The existence and uniqueness of the solution for uncertain functional differential equations

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Abstract

The uncertain functional differential equations are an important tool to deal with dynamic systems including the past states in uncertain environments. The contribution of this paper to the uncertain functional differential equation theory is to provide an existence and uniqueness theorem under the weak Lipschitz condition and the linear growth condition.

1 Introduction

Most of phenomena and events in the real world occur unexpectedly among which are the changes in economic and political systems, collapse of governments, conflicts between tribes, wars, terrorist attacks. Thus, it is not possible to anticipate or estimate, the price of stocks, valuable papers, monetary units and precious metals accurately. Therefore, the only way find out how this factor can affect the growth or drop in the value of companies is focusing on the price of stocks. Investigation on effects of the factors along with uncertainty theory can help better understanding and more exact modeling of these phenomena. The uncertainty theory was first introduced by Liu who then presented the concept of uncertainty measure which is powerful tool for dealing with uncertain phenomena, to facilitate measuring of uncertain events that are based on normality, monotonicity, self-duality, and maximality axioms.

Then the concept of uncertain process was proposed by Liu, [7], that introduces a particular uncertain process with stationary and independent increment named canonical Liu's process which is just like a stochastic process described by Brownian motion. Since then some literatures have been published on the canonical Liu's process and its applications in other sciences, such as economics and optimal control have been published [27]. Then Liu was inspired by stochastic notions and ito process to introduce uncertain differential equations [10] which were driven by canonical Liu's process for better understanding the uncertain phenomena.

This paper concerns a class of uncertain functional differential equations (UFDEs). A prototype for this class is the equation,

$$dX(t) = f(X_t, t)dt + g(X_t, t)dC(t), \quad t_0 \le t \le T, \tag{1.1}$$

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where $X_t = \{X(t+\theta) \mid -\infty < \theta \leq 0\}$ can be regarded as a $C((-\infty,0],R^d)$-value uncertain process, where $f : C((-\infty,0],R^d) \times [t_0,T]\square \to R^d$ and $g : C((-\infty,0],R^d) \times [t_0,T]\square \to R^{d\times m}$ be Borel measurable, and $x(t)$ is the solution to the Eq. (1.1) and is a function of a uncertain process. Regarding to the importance of existence and uniqueness of a solution to uncertain differential equations driven by canonical Liu's process, Liu investigated the existence and uniqueness of solution to the uncertain differential equations by employing Lipschitz and Linear growth conditions [4], and stability analysis of uncertain differential equations was given by Yao et al.[23]. Many researchers managed to find analytic solutions for some special types of uncertain differential equations such as Chen and Liu [1].

The main goal of this paper is the prove weaker conditions to study of existence and uniqueness of solution to the uncertain functional differential equations. In this regard, we prove a new existence and uniqueness theorem under the weak Lipschitz condition and the linear growth condition.

This paper is arranged as follows: Section 2 is intended to introduce some basic concepts and theorems in uncertainty theory and uncertain differential equation. In section 3, we will focus on the main results including a new existence and uniqueness theorem for the solution of a UFDE under a weak condition. Finally, conclusion is drawn.

2 Preliminaries

This section will briefly introduce some basic concepts in uncertainty theory.

Let $\|\cdot\|$ denote Euclidean norm in R^n. If A is a vector or a matrix, its transpose is denoted by A^T, if A is a matrix, its trace norm is represented by $\|A\| = \sqrt{trace(A^TA)}$.

Definition 2.1 [1]: Let L be a σ-algebra on a nonempty set Γ. A set function $M : L \to [0,1]$ is called an uncertain measure if it satisfies the following axioms:

1. Axiom (Normality) $M\{\Gamma\} = 1$.
2. Axiom (Duality) $M\{A\} + M\{A^c\} = 1$ for any event A.
3. Axiom (Subadditivity) For every countable sequence of events $A_1, A_2, \cdots$, we have

$$M\{\cup_{i=1}^{\infty} A_i\} \leq \Sigma_{i=1}^{\infty} M\{A_i\}$$

Besides, the product uncertain measure on the product σ-algebra L is defined by Liu [10] as follows:

4. Axiom (Product) Let (Γ_k, L_k, M_k) be uncertainty spaces for $k = 1, 2, \cdots$ Then the product uncertain measure M on the product σ-algebra satisfies

$$M\{\Pi_{i=1}^{\infty} A_k\} = \Lambda_{k=1}^{\infty} M_k\{A_k\}$$

where A_k are arbitrarily chosen events from L_k for $k = 1, 2, \cdots$, respectively. In order to represent the quantities with uncertainty, an uncertain variable was proposed as a real valued function on an uncertainty space.

