Theory of Approximation and Applications, 2020, Volume 14, No. 1: 27-43, DOR: 20.1001.1.25382217.2020.14.1.5.7

Simulation Functions and Interpolative Contractions

Andreea Fulga

Department of Mathematics and Computer Science, Transilvania University of Brasov, Brasov, Romania

Article Info

Keywords
Metric spaces
fixed point
simulation function.

Article History
Received: 2019 June 15
Accepted: 2020 February 11

Abstract

In this manuscript, we consider the interpolative contractions mappings via simulation functions in the setting of complete metric space. We also express an illustrative example to show the validity of our presented results.

1 Introduction and Preliminaries

In this section, we will sum up some basic notations, concepts and definitions, which we will use later on.

Definition 1.1. *[15] A mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:*

((ζ_1)) $\zeta(0,0) = 0$*;*

(ζ_2) $\zeta(u,v) < v - u$ *for all* $u, v > 0$*;*

(ζ_3) *if* $\{u_n\}, \{v_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n > 0$*, then*

$$\limsup_{n\to\infty} \zeta(u_n, v_n) < 0. \tag{1.1}$$

is called simulation function.

We denote by $\mathcal{Z}$ the family of all simulation functions $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$. In [7], observing that in fact in the proof of the main result in [15] the presumption (ζ_1) was not used they proposed a slightly modified simulation function definition by removing the condition (ζ_1). So the following notion can be used:

Definition 1.2. *[7] A simulation function is a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the conditions (ζ_2) and (ζ_3).*

Certainly, the class of simulation functions in the sense of Definition 1.2 is wider than the class of simulation functions in the original sense. To illustrate this Argoubi *et all* gave the following example.

* Corresponding Author's E-mail: afulga@unitbv.ro

Example 1.1. *[7] Let $k \in (0,1)$ and $\zeta_k : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be the function defined by*

$$\zeta_k(u,v) = \begin{cases} 1 & \text{if } (u,v) = (0,0) \\ k\ v - u, & \text{otherwise}. \end{cases}$$

Then ζ_k satisfies , (ζ_2) and (ζ_3), but $\zeta_k(0,0) = 1 > 0$.

Later, the family of all simulation functions was again enlarged. In [21], the authors have observed that the third condition is symmetric in both arguments of ζ which is not necessary in proofs. So, they proposed a refinement of this notion.

Definition 1.3. *[21] A mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:*

((ζ_1)) $\zeta(0,0) = 0$*;*

(ζ_2) $\zeta(u,v) < v - u$ *for all* $u, v > 0$*;*

(ζ_3) *if $\{u_n\}, \{v_n\}$ are sequences in $(0,\infty)$ such that $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} v_n > 0$, and $u_n < v_n$ for all $n \in \mathbb{N}$, then* $\limsup_