t_{n+1} = b$,

(3) On subinterval of $[t_{k-1}, t_{k+1}]$, for $k = 2, \cdots, n-1$, $B_k(t)$, is certainly an increasing function on $[t_{k-1}, t_k]$ and decreasing function on $[t_k, t_{k+1}]$.
The membership functions $B_1, B_2, \cdots, B_n$ are named basic functions (BFs).

The next formulas give the standard display of such triangular membership functions:

$$
\begin{aligned}
B_1(t) &= \begin{cases} 1-\frac{t-t_1}{h_1}, & t_1 \le t \le t_2 \\ 0, & \textit{otherwise}, \end{cases} \\
B_k(t) &= \begin{cases} \frac{t-t_{k-1}}{h_{k-1}}, & t_{k-1} \le t \le t_k \\ 1-\frac{t-t_k}{h_k}, & t_k \le t \le t_{k+1},\ k=2,3,\cdots,n-1, \\ 0, & \textit{otherwise}, \end{cases} \\
B_n(t) &= \begin{cases} \frac{t-t_{n-1}}{h_{n-1}}, & t_{n-1} \le t \le t_n, \\ 0, & \textit{otherwise}. \end{cases}
\end{aligned}
\tag{3.1}
$$

The formulas that follow for $k = 2, \cdots, n-1$*, give the standard display of such sinusoidal membership functions:*

$$
\begin{aligned}
B_1(t) &= \begin{cases} o.5\left(1+cos\frac{\pi}{h}(t-t_1)\right), & t_1 \le t \le t_2 \\ 0, & \textit{otherwise}, \end{cases} \\
B_k(t) &= \begin{cases} o.5\left(1+cos\frac{\pi}{h}(t-t_k)\right), & t_{k-1} \le t \le t_{k+1},\ \ k=2,3,\cdots,n-1, \\ 0, & \textit{otherwise}, \end{cases} \\
B_n(t) &= \begin{cases} o.5\left(1+cos\frac{\pi}{h}(t-t_n)\right), & t_{n-1} \le t \le t_n \\ 0, & \textit{otherwise}, \end{cases}
\end{aligned}
\tag{3.2}
$$

in which $h_k = t_{k+1} - t_k$ *for* $k = 1, \cdots, n-1$. *It can be stated that fuzzy partition of* $[a, b]$ *is uniform if* $t_{k+1} - t_k = h = \frac{b-a}{n-1}$ *and if two additional properties coincide:*

(4) $B_k(t_k - t) = B_k(t_k + t)$*, for all* $t \in [0, h]$*, for* $k = 2, \cdots, n-1$*,*

(5) $B_k(t) = B_{k-1}(t-h)$ *and* $B_{k+1}(t) = B_k(t-h)$*, for* $k = 2, \cdots, n-1$*, and* $t \in [t_k, t_{k+1}]$.

Definition 3.2. *[24] Let* f *be any function belonging to* $C([a, b])$ *and* $B_1, B_2, \cdots, B_n$*, be the BFs which compose a fuzzy partition of* $[a, b]$. *We define the* n*-tuple* $[F_1, F_2, \cdots, F_n]$ *of real numbers given by*

$$
F_k = \frac{\int_a^b f(t)B_k(t)dt}{\int_a^b B_k(t)dt}, \quad k = 1, 2, \cdots, n,
\tag{3.3}
$$

as the F*-transform of* f *in relation to* $B_1, B_2, \cdots, B_n$.

Definition 3.3. *[24] Let* $[F_1, F_2, \cdots, F_n]$ *be the* F*-transform of function* f *relative to BFs,* $B_1, B_2, \cdots, B_n$. *Then,*

$$
f_n(t) = \sum_{k=1}^{n} F_k B_k(t),
$$

which is named the inverse F*-transform* (IFT) *of function* f *on* $[a, b]$.