Definition 2.2 [1] : An uncertain variable ξ is a measurable function from an uncertainty space (Γ, L, M) to the set of real numbers with a filtration $\{L_t\}_{t\geq t_0}$ satisfying the usual conditions (i.e. it is right continuous and L_{t_0} contains all M-null sets).

Let T be an index set and let (Γ, L, M) be an uncertainty space. An uncertain process is a measurable function

from $T \times (\Gamma, L, M)$ to the set of real numbers, (i.e., for each $t \in T$ and any Borel set B of real numbers, the set $\{x_t \in B\} = \{\gamma \in \Gamma | x_t(\gamma) \in B\}$ is an event.
Now, let us introduce a special uncertain process called the canonical process, that plays the role of counterpart of Brownian motion.
Definition 2.3 [6]: A uncertain process C_t is said to be a canonical process if:

- (i) $C_0 = 0$,
- (ii) C_t has stationary and independent increments,
- (iii) every increment $C_{t+s} - C_s$ is a normally distributed uncertain variable $\aleph(0, t)$ with expected value 0 and variance t^2 whose uncertainty distribution is

$$\phi(x) = (1 + \ exp\,(\tfrac{\pi x}{\sqrt{3}t}))^{-1}. \quad x \in \Re$$

Based on the canonical process, Liu [6] proposed an uncertain integral of an uncertain process with respect to the canonical process and thus founded a theory of uncertain calculus.
Definition 2.4: [6] Let X_t be an uncertain process and C_t be a canonical process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|$$

then the uncertain integral of X_t with respect to C_t is

$$\int_a^b X_t dC_t = \lim_{\Delta \longrightarrow 0} \sum_{i=1}^{k} X_{t_i}.(C_{i+1} - C_{t_i}) \tag{2.1}$$

provided that the limit exists almost surely and is finite.
Definition 2.5:[6] The uncertainty distribution ϕ of an uncertain variable is defined by
$\phi(x) = M\{\xi \leq x\}$ for any real number x. The expected value of an uncertain variable ξ is

$$E[\xi] = \int_0^{+\infty} M\{\xi \geq r\}dr - \int_{-\infty}^0 M\{\xi \leq r\}dr$$

provided that at least one of the two integrals is finite.
Theorem 2.1(Holder□s Inequality)[15]: Let p and q be two positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$, and ξ and η be independent uncertain variables with $E[\|\xi\|^p] \leq +\infty$ and $E[\|\eta\|^q] \leq +\infty$. We have

$$E[\|\xi\eta\|] \leq \sqrt[p]{E[\|\xi\|^p]}\,\sqrt[q]{E[\|\eta\|^q]}.$$

Theorem 2.2(Chebychev's Inequality):[15] Let $\xi : \theta \rightarrow R^n$ be uncertain variable such that $E[\|\xi\|^p] \leq +\infty$ for some $p, 0 \leq p \leq \infty$.
Then Chebychev's inequality is as follows:

$M[\|\xi\| \geq \lambda] \leq \frac{1}{\lambda^p} E[\|\xi\|^p]\ for\ all\ \lambda \geq 0.$

Definition 2.6:[6] Assume that C(t) is an m-dimensional canonical process defined on uncertainty space, that is $C(t) = (C_1(t), C_2(t), \cdots, C_m(t))^T$.
Definition 2.7:[6] Let $C((-\infty, 0], R^d)$ denote the family of bounded continuous R^d -value functions x defined on $(\infty, 0]$ with norm

$$\| x \| = \sup_{-\infty \le \theta \le 0} \|x(\theta)\|.$$

Notation 1: $\ell^{\mathbf{p}}(\theta, \mathbf{R^d})$ the family of $\mathbf{R^d}$-valued fuzzy variables ξ with $\mathbf{E}|\xi|^{\mathbf{p}} < \infty$.

Notation 2: $\ell^{\mathbf{p}}([a,b], \mathbf{R^d})$ the family of $\mathbf{R^d}$-valued $\mathcal{P}_t$-adapted processes $\{f(t)\}_{a\le t\le b}$ such that $\int_a^b |f(t)|^{\mathbf{p}} \mathrm{d}t < \infty$ almost surely.
Notation 3: $M^{\mathbf{p}}([a,b], \mathbf{R^d})$ the family of processes $\{f(t)\}_{a\le t\le b}$ in $\ell^{\mathbf{p}}([a,b], \mathbf{R^d})$ such that $\int_a^b |f(t)|^{\mathbf{p}} dt < \infty$.
Notation 4: $\ell^{\mathbf{p}}(\mathbf{R}_+, \mathbf{R^d})$ the family of processes $\{f(t)\}_{t>0}$ such that for every $T > 0$, $\{f(t)\}_{a\le t\le T} \in \ell^{\mathbf{p}}([0,T], \mathbf{R^d})$.

