Interval Economic Efficiency Measures in Data Envelopment Analysis

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Abstract

One of the most essential pieces of information given by DEA models is the cost efficiency of decision-making units (DMUs). Cost efficiency (CE) is defined as the ratio of minimum costs to current costs and in fact, evaluates the ability to produce current outputs at a minimal cost. While the traditional cost efficiency models require the values for all data to be known exactly, in real-world problems the exact values of input prices are unknown, and only the maximum and minimum bounds of input prices can be estimated for each DMU. Hence, the main aim of the current paper is to develop a pair of two-level mathematical programming problems, whose optimal values represent the optimistic and pessimistic cost efficiency measures. The two-level nonlinear program for the optimistic cost efficiency measure is then transformed into a one-level linear program. In this regard, we provide an explicit formula for measuring the pessimistic CE measure.

1 Introduction

Data Envelopment Analysis (DEA) is a non-parametric technique for evaluating decision making units (DMUs) based on the production possibility set. Also, DEA is a methodology for measuring the efficiency of a set of DMUs where the input and output data of the DMUs are known exactly [5]. Zhu [25,26], Wang et al. [24], Kao [17], Entani et al.[7], and Despotis and Smirlis [6] developed the theory of efficiency measurement where the data are imprecise (see [1, 8, 10, 12]). In these references there is no discussion concerning cost efficiency with imprecise data. In fact these references focused on the technical-physical aspects of production for use in situations where unit price and unit cost information are not available, or where their uses are limited because of variability in the prices and costs that might need to be considered. It is worthwhile to note that in many real application of DEA, the cost efficiency analysis is required [3, 11, 13, 21, 22, 23]. There are some DEA models that deal with cost efficiency (CE) analysis when the data are known exactly. In fact, cost efficiency evaluates the ability to produce current outputs at minimal cost. See, e.g., [9,15,16,23] for more details concerning cost efficiency analysis with deterministic data. In these references, there is no discussion concerning imprecise data, whereas many research situations are best described by the intermediate case. One of these cases is when the input-output data are imprecise in the form of intervals. One another case is when the input-output data as well as input prices are in the interval form, due to incomplete price information: exact knowledge of prices is difficult and prices may be subject to variations in the short term. Estimation of cost efficiency is one of the vital topics in DEA. Although there are many papers for estimating cost efficiency in DEA models (see, for example, [9,15,16,23]), there are only few papers which concern the estimation of cost efficiency in the presence of imprecise data: Jahanshahloo et al.[14], Kuosmanen and Post

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[18,19], and Camanho and Dyson [4], for instance Jahanshahloo et al.[14] provide some models for the treatment of ordinal data in cost efficiency analysis. In [14] the models have multiplier forms with additional weight restrictions. The main idea in constructing these models is based on the weighted enumeration of the number of inputs/outputs of each unit which are categorized on the same scale rate. Kuosmanen and Post [18,19] derived upper and lower bounds for overall cost efficiency assuming incomplete price data in the form of a convex polyhedral cone. In their method finding the lower bound of cost efficiency is difficult or impractical, while in the current paper we provide a explicit formula for finding the lower bound of cost efficiency (the pessimistic point of view). Camanho and Dyson [4] discussed the assessment of CE in complex scenarios of price uncertainty. They assumed that input prices appear in the form of intervals. The upper bound of the CE estimate is obtained with the incorporation of weight restrictions in a standard DEA model, while by solving the model which they provided (see Model (7) in [4]) an upper bound of CE is obtained, which is not the best upper bound of the CE measure, because they relaxed the weight restrictions. In order to obtain the pessimistic CE measure for n DMUs, as they mentioned, it is required to solve n^2 linear programming models. Note that the proposed model may be infeasible, and also computationally expensive.

In this paper, we propose a pair of two-level mathematical programming problems to obtain the optimistic and pessimistic cost efficiency measure when input prices appear in the form of intervals, and we obtain the measure of cost efficiency in the pessimistic and optimistic point of view. In turns, the optimistic model is transformed to equivalent linear one. Also, we present an explicit formulae for computing the pessimistic CE measure. The rest of the paper unfolds as follows: in section 2, some DEA CE models are reviewed and a CE model in the multiplier form is provided, which is necessary in the next sections. Section 3, includes the main results. In fact, in this section the theory of CE is generalized to the situations in which the input prices are imprecise in the form of intervals. Section 4 gives some conclusions.

2 Preliminaries

Assume that we deal with a set of DMUs consisting of DMU$_j$; $j = 1, \ldots, n$, with input-output vectors (x_j, y_j); $j = 1, \ldots, n$, in which $x_j = (x_{1j}, \ldots, x_{mj})^T$ and $y_j = (y_{1j}, \ldots, y_{sj})^T$. Define $X = [x_1, x_2, \ldots, x_n]$ and $Y = [y_1, y_2, \ldots, y_n]$ as $m \times n$ and $s \times n$ matrices of inputs and outputs, respectively.