Theorem 3.1. *[24] Let* f *be a continuous function on* $[a, b]$ *and* $B_1, B_2, \cdots, B_n$ *be the BFs which form a fuzzy partition of* $[a, b]$. *Then, the* kth *component of the integral* F*-transform signified over* $[f(a), f(b)]$*, gives the min-*

imum to the function

$$\phi(y) = \int_a^b (f(t) - y)^2 B_k(t) dt,$$

for $k = 1, 2, ..., n$.

Lemma 3.1. *[24] (Convergence) Let f be a continuous function on $[a, b]$. Thus, for any $\epsilon > 0$, there exist n_ϵ and a fuzzy partition $B_1, \cdots, B_{n_\epsilon}$ of $[a, b]$ such that for all $t \in [a, b]$*

$$|f(t) - f_{n_\epsilon}(t)| \leq \epsilon. \tag{3.4}$$

4 Description of the new approach

Let $u(t)$ be the continuous solution of (1.1) on $[0, T]$ satisfying. Also, $U_1, \cdots, U_n$ of F-transform $u(t)$, calculated by using *BFs* $B_0, B_1, \cdots, B_n$ in $[0, T]$ regarding (3.2) with $t_{j+1} - t_j = \tau$ which are uniform fuzzy partitions. Now with applying *IFT* on the function $u(t)$, the approximation $u_n(x)$ is obtained based on the following formula:

$$u_n(t) = \sum_{k=0}^{n} U_k B_k(t), \;\; t \in [0, T]. \tag{4.1}$$

Hence for approximate solution, we can calculate U_k for $k = 0, 1, 2, \cdots, n$. In the next proposition the discretization of the Caputo derivative for $u_n(t)$ for Eq.(2.1) is presented. With substituting $u_n(t)$ in Eqs.(2.1), (2.2), we will have the next equations, respectively:

$$D^\alpha u_{n,q}(t_{k+1}) \approx \frac{1}{\tau^\alpha \Gamma(2-\alpha)} \sum_{j=0}^{k} (U_{j+1,q} - U_{j,q}) \left((k-j+1)^{1-\alpha} - (k-j)^{1-\alpha}\right), \;\; 0 < \alpha \leq 1, \tag{4.2}$$

$$\begin{aligned} D^\alpha u_{n,q}(t_{k+1})) \approx \frac{1}{\tau^\alpha \Gamma(3-\alpha)} \sum_{j=0}^{k} (U_{j+1,q} - 2U_{j,q} + U_{j-1,q}) \times \\ \left((k-j+1)^{2-\alpha} - (k-j)^{2-\alpha}\right), \;\; 1 < \alpha \leq 2, \end{aligned} \tag{4.3}$$

where $u(t_0)$ and $u'(t_0)$ are known of initial conditions, $U_0 = u(t_0)$ and $U_{-1} = u(t_0) - \tau\, u'(t_0)$.

In order to gain the approximate solution of the problem (1.1), we use $u_n(t)$, hence

$$D^\alpha u_{n,q}(t) + \mathfrak{N}_q\left(u_{n,1}(t), u_{n,2}(t), \ldots, u_{n,p}(t)\right) = h_{n,q}(t), \; q = 1, 2, \ldots, p, \tag{4.4}$$

in which $n < \alpha \leq n+1, \;\; q = 1, 2, \ldots, p$ and $0 < t \leq T$, and by putting $t = t_{k+1}$ in the formula 4.4, we have

$$D^\alpha u_{n,q}(t_{k+1}) + \mathfrak{N}_q\left(u_{n,1}(t_{k+1}), u_{n,2}(t_{k+1}), \ldots, u_{n,p}(t_{k+1})\right) = h_{n,q}(t_{k+1}), \tag{4.5}$$

in which $k = 0, 1, \ldots, n-1$ and $\; q = 1, 2, \ldots, p$.

Considering Caputo's derivative and using Eqs. (4), (4.3), Eqs. (4.5) converts to the following form for $q = 1, 2, \ldots, p$.