3 The existence-and-uniqueness theorem

Consider a d-dimensional uncertain functional differential equations(UFDE)

$$dX(t) = f(X_t, t)dt + g(X_t, t)dC(t), \quad t_0 \le t \le T, \tag{3.1}$$

where $X_t = \{X(t+\theta) \mid -\infty < \theta \le 0\}$ can be regarded as a $C((-\infty,0], R^d)$-value uncertain process, where $f : C((-\infty,0], R^d) \times [t_0, T] \to R^d$ and $g : C((-\infty,0], R^d) \times [t_0, T]\square \to R^{d\times m}$ be Borel measurable.
First, we propose the definition of the solution of (2.3), then existence, uniqueness theorem is brought.

Definition 3.1: An R^n-value uncertain process X_t on $t_0 \le t \le T$ is called a solution to equation (2.3) with initial data (2.4) if it has the following properties:

1. it is continuous and $\{X_t\}_{t_0\le t\le T}$ is P_t-adapted
2. $\int_{t_0}^T |f(X,t)|dt < \infty$ and $\int_{t_0}^T |g(X,t)|dt < \infty$
3. $X_{t_0} = \beta$ and, for every $t_0 \le t \le T$,

$$X(t) = \beta(0) + \int_{t_0}^t f(X_s, s)ds + \int_{t_0}^t g(X_s, s)dC(s) \qquad a.s.$$

We consider two conditions A_1 and A_2 as follows

- A_1: (uniform Lipschitz condition) For any $x, y \in C((-\infty,0], R^d)$ and $t \in [t_0, T]$, it follows

$$|f(x,t) - f(y,t)|^2 \vee |g(x,t) - g(y,t)|^2 \le \overline{K}\|x-y\|^2; \tag{3.2}$$

- A_2: For any $t \in [t_0, T]$, it follows that $f(0,t), g(0,t) \in L^2$ such that

$$|f(0,t)|^2 \vee |g(0,t)|^2 \le K. \tag{3.3}$$

where $\overline{K}$ and K are two positive numbers.

Lemma 3.1. If $p \ge 2$, $g\square \in M^2([t_0, T], R^{d\square\times m})$ such that

$$E\int_{t_0}^T |g(s)|^p ds < \infty,$$

then

$$E|\int_{t_0}^T g(s)dC(s)|^p \le (\tfrac{p(p-1)}{2})^{\frac{p}{2}} T^{\frac{p-2}{2}} E\int_{t_0}^T |g(s)|^p ds$$

Lemma 3.2: Let A_1 is hold. If $X(t)$ is the solution of (2.3) with initial data (2.4), then

$$E(\sup_{-\infty<t\leq T}|X(t)|)^2 \leq E\|\xi\|^2 + Ce^{6\overline{K}(T-t_0+1)(T-t_0)} \tag{3.4}$$

$$\text{where } C = 3E\|\xi\|^2 + 6(T-t_0+1)(T-t_0)(K + \square\overline{K}E\|\xi\|^2).$$

In addition, $X(t) \in M^2((-\infty,T],R^d)$.
Proof: For each number $n \geq 1$, define the stopping time

$$\tau_n = T \wedge \inf\{t \in [t_0,T] : \|X_t\| \geq n\}.$$

Obviously, as $n\square \to \square\infty, \tau_n \uparrow T a.s.$ Let $X^n(t) = X(t\wedge\tau_n), t \in [t_0,T]$. Then $X^n(t)$ satisfy the following equation

$$X^n(t) = \xi(0) + \int_{t_0}^t f(X_s^n,s)I_{[t_0,\tau_n]}(s)ds + \int_{t_0}^t g(X_s^n,s)I_{[t_0,\tau_n]}(s)dC(s)$$

by the elementary inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, one gets

$$|X^n(t)|^2 \leq 3|\xi(0)|^2 + 3|\int_{t_0}^t f(X_s^n,s)I_{[t_0,\tau_n]}(s)ds|^2 + 3|\int_{t_0}^t g(X_s^n,s)I_{[t_0,\tau_n]}(s)dC(s)|^2 \tag{3.5}$$

Taking the expectation on both sides, and by the Holder inequality and (3.5) thus we have

$$E|X^n(t)|^2 \leq 3E|\xi(0)|^2 + 3E|\int_{t_0}^t f(X_s^n,s)I_{t_0,\tau_n}(s)ds|^2 + 3E|\int_{t_0}^t g(X_s^n,s)I_{[t_0,\tau_n]}(s)dC(s)|^2$$

$$\leq 3E\|\xi\|^2 + 3(t-t_0)E\int_{t_0}^t |f(X_s^n,s)|^2 ds + 3E\int_{t_0}^t |g(X_s^n,s)|^2 I_{[t_0,\tau_n]}(s)ds.$$