In order to obtain a measure of cost efficiency, when the input and output data are known exactly, F$\ddot{a}$re et al.[9] provide the following LP model:

$$\theta_o = \min\{\frac{w_o x}{w_o x_o} : \ X\lambda = x, \ \ Y\lambda \geqslant y_o, \ \ \lambda \geqslant 0\}. \qquad (1)$$

In the above model $w_o \in \mathbb{R}^m_+$ is a user-specified row vector of the prices of the inputs of DMU$_o$, the unit under assessment. The variables of Model (1) are x and λ. ($\lambda_o = 1, \ \lambda_j = 0; j \neq o, \ x = x_o$) is a feasible solution to (1) which implies that this model is feasible and bounded, and $\theta_o \in (0, 1]$. Note that model (1) has $m + s$ constraints where the RHS values of m constraints are zero, and this can lead to strong degeneracy and hence to great complexity. Regarding part (iii) of the main theorem in [15] one can use the following model instead of model (1), to determine the cost efficiency:

$$\vartheta_o = \min\{\frac{w_o X\lambda}{w_o x_o} : Y\lambda \geqslant y_o, \ \lambda \geqslant 0\}. \qquad (2)$$

Alternatively, the measure of cost efficiency can be obtained by solving the dual of Model (2) as follows (see [16]):

$$\max\{\ u^T y_o :\ u^T Y \leqslant \frac{w_o X}{w_o x_o},\ u \geqslant 0\}. \qquad (3)$$

In Model (3) the variable is u vector. Regarding the constraints, the optimal objective value of this model is not greater than one.

3 Cost efficiency with incomplete price information

3.1 The optimistic cost efficiency model

In the current section, we generalize the theory of cost efficiency to situations in which input prices appear in the form of intervals, due to incomplete price information, represented by intervals $[w_o^L, w_o^U]$, in which $w_o^L = (w_{1o}^L, \ldots, w_{mo}^L)$ and $w_o^U = (w_{1o}^U, \ldots, w_{mo}^U)$. Here we assume that $w_o^L > 0$. In this case, we propose the following model to obtain the upper bound of cost efficiency.

$$CE_o^U = \max_{w_o^L \leqslant w_o \leqslant w_o^U} \max\{u^T y_o : u^T y_j \leqslant \frac{w_o x_j}{w_o x_o}; for\ all\ j,\ u \geqslant 0\}. \ (4)$$

Since the inner program and outer program have the same objective of maximization, the above program is equivalent to the following one-level program:

$$\max\{u^T y_o : w_o^L \leqslant w_o \leqslant w_o^U,\ u^T y_j \leqslant \frac{w_o x_j}{w_o x_o}; j \neq o,\ u^T y_o \leqslant 1,\ u \geqslant 0\}. \ (5)$$

The following theorem provides a linear programming problem for obtaining the upper bound of cost efficiency in the presence of uncertainty:

Theorem 1. *When the input prices are in the form of intervals, the optimistic cost efficiency measure is equal to* *the optimal value of the following linear program:*

$$
\begin{aligned}
\psi_o^* = \max \quad & \sum_{r=1}^{s} u_r y_{ro} && (6)\\
s.t. \quad & \\
& \sum_{i=1}^{m} \hat{w}_{io} x_{io} = 1, \\
& \sum_{r=1}^{s} u_r y_{ro} \leqslant 1, \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} \hat{w}_{io} x_{ij} \leqslant 0, && j \neq o, j = 1, 2, \ldots, n, \\
& \eta w_{io}^L \leqslant \hat{w}_{io} \leqslant \eta w_{io}^U, && i = 1, 2, \ldots, m, \\
& u_r \geqslant 0, && r = 1, 2, \ldots, s, \\
& \eta \geqslant 0,
\end{aligned}
$$