Case 1. Considering Caputo's derivative for $0 < \alpha \leq 1$

$$\frac{1}{\tau^{\alpha}\Gamma(2-\alpha)}\sum_{j=0}^{k}(U_{j+1,q}-U_{j,q})\left((k-j+1)^{1-\alpha}-(k-j)^{1-\alpha}\right)+ \mathfrak{N}_q\left(U_{k+1,1},U_{k+1,2},\ldots,U_{k+1,p}\right)=h_q(t_{k+1}),\ k=0,1,\ldots,n-1. \tag{4.6}$$

Case 2. Considering Caputo's derivative for $1 < \alpha \leq 2$

$$\frac{1}{\tau^{\alpha}\Gamma(3-\alpha)}\sum_{j=0}^{k}(U_{j+1,q}-2U_{j,q}+U_{j-1,q})\left((k-j+1)^{2-\alpha}-(k-j)^{2-\alpha}\right)+ \mathfrak{N}_q\left(U_{k+1,1},U_{k+1,2},\ldots,U_{k+1,p}\right)=h_q(t_{k+1}),\ k=0,1,\ldots,n-1. \tag{4.7}$$

Now, consider boundary conditions $U_{0,q} = u_q(t_0)$ and $U_{-1,q} = u_q(t_0) - \tau\, u'_q(t_0)$. we can calculate $U_{k,q}$, for $k = 0, 1, 2, \cdots n$, by the obtained recursive equation (4.6) and (4.7); then by IFT, we can gain the approximate solution $u(t) \approx u_{n,q}(t)$ for Eq.(1.1).

An algorithm for approximation of FSDEs by this method is stated in the next Algorithm.

Algorithm *1.* An algorithm for approximation of FSDEs

Step 1. Input p, n, $U_0 = u(0)$ and T.

Step 2. Set $\tau \leftarrow \frac{T}{m}$.

Step 3. Locate $t_k \leftarrow k\,\tau,\ \ k = 0, 1, 2, \cdots, n$.

Step 4. Choose sinusoidal *BFs* related to B_k for $k = 0, 1, 2, \cdots, n$.

Step 5. Set recursive equation for $q = 1, 2, \ldots, p$.

Case 1. For $0 < \alpha \leq 1$:

$$\frac{1}{\tau^{\alpha}\Gamma(2-\alpha)}\sum_{j=0}^{k}(U_{j+1,q}-U_{j,q})\left((k-j+1)^{1-\alpha}-(k-j)^{1-\alpha}\right)+ \mathfrak{N}_q\left(U_{k+1,1},U_{k+1,2},\ldots,U_{k+1,p}\right)=h_q(t_{k+1}),\ k=0,1,\ldots,n-1. \tag{4.8}$$

Case 2. For $1 < \alpha \leq 2$:

$$\frac{1}{\tau^{\alpha}\Gamma(3-\alpha)}\sum_{j=0}^{k}(U_{j+1,q}-2U_{j,q}+U_{j-1,q})\left((k-j+1)^{2-\alpha}-(k-j)^{2-\alpha}\right)+ \mathfrak{N}_q\left(U_{k+1,1},U_{k+1,2},\ldots,U_{k+1,p}\right)=h_q(t_{k+1}),\ k=0,1,\ldots,n-1. \tag{4.9}$$

Regarding the boundary conditions $U_{0,q} = u_q(t_0)$ and $U_{-1,q} = U_{0,q} - \tau\, u'_q(t_0)$.

Step 6. Calculate every $U_{k,q}$, $p = 0, 1, 2, \cdots, n$, $k = 0, 1, 2, \cdots, n-1$.

Step 7. The approximate solution with IFT is

$$u_{n,q}(t) = \sum_{k=0}^{n} U_{k,q}(x) B_k(t),$$

for $q = 1, 2, \ldots, p$.