One further obtains that

$$E((\sup_{t_0<s\leq t}|X^n(t)|^2) \leq 3E\|\xi\|^2 + 3(t-t_0)E\int_{t_0}^t |f(X_s^n,s)|^2 ds|^2 + 3E|\int_{t_0}^t 2|g(X_s^n,s)|^2 d(s)$$

$$\leq 3E\|\xi\|^2 + 6(t-t_0+1)E\int_{t_0}^t (\overline{K}\|X_s^n\|^2 + K)ds$$

$$\leq C_1 + 6\overline{K}(T-t_0+1)\int_{t_0}^t E(\|\xi\|^2 + \sup_{t_0<r\leq s}|X_r^n|^2)ds$$

$$\leq C_2 + 6\overline{K}(T-t_0+1)\int_{t_0}^t E(\sup_{t_0<r\leq s}|X_r^n|^2)ds,$$

where $C_1 = 3E\|\xi\|^2 + 6(t-t_0+1)(T-t_0)$, $C_2 = C_1 + 6\overline{K}(T-t_0+1)(T-t_0)E\|\xi\|^2$,
By the Gronwall inequality,

$$E(\sup_{t_0<s\leq t}|X^n(s)|^2) \leq Ce^{6\overline{K}(T-t_0+1)(T-t_0)}, \qquad t_0 \leq t \leq T.$$

Noting the fact that

$$(\sup_{-\infty<t\leq T}|X(s)|)^2 \leq \|\xi\|^2 + (\sup_{t_0<s\leq t}|X(s)|)^2,$$

therefore

$$E(\sup_{-\infty<s\leq t}|X^n(s)|^2) \leq E\|\xi\|^2 + E(\sup_{t_0<s\leq t}|X^n(s)|^2)$$

$$\leq E\|\xi\|^2 + Ce^{6\overline{K}(T-t_0+1)(T-t_0)}.$$

Letting $t = T$, it then follows that

$$E(\sup_{-\infty<s\leq T}|X^n(s)|^2) \leq E\|\xi\|^2 + Ce^{6\overline{K}(T-t_0+1)(T-t_0)},$$

that is

$$E(\sup_{-\infty<s\leq T}|X(s\wedge\tau_n)|^2) \leq E\|\xi\|^2 + Ce^{6\overline{K}(T-t_0+1)(T-t_0)}.$$

Consequently

$$E(\sup_{-\infty<s\leq \tau_n}|X(s)|^2) \leq E\|\xi\|^2 + Ce^{6\overline{K}(T-t_0+1)(T-t_0)}.$$

Letting $n\Box \to \infty$, it then implies the following inequality

$$E(\sup_{-\infty<s\leq T}|X(s)|^2) \leq E\|\xi\|^2 + Ce^{6\overline{K}(T-t_0+1)(T-t_0)} \quad \Box$$

Theorem 3.1: Let conditions A_1 and A_2 are holds. Then initial value problem (2.3)-(2.4) has a unique solution $X(t)$. Moreover, $X(t) \in M^2((-\infty, T], R^d)$.

Proof. Let X_t and $\Box\overline{X}_t$ are solutions of equation (2.3),
put $a(s,w) = f(s,X_s) - f(s,\overline{X}_s)$ and $b(s,w) = g(s,X_s) - g(s,\overline{X}_s)$ where $w \in \theta$. Then

$$X_t - \overline{X}_t = \int_{t_0}^t ads + \int_{t_0}^t bdC_s.$$

Using Hölder inequality and Lipschitz condition, we obtain

$$|X_t - \overline{X}_t|^2 \leq 2|\int_{t_0}^t ads|^2 + 2|\int_{t_0}^t bdC_s|^2 \leq 2(t-t_0)\int_{t_0}^t K|X_s - \overline{X}_s|^2 ds + 2|\int_{t_0}^t bdC_s|^2.$$

Thus, we get

$$\sup_{t_0\leq s\leq t}|X_s - \Box\overline{X}_s|^2 \leq 2K(T-t_0)\int_{t_0}^t |X_s - \Box\overline{X}_s|^2 ds + 2\sup_{t_0\leq s\leq t}|\int_{t_0}^t bdC_s|^2.$$

Taking the expectation and noting Doob inequality we may deduce that

$$E(\sup_{t_0\leq s\leq t}|X_s - \overline{X}_s|^2) \leq 2K(T+4)\int_{t_0}^t E(\sup_{t_0\leq r\leq s}|X_r - \overline{X}_r|^2)ds.$$