Proof. First we set

$$\eta = \frac{1}{w_o x_o}\ . \tag{7}$$

We have

$$\eta w_o x_o = 1 \; and \; \eta > 0.$$

Equivalently, this can be expressed as follows:

$$\sum_{i=1}^{m} \eta w_{io} x_{io} = 1. \tag{8}$$

Now, we set $\hat{w}_{io} = \eta w_{io}, \quad i = 1, 2, \ldots, m$. Considering this variable alteration and (8), the constraints of Model (5) are transformed to the following constraints:

$$w_{io}^L \leqslant w_{io} \leqslant w_{io}^U \Longleftrightarrow \eta w_{io}^L \leqslant \eta w_{io} \leqslant \eta w_{io}^U \Longleftrightarrow \eta w_{io}^L \leqslant \hat{w}_{io} \leqslant \eta w_{io}^U, \tag{9}$$

and

$$u^T y_j \leqslant \frac{w_o x_j}{w_o x_o} \Longleftrightarrow u^T y_j - \eta w_o x_j \leqslant 0 \Longleftrightarrow u^T y_j - \hat{w}_o x_j \leqslant 0; j \neq o, \tag{10}$$

and

$$\sum_{i=1}^{m} \eta w_{io} x_{io} = 1 \Longleftrightarrow \sum_{i=1}^{m} \hat{w}_{io} x_{io} = 1 \tag{11}$$

Constraint $u^T y_o \leqslant 1$ and the objective function remain unchanged. It is clear that if $(\dot{u}, \dot{w}_o)$ is an optimal solution to (5), then

$$(\eta = \frac{1}{\sum_{i=1}^{m} \dot{w}_{io} x_{io}}, \widehat{w}_{io} = \eta \dot{w}_{io}, u = \dot{u})$$

is a feasible solution to (6) and the objective value corresponding to this solution is $\dot{u}^T y_o^U$. So $\psi_o^* \geqslant CE_o^*$. On the other hand if $(\ddot{u}, \ddot{\widehat{w}}_o, \ddot{\eta})$ is an optimal solution to (6), then $(w = \frac{\ddot{\widehat{w}}_o}{\ddot{\eta}}, u = \ddot{u})$ is a feasible solution to (5) and the objective value corresponding to this feasible solution is $\ddot{u}^T y_o^U = \psi_o^*$. So $\psi_o^* \leqslant CE_o^U$. Now, regarding the above mentioned point and constraints (9)-(11), Models (5) and (6) are equivalent, and the proof is complete. ■

Model (6) has $2m + 1$ constraints more than Model (3), and this is the penalty that we should pay to linearize the upper bound of the cost efficiency model when the input prices are imprecise.
If an optimal solution to Model (6) is $(\hat{u}^*, \hat{w}_o^*, \hat{\eta}^*)$, then we have an optimal solution to Model (5) as: $(u^* = \hat{u}^*, w_o^* = \hat{w}_o^*/\hat{\eta}^*,)$. Note that $\eta > 0$ in all of the feasible solutions of Model (6), because for all i, $\hat{w}_{io} \leqslant \eta w_{io}^U$, and $\sum_{i=1}^{m} \hat{w}_{io} x_{io} = 1$. So, in (6) we just put $\eta \geqslant 0$.

3.2 The pessimistic cost efficiency model

In this subsection we use the following notations. For vectors a^+, a^- defined by $a^+ = \max\{a, 0\}, a^- = \max\{-a, 0\}$, here max is pointwise. We have $a = a^+ - a^-, |a| = a^+ + a^-, a^+ > 0, a^- > 0$. $e = (1, 1, \ldots, 1)^T$ is a vector of all ones. In our description to follow, an important role is played by the set V_m of all ± 1-vectors in R^m; i.e., $V_m = \{v \in R^m : |v| = e\}$. Obviously, the cardinality of V_m is 2^m. For a given vector $v \in R^m$ we denote

$$P_v = diag(v_1, v_2, \ldots, v_m) = \begin{pmatrix} v_1 & 0 & \ldots & 0 \\ 0 & v_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & v_m \end{pmatrix}.$$

An algorithm for generating V_m
It will be helpful at a sequel stage generating all the ± 1-vectors forming set V_m systematically one-by-one in such a way that any two successive vectors differ in exactly one entry.
Algorithm
Set $z := 0 \in R^m$; select $v \in V_m; V_m := \{v\}$
while $z \neq e$
$k = \min\{i : z_i = 0\}$,
for i:= 1 to k-1, $z_i = 0$;end
$z_k := 1; v_k := -v_k;$
$V_m := V_m \cup \{v\};$
end,
$V = V_m$.
Consider $W_o = \{w_o : w_o^L \leqslant w_o \leqslant w_o^U\}$, where $w_o^L, w_o^U \in R^m$. As shown later, in many cases it is more advantageous

to express the data of the input prices in terms of the center vector

$$w_o^c = \frac{1}{2}(w_o^L + w_o^U)$$

and the nonnegative radius vector

$$\chi = \frac{1}{2}(w_o^U - w_o^L)$$

and we employ both forms $W_o = [w_o^L, w_o^U] = [w_o^c - \chi, w_o^c + \chi]$. For an m-dimensional interval vector $W_o = [w_o^c - \chi, w_o^c + \chi]$ we define vectors $w_o^v = w_o^c + P_v\chi$ for each $v \in V_m$. Then for any such v we have

$$(w_o^v)_i = (w_o^c)_i + v_i\chi_i = \begin{cases} w_{io}^L & \text{if } v_i = -1, \\ w_{io}^U & \text{if } v_i = 1. \end{cases}$$