5 Test examples

Now in this section, we present various examples to illustrate FTM for FSDEs. In all these examples, we have used mathematical software $Mathematica$.

t	$Approx$	$Exact$	*Absolute error*
0.3	-0.682918	-0.683348	429.823×10^{-6}
0.5	-0.498169	-0.498529	359.979×10^{-6}
0.7	-0.320889	-0.321169	280.104×10^{-6}
0.9	-0.148699	-0.148879	179.367×10^{-6}

(a) The exact and estimate results of u_1.

t	$Approx$	$Exact$	*Absolute error*
0.3	0.682785	0.683348	562.678×10^{-6}
0.5	0.497942	0.498529	586.246×10^{-6}
0.7	0.320551	0.321169	618.602×10^{-6}
0.9	0.1482195	0.148879	659.671×10^{-6}

(b) The exact and estimate results of u_2.

Table 1: Absolute error in different values of t for the test example 5.1 with $\alpha = \beta = 0.9$.

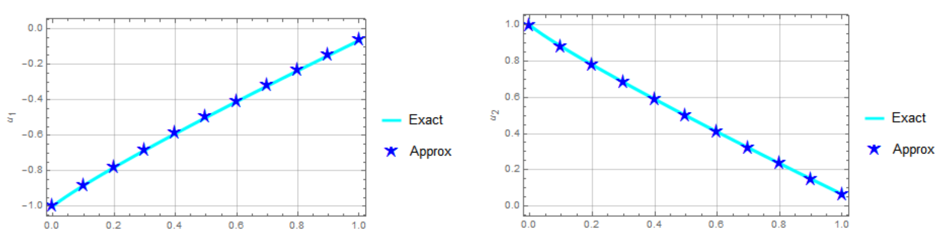
Example 5.1. *For the first example, we propose the coupled system of fractional differential equations:*

$$\begin{cases} D^{\alpha} u_1 - u_1 - u_2 = \alpha + \frac{t^{\beta}}{\Gamma(\beta)} - \frac{t^{\alpha}}{\Gamma(\alpha)}, & 0 < \alpha \le 1 \\ D^{\beta} u_2 - u_1 + u_2 = 2 - \beta - \frac{t^{\beta}}{\Gamma(\beta)} - \frac{t^{\alpha}}{\Gamma(\alpha)}, & 0 < \beta \le 1 \end{cases} \tag{5.1}$$

with the solutions $u_1(t) = \frac{t^{\alpha}}{\Gamma(\alpha)} - 1$ *and* $u_2(t) = \frac{t^{\beta}}{\Gamma(\beta)} + 1$ *and the primary conditions:*

$$u_1(0) = -1, \quad u_2(0) = 1. \tag{5.2}$$

In Table 1, we can see the estimated solutions toward $\alpha = \beta = 0.9$*, which is derived for various values of* t*,*



(a) The exact and estimate figure of u_1. (b) The exact and estimate figure of u_2.

Figure 1: The exact and estimate solution for $\alpha = \beta = 0.9$ for test example 5.1.

applying FTM. In figure 1, we can view the precise and estimate answers featuring $\alpha = \beta = 0.9$ *and* $\tau = 0.002$.

Example 5.2. *For the second example, we propound the system of nonlinear fractional differential equations:*

$$\begin{cases} D^{\alpha}u_1 - u_1 - 2u_1\,u_2 = \frac{t^{\alpha}}{\Gamma(\alpha+2)}\left(t^{\alpha} - \frac{4^{\alpha}\Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}} + \frac{2t^{\alpha+3\beta}}{\Gamma(\beta+3)}\right), & 1<\alpha\leq 2 \\ D^{\beta}u_2 - u_1^2 + u_2 - 2u_3 = \frac{2t^{4\gamma}}{\Gamma(\gamma+1)} - \frac{t^{4\alpha}}{\Gamma(\alpha+2)^2} + \frac{1}{\Gamma(\beta+3)}\left(t^{3\beta} + \frac{3\Gamma(3\beta)t^{2\beta}}{2\Gamma(2\beta)}\right), & 1<\beta\leq 2 \\ D^{\gamma}u_3 - u_1\,u_2 = \frac{t^{2\alpha+3\beta}}{\Gamma(\alpha+2)\Gamma(\beta+3)} - \frac{4\Gamma(4\gamma)t^{3\gamma}}{\Gamma(\gamma)\Gamma(3\gamma+1)}, & 1<\gamma\leq 2 \end{cases} \tag{5.3}$$