According to Gronwall inequality, we have

$$E(\sup_{t_0\leq t\leq T}|X_t - \overline{X}_t|^2) = 0. \tag{3.6}$$

The above expression means that $X(t) = \Box\overline{X}_t$ for $t_0 \leq t \leq T$.
Therefore, for all $-\infty < t \leq T, X_t = \Box\overline{X}_t$a.s. The proof of uniqueness is complete.
Next to check the `existence`, define $X^0_{t_0} = \xi$ and $X^0(t) = \xi(0)$, for $t_0 \leq t \leq T$.
Let $X^n_{t_0} = \xi$, $n = 1, 2, \cdots$, and define Picard sequence

$$X^n_t = \xi_0 + \int_{t_0}^t f(X^{n-1}_s, s)ds + \int_{t_0}^t g(X^{n-1}_s, s)dC(s). \tag{3.7}$$

Clearly $X^0(t) \in M((-\infty, T], R^d)$. By induction $X^n(t) \in M^2((-\infty, T], R^d)$. In fact

$$|X^n(t)|^2 \leq 3|\xi(0)|^2 + 3|\int_{t_0}^t f(X^{n-1}_s, s)ds|^2 + 3|\int_{t_0}^t g(X^{n-1}_s, s)dC(s)|^2. \tag{3.8}$$

From the Hölder inequality, we have

$$E|X^n(t)|^2 \leq 3E|\xi(0)|^2 + 3E|\int_{t_0}^t f(X_s^{n-1},s)ds|^2 + 3E|\int_{t_0}^t g(X_s^{n-1},s)dC(s)|^2$$

$$\leq 3E\|\xi\|^2 + 3(t-t_0)E\int_{t_0}^t |f(X_s^{n-1},s) - f(0,s) + f(0,s)|^2 ds$$

$$+3E\int_{t_0}^t |g(X_s^{n-1},s) - g(0,s) + g(0,s)|^2 ds.$$

Again the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$ and (3.6) imply that

$$E|X^n(t)|^2 \leq 3E\|\xi\|^2 + 3(t-t_0)E\int_{t_0}^t 2|f(X_s^{n-1},s) - f(0,s)|^2 + 2|f(0,s)|^2 ds$$

$$+3E\int_{t_0}^t 2|g(X_s^{n-1},s) - g(0,s)|^2 + 2|g(0,s)|^2)ds$$

$$\leq 3E\|\xi\|^2 + 3(t-t_0+1)E\int_{t_0}^t (2\overline{K}\|X_s^{n-1}\|^2 + 2K)ds$$

$$\leq C_1 + 6\overline{K}(T-t_0+1)\int_{t_0}^t E(\sup_{t_0<r\leq s}|X_r^{n-1}|^2)ds$$

$$\leq C_1 + 6\overline{K}(T-t_0+1)\int_{t_0}^t E(|X_s^{n-1}|^2)ds,$$

where $C_1 = 3E\|\xi\|^2 + 6(t-t_0+1)(T-t_0)$. Hence for any $k \geq 1$, one can derive that

$$\max_{1\leq n\leq k} E|X^n(t)|^2 \leq C_1 + 6\overline{K}(T-t_0+1)\int_{t_0}^t \max_{1\leq n\leq k} E(|X_s^{n-1}|^2)ds.$$

Note that

$$\max_{1\leq n\leq k} E|X^{n-1}(s)|^2 = \max\{E|\xi(0)|^2, E|X^1(s)|^2, \cdots E|X^{k-1}(s)|^2\}$$

$$\leq \max\{E\|\xi\|^2, E|X^1(s)|^2, \cdots E|X^{k-1}(s)|^2, E|X^k(s)|^2\}$$

$$= \max\{E\|\xi\|^2, \max_{1\leq n\leq k} E|X^n(s)|^2\}$$

$$\leq E\|\xi\|^2 + \max_{1\leq n\leq k} E|X^n(s)|^2,$$

therefore

$$\max_{1\leq n\leq k} E|X^n(t)|^2 \leq C_1 + 6\overline{K}(T-t_0+1)\int_{t_0}^t (E\|\xi\|^2 + \max_{1\leq n\leq k} E(|X_s^n|^2))ds$$

$$\leq C_2 + 6\overline{K}(T-t_0+1)E\int_{t_0}^t E(\max_{1\leq n\leq k}|X_s^n|^2)ds,$$

where $C_2 = C_1 + 6\overline{K}(T-t_0+1)(T-t_0)E\|\xi\|^2$.