When the input prices are imprecise in the form of intervals, we propose the following model to obtain the pessimistic cost efficiency measure:

$$CE_o^L = \min_{w_o^L \leqslant w_o \leqslant w_o^U,} \min\{\frac{w_o x}{w_o x_o} : \ X\lambda = x, \ Y\lambda \geqslant y_o, \ \ \lambda \geqslant 0\}. \qquad (12)$$

The following theorem gives an explicit formula for computing the pessimistic cost efficiency measure.

Theorem 2. *We have:*

$$\varphi_o^* = \min_{v \in V_m} \{\min \ \frac{w_o^v x}{w_o^v x_o} \qquad (13)$$
$$s.t. \quad \sum_{j=1}^{n} \lambda_j x_j = x, \quad \sum_{j=1}^{n} \lambda_j y_j \geqslant y_o, \quad \lambda_j \geqslant 0, \qquad j = 1, \ldots, n.\}$$

Comment. By using (6), solving only one linear programming problem is needed to evaluate CE^U, whereas up to 2^m LPs are to be solved to compute CE^L by (13).

Proof. It is clear that the set $\{w_o^v : \ v \in V_m\}$ is the set of all extreme points of the bounded set $W_o = \{w_o : \ w_o^L \leqslant w_o \leqslant w_o^U\}$. To prove the theorem, it is sufficient to prove the following assertion: at least one optimal solution of Model (12) occurs at an extreme point of W_o. Assume that Model (12) has an optimal solution, say (w_o^*, λ^*, x^*), and hence, considering the representation theorem (see Theorem 2.1 on p.69 of [1]) we have

$$w_o^* = \sum_{v \in V_m} \mu_v w_o^v, \ \ \sum_{v \in V_m} \mu_v = 1.$$

If there exists a $v \in V_m$ such that

$$\frac{w_o^v x^*}{w_o^v x_o} = \frac{w_o^* x^*}{w_o^* x_o},$$

then the proof is at hand. Now, by contradiction let $\frac{w_o^v x^*}{w_o^v x_o} > \frac{w_o^* x^*}{w_o^* x_o}$, for each $v \in V_m$, then we have $(w_o^v x^*)(w_o^* x_o) > (w_o^v x_o)(w_o^* x^*)$. By multiplying both sides of the above inequality by $\mu_v; v \in V_m$ and summation on $v \in V_m$ we have $\sum_{v \in V_m} \mu_v (w_o^v x^*)(w_o^* x_o) > \sum_{v \in V_m} \mu_v (w_o^v x_o)(w_o^* x^*)$. This in turn implies

$$\frac{w_o^* x^*}{w_o^* x_o} < \frac{\sum_{v \in V_m} \mu_v (w_o^v x^*)}{\sum_{v \in V_m} \mu_v (w_o^v x_o)} = \frac{\sum_{v \in V_m} \mu_v w_o^v x^*}{\sum_{v \in V_m} \mu_v w_o^v x_o} = \frac{w_o^* x^*}{w_o^* x_o}.$$

This is obviously a contradiction, and completes the proof. ■

Remark.

In the current study we considered the estimation of cost efficiency with incomplete input price information, and we provided the pessimistic and optimistic cost efficiency measures with respect to solving a set of linear programming problems. As a complementary discussion we can also study revenue efficiency and profit efficiency (economic efficiency) when the input prices and output prices appear in the form of ranges. This complementary discussion is very similar to the provided results.

4 Conclusions

This study develops a new idea for cost efficiency analysis dealing with interval data. In fact, when the data are imprecise in the form of intervals, the cost efficiency measure calculated from the data should be interval, as well. So, a pair of two-level mathematical programming problems were provided to obtain the pessimistic and optimistic CE measures in cases of uncertainty. The provided models are very easy to understand and convenient to use. The resulting two-level mathematical programs are nonlinear and solving them is difficult. In turn, these programs are transformed into equivalent linear ones. In some cases, the input prices of all DMUs are known exactly but the prices differ from one DMU to another. In such cases, for comparing the performance of all DMUs based on their CE measure, it seems in order if we consider the minimum value of each input price, among all DMUs, as the lower bound of that respective input price; and the maximum value of each input price, among all DMUs, as the upper bound of that respective input price. Now, we can use Models (6) and (13) with exact inputs and outputs to obtain the range of CE, with the same intervals for the input prices.

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