given that the primary conditions:

$$u_1(0) = -1,\ \frac{d\,u_1}{dt}(0) = 0,\ u_2(0) = 0,\ \frac{d\,u_2}{dt}(0) = 0,\ u_3(0) = 0,\ \frac{d\,u_3}{dt}(0) = 0. \tag{5.4}$$

In Table 2 and in figure 2, we can view the precise and estimate answers featuring $\tau = 0.002$ *and* $\alpha = \beta = \gamma = 1.9$ *through applying FTM for various values of* t.

t	$Approx$	$Exact$	Absolute error
0.3	-0.00197391	-0.00194464	29.2632×10^{-6}
0.5	-0.0136694	-0.0135477	121.728×10^{-6}
0.7	-0.0489704	-0.0486577	312.664×10^{-6}
0.9	-0.12708	-0.126445	635.314×10^{-6}

(a) The exact and estimate results of u_1.

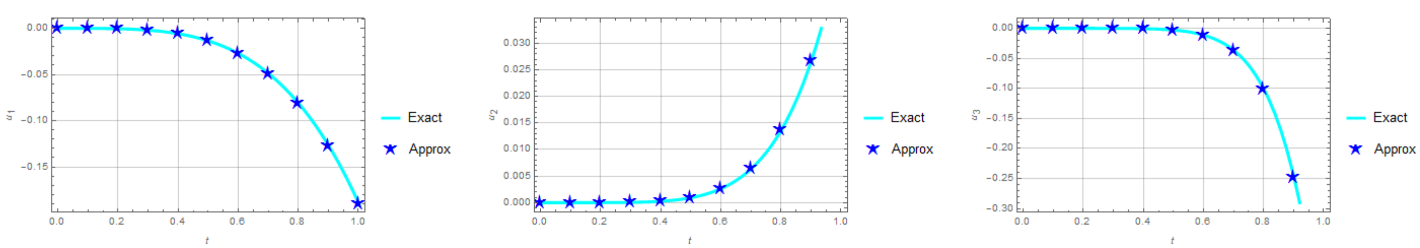
t	$Approx$	$Exact$	Absolute error
0.3	0.0000517646	0.0000506181	1.14657×10^{-6}
0.5	0.000943059	0.000930772	12.2866×10^{-6}
0.7	0.00639178	0.00633539	56.3927×10^{-6}
0.9	0.0267023	0.0265397	162.549×10^{-6}

(b) The exact and estimate results of u_2.

t	$Approx$	$Exact$	Absolute error
0.3	-0.0000598939	-0.0000581164	1.77756×10^{-6}
0.5	-0.00287175	-0.00282065	51.0977×10^{-6}
0.7	-0.0368524	-0.036385	467.44×10^{-6}
0.9	-0.248152	-0.245708	2.44329×10^{-6}

(c) The exact and estimate results of u_3.

Table 2: Absolute error in different values of t for the test example 5.2 with $\alpha = \beta = \gamma = 1.9$.



(a) The exact and estimate figure of u_1. (b) The exact and estimate figure of u_2. (c) The exact and estimate figure of u_3.

Figure 2: The exact and estimate solution for $\alpha = \beta = \gamma = 1.9$ for test example 5.2.

With the knowledge that $\alpha = \beta = \gamma = 1.9$*, the estimate solution obtained by the proposed method corresponds to the precise solutions* $u_1(t) = -\frac{t^{2\alpha}}{\Gamma(\alpha+2)}$, $u_2(t) = \frac{t^{3\beta}}{\Gamma(\beta+3)}$ *and* $u_3(t) = -\frac{t^{4\gamma}}{\Gamma(\gamma+1)}$.