From the Gronwall inequality, one gets that

$$\max_{1\leq n\leq k} E|X^n(s)|^2 \leq C_2 e^{6\overline{K}(T-t_0+1)(T-t_0)}$$

Since k is arbitrary,

$$E|X^n(s)|^2 \leq C_2 e^{6\overline{K}(T-t_0+1)(T-t_0)}, \qquad t_0 \leq t \leq T \qquad n \geq 1 \tag{3.9}$$

From the Hölder inequality and (3.6), one then has

$$E|X^1(t)-X^0(t)|^2 \leq 2E|\int_{t_0}^t f(X_s^0,s)ds|^2 + 2E|\int_{t_0}^t g(X_s^0,s)dC(s)|^2$$

$$\leq 2(t-t_0)E\int_{t_0}^t |f(X_s^0,s)|^2ds + E\int_{t_0}^t |g(X_s^0,s)|^2ds$$

$$\leq 2(t-t_0+1)E\int_{t_0}^t (2\overline{K}\|X_s^0\|^2+2K)ds$$

$$\leq 4K(t-t_0+1)(t-t_0) + 4\overline{K}(t-t_0+1)(t-t_0)E\|\xi\|^2,$$

that is

$$E(\sup_{t_0\leq s\leq t}|X^1(t)-X^0(t)|^2) \leq 4K(t-t_0+1)(t-t_0) + 4\overline{K}(t-t_0+1)(t-t_0)E\|\xi\|^2.$$

Taking $t = T$, then

$$E(\sup_{t_0\leq s\leq t}|X^1(t)-X^0(t)|^2) \leq 4K(T-t_0+1)(T-t_0) + 4\overline{K}(T-t_0+1)(T-t_0)E\|\xi\|^2 := C.$$

By the same ways as above, we compute

$$E|X^1(t)-X^0(t)|^2 \leq 2E|\int_{t_0}^t [f(X_s^1,s)-f(X_s^0,s)]ds|^2$$

$$+2E|\int_{t_0}^t [g(X_s^1,s)-g(X_s^0,s]dC(s)|^2$$

$$\leq 2(t-t_0)E\int_{t_0}^t |f(X_s^1,s)-f(X_s^0,s)|^2ds + E\int_{t_0}^t |g(X_s^1,s)-g(X_s^0,s)|^2ds$$

thus we derive that

$$E(\sup_{t_0\leq r\leq s}|X^2(t)-X^1(t)|^2) \leq ME\int_{t_0}^t \|X^1(s)-X^0(s)\|^2ds$$

$$\leq M\int_{t_0}^t E(\sup_{t_0\leq r\leq s}|X^1(r)-X^0(r)|^2)ds \leq M(t-t_0)C,$$

where $M = 2\overline{K}(T-t_0+1)$. Similarly,

$$E(\sup_{t_0\leq s\leq t}|X^3(t)-X^2(t)|^2) \leq M\int_{t_0}^t E(\sup_{t_0\leq r\leq s}|X^2(r)-X^1(r)|^2)ds$$

$$\leq M\int_{t_0}^t M(s-t_0)Cds = \frac{C[M(t-t_0)]^2}{2},$$

continuing this process to find that

$$E(\sup_{t_0\leq s\leq t}|X^4(t)-X^3(t)|^2) \leq M\int_{t_0}^t E(\sup_{t_0\leq s\leq t}|X^3(r)-X^2(r)|^2)ds$$

$$\leq M\int_{t_0}^t \frac{[M(s-t_0)]^2C}{2}ds = \frac{C[M(t-t_0)]^3}{6}.$$

Now we claim that for all $n \geq 0$,

$$E(\sup_{t_0\leq s\leq t}|X^{n+1}(s)-X^n(s)|^2) \leq \frac{C[M(t-t_0)]^n}{n!} \qquad t_0\leq t\leq T. \tag{3.10}$$

When $n = 0,1,2,3,$ inequality (3.12) holds. We suppose that (3.12) holds for some n, now to check (3.12) for $n+1$. In fact,

$$E(\sup_{t_0\le s\le t}|X^{n+2}(s)-X^{n+1}(s)|^2)\le 2\overline{K}(t-t_0+1)\int_{t_0}^{t}E\|X^{n+1}(s)-X^n(s)\|^2ds$$

$$\le M\int_{t_0}^{t}E(\sup_{t_0\le r\le s}|X^{n+1}(r)-X^n(r)|^2)ds$$

By induction and (3.12),

$$E(\sup_{t_0\le s\le t}|X^{n+2}(s)-X^{n+1}(s)|^2)\le M\int_{t_0}^{t}\frac{[M(s-t_0)]^nC}{n!}ds=\frac{C[M(t-t_0)]^{n+1}}{(n+1)!}.$$

It is easy to see that (3.12) holds for $n+1$. Therefore, by induction, (3.12) holds for all $n\ge 0$.
Next to verify $X^n(t)$ converge to $X(t)$ at the sense of L^2 and uncertainty on $M^2((-\infty,T],R^d)$, moreover, $X(t)$ is the solution of (2.3) with initial data (2.4). For (3.12), taking $t=T$,

$$E(\sup_{t_0\le t\le T}|X^{n+1}(t)-X^n(t)|^2)\le\frac{C[M(T-t_0)]^n}{(n)!}.$$

By the Chebyshev inequality,

$$M\{\sup_{t_0\le t\le T}|X^{n+1}(t)-X^n(t)|^2>\frac{1}{2^n}\}\le\frac{C[4M(T-t_0)]^n}{(n)!}.$$