Example 5.3. *For the fourth example, we propose the system of linear fractional differential equations:*

$$\begin{cases} D^{\alpha}u_1 - u_1 + 2u_2 = \frac{t^{\alpha}}{\Gamma(\alpha+2)}\left(t^{\alpha} - \frac{4^{\alpha}\Gamma\left(\alpha+\frac{1}{2}\right)}{\sqrt{\pi}} + \frac{2t^{\alpha+3\beta}}{\Gamma(\beta+3)}\right), & 0<\alpha\leq 1 \\ D^{\beta}u_2 - u_1 + u_2 - 2u_3 = \frac{2t^{4\gamma}}{\Gamma(\gamma+1)} - \frac{t^{4\alpha}}{\Gamma(\alpha+2)^2} + \frac{1}{\Gamma(\beta+3)}\left(t^{3\beta} + \frac{3\Gamma(3\beta)t^{2\beta}}{2\Gamma(2\beta)}\right), & 1<\beta\leq 2 \\ D^{\gamma}u_3 - u_2 = \frac{t^{2\alpha+3\beta}}{\Gamma(\alpha+2)\Gamma(\beta+3)} - \frac{4\Gamma(4\gamma)t^{3\gamma}}{\Gamma(\gamma)\Gamma(3\gamma+1)}, & 0<\gamma\leq 1 \end{cases} \tag{5.5}$$

with the solutions $u_1(t) = \frac{t^{2\alpha}}{\Gamma(\alpha+1)}$, $u_2(t) = -\frac{t^{\beta}}{\Gamma(\beta+2)}$ *and* $u_3(t) = -\frac{t^{3\gamma}}{\Gamma(\gamma+3)}$ *and the primary conditions:*

$$u_1(0) = -1,\ u_2(0) = 0,\ \frac{d\,u_2}{dt}(0) = 0,\ u_3(0) = 0. \tag{5.6}$$

t	$Approx$	$Exact$	*Absolute error*
0.3	0.119669	0.119055	613.253×10^{-6}
0.5	0.299527	0.298591	935.64×10^{-6}
0.7	0.548404	0.547151	1.25308×10^{-6}
0.9	0.861718	0.860136	1.582×10^{-6}

(a) The exact and estimate results of u_1.

t	$Approx$	$Exact$	*Absolute error*
0.3	-0.0192539	-0.0191562	97.663×10^{-6}
0.5	-0.0507081	-0.0505617	146.325×10^{-6}
0.7	-0.0960069	-0.095822	184.893×10^{-6}
0.9	-0.154681	-0.154468	212.42×10^{-6}

(b) The exact and estimate results of u_2.

t	$Approx$	$Exact$	*Absolute error*
0.3	-0.00737048	-0.0073115	58.9837×10^{-6}
0.5	-0.029178	-0.0290401	137.893×10^{-6}
0.7	-0.0722749	-0.072035	239.833×10^{-6}
0.9	-0.142343	-0.141982	360.384×10^{-6}

(c) The exact and estimate results of u_3.

Table 3: Absolute error in different values of t for the test example 5.3 with $\alpha = \gamma = 0.9$ and $\beta = 1.9$.

In Table 3, we can see the estimated solutions toward $\alpha = \gamma = 0.9$ *and* $\beta = 1.9$*, which is derived for various values of* t *applying FTM.*

In all examples, the length of the step is $\tau = 0.002$ on $t \in [0,\ 1]$. It is obvious that if the step length is smaller the results will be better.

6 Conclusion

We have successfully applied FTM to obtain estimate solution of the linear and non linear system differential equations featuring fractional derivative. The result indicate that a few iteration of FTM will results in some useful solutions. Finally, it should be added that the suggested technique has the potentials to be applied in solving other similar nonlinear and linear problems in partial differential equations featuring fractional derivative.

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