From the fact $\sum_{n=0}^{\infty}C[4M(T-t_0)]^n/n!<\Box\infty$, and by the Borel-Cantelli lemma, for almost all $w\in\Omega\Box$, there exists a positive integer $n_0=n_0(w)$ such that

$$\sup_{t_0\le t\le T}|X^{n+1}(t)-X^n(t)|^2\le\frac{1}{2^n}\quad as\quad n\ge n_0.$$

$$X^0(t)+\sum_{i=1}^{n}[X^i(t)-X^{i-1}(t)]=X^n(t)$$

is the partial sum of function series

$$X^0(t)+[X^1(t)-X^0(t)]+\cdots+[X^n(t)-X^{n-1}(t)]+\cdots \tag{3.11}$$

by the second item of series (3.13), the absolute value of every item of (3.13) is less than the corresponding item of positive series

$$1+\frac{1}{2}+\cdots+\frac{1}{2^n}+\cdots,$$

moreover, the positive series is convergent, further, series (3.13) is convergent on $(-\infty,T]$, furthermore, it is uniformly convergent on $(-\infty,T]$. Let sum function be $X(t)$, therefore approximate sequence $X^n(t)$ uniformly converge to $X(t)$ on $(-\infty,T]$. Since $X^n(t)$ is continuous on $(-\infty,T]$ and F_t- adapted, hence $X(t)$ is also continuous and F_t adapted.
On the other hand, (3.12) implies that for each t , sequence $\{X^n(t)\}$ is also a Cauchy sequence in L^2. Hence, as $n\Box\to\infty, X^n(t)\to^{L^2}X(t)$, that is $E|X^n(t)-X(t)|^2\to 0$. Letting $n\Box\to\infty$ in (3.11) then yields that

$$E|X^n(s)|^2\le C_2e^{6\overline{K}(T-t_0+1)(T-t_0)}\quad for\ \ all\ \ t_0\le t\le T,$$

where $C_2=C_1+6\overline{K}(T-t_0+1)(T-t_0)E\|\xi\|^2$.
Therefore, by use of the above result, we obtain that

$$E\int_{-\infty}^{T}|X(s)|^2ds=E\int_{-\infty}^{t_0}|X(s)|^2ds+E\int_{t_0}^{T}|X(s)|^2ds$$

$$\le E\int_{-\infty}^{0}|\xi(s)|^2ds+\int_{t_0}^{T}C_2e^{6\overline{K}(T-t_0+1)(T-t_0)}ds<\infty,$$

that is $X(t) \in M^2((-\infty, T], R^d)$. Now to show that $X(t)$ satisfy (2.1).

$$E|\int_{t_0}^t [f(X_s^n, s) - f(X_s, s)]ds|^2 + E|\int_{t_0}^t [g(X_s^n, s) - g(X_s, s)]dC(s)|^2$$

$$\leq 2E|\int_{t_0}^t [f(X_s^n, s) - f(X_s, s)]ds|^2 + 2E|\int_{t_0}^t [g(X_s^n, s) - g(X_s, s)]dC(s)|^2$$

$$\leq 2(t-t_0)E|\int_{t_0}^t [f(X_s^n, s) - f(X_s, s)]ds|^2 + 2E|\int_{t_0}^t [g(X_s^n, s) - g(X_s, s)]dC(s)|^2$$

$$\leq ME\int_{t_0}^t \|X_s^n - X_s\|^2 ds \leq M\int_{t_0}^t E(\sup_{t_0 \leq r \leq s} |X_r^n - X_r|^2)ds$$

$$\leq M\int_{t_0}^T E(|X_s^n - X_s|^2)ds.$$

Noting that sequence $X^n(t)$ is uniformly converge on $(-\infty, T]$, it means that for any given $\epsilon > 0$, there exists an n_0 such that as $n \leq n_0$, for any $t \in (-\infty, T]$, one then deduces that $E(|X_t^n - X_t|^2 \leq \epsilon$, further,

$$\int_{t_0}^T E(|X_s^n - X_s|^2)ds < (T-t_0)\epsilon.$$

In other words, for $t \in [t_0, T]$ one has

$$\int_{t_0}^t f(X_s^n, s)ds \to^{L^2} \int_{t_0}^t f(X_s, s)ds, \qquad \int_{t_0}^t g(X_s^n, s)dCs \to^{L^2} \int_{t_0}^t g(X_s, s)dCs$$

For $t_0 \leq t \leq T$, taking limits on both sides of (3.10),

$$\lim_{n\to\infty} X^n(t) = \xi(0) + \lim_{n\to\infty}\int_{t_0}^t f(X_s^{n-1}, s)ds + \lim_{n\to\infty}\int_{t_0}^t g(X_s^{n-1}, s)dCs$$

that is

$$X(t) = \xi(0) + \int_{t_0}^t f(X_s, s)ds + \int_{t_0}^t g(X_s, s)dCs \qquad t_0 \leq t \leq T$$

The above expression demonstrates that $X(t)$ is the solution of (2.3). So far, the existence of theorem is complete.■

4 Conclusion

In this paper, the existence and uniqueness of uncertain functional differential equations with infinite delay are discussed by using uncertain space axioms. In this work, existence and uniqueness theorem under the weak Lipschitz condition and the linear growth condition is proved.

References

[1] X. Chen, X. Qin. A new existence and uniqueness theorem for fuzzy differential equations. International Journal of Fuzzy Systems., 2013, 13(2): 148-151.

[2] X. Chen, D. A. Ralescu. Liu process and uncertain calculus. J. Uncertain. Anal. Appl., 2013, 3.

[3] W. Fei. Uniqueness of solutions to fuzzy differential equations driven by Lius process with non-Lipschitz coefficients. Internatinal Confrance On Fuzzy and Knowledge Discovery., 2009: 565-569.

[4] Y. Gao. Existence and uniqueness theorem on uncertain differential equation with local Lipschitz codition. . Uncertain Syst., 2012, 6(3): 223-232.

[5] J. Gao. Credibilistic Option Pricing. Journal of Uncertain Systems., 2008, 2(4): 243-247.

[6] X. Li, B. Liu. A suffcient and necessary condition for credibility measures. International Journal of Uncertainty, Fuzziness and Knowledge Based Systems., 2004, 14(5): 527-535.

[7] B. Liu. Uncertainty Theory. Springer-Verlag. Berlin., 2004.

[8] B. Liu. A survey of credibility theory. J. Fuzzy Optimization and Decision Making., 2006, 5(4): 387-408.

[9] B. Liu. Uncertainty Theory, 2nd ed. Springer-Verlag. Berlin., 2007.

[10] B. Liu. Fuzzy process, hybrid process and uncertain process. Journal of Uncertain Systems., 2008, 2(1): 3-16.

[11] B. Liu, Some research problems in uncertainty theory. J. Uncertain Syst., 2009, 3(1): 3-10.

[12] B. Liu. Uncertainty Theory. A Branch of Mathematics for Modeling Human Uncertainty., Springer-Verlag, Berlin, 2010.

[13] B. Liu. Why is there a need for uncertainty theory?. J. Uncertain Syst., 2012, 6(1): 3-10.

[14] B. Liu. Toward uncertain finance theory.J. Uncertain. Anal. Appl., 2013.

[15] B. Liu. Uncertainty Theory, 4th edition. Springer-Verlag, Berlin., 2015.

[16] H. Liu, H. Ke, W. Fei. Almost sure stability for uncertain differential equation. Fuzzy Optim. Decis. Mak., 2014, 13(4): 463-473.

[17] Y. Liu.An analytic method for solving uncertain differential equations. J. Uncertain Syst., 2012, 6(4): 244-249.

[18] Z. Peng, K. Iwamura. A sufficient and necessary condition of uncertainty distribution. J. Interdiscip. Math., 2010, 13(3): 277-285.

[19] Y. Shen, K. Yao. Runge-Kutta method for solving uncertain differential equations. Journal of Uncertainty Analysis and Applications., 2015, 2(1): 3-17.

[20] X. Wanga,Y. Tauqir, A. Moughal, X. Chena. Adams Simpson method for solving uncertain differential equation., Applied Mathematics and Computation., 2015, 271: 209-219.

[21] X. Yang, Dan A. Ralescu. Adams method for solving uncertain differential equations. Applied Mathematics and Computation., 2015, 270: 993-1003.

[22] K. Yao, X. Chen. A numerical method for solving uncertain differential equations, J. Intell. Fuzzy Syst., 2013, 25(3): 825-832.

[23] K. Yao, J. Gao, Y. Gao. Some stability theorems of uncertain differential equation. Fuzzy Optim.,2013, 12(1): 3-13. C. You. Multi-dimensional Liu process, differential and integral. Proceedings of the First Intelligent Computing Conference. China., 2007: 153-158.

[24] K. Yao. A type of uncertain differential equations with an alytic solution. J. Uncertain. Anal. Appl., 2013, 8(1).

[25] K. Yao, H. Ke, Y. Sheng. Stability in mean for uncertain differential equation. Fuzzy Optim. Decis. Mak., 2015, 14(3): 365-379.

[26] C. You, W. Wang, H. Huo. Existence and Uniqueness Theorems for Fuzzy Differential Equations. Journal of Uncertain Systems., 2013, 7(4): 303-315.

[27] Y. Zhu. Uncertain optimal control with application to a portfolio selection model, Cybernet. Syst., 2010, 41(7): 535-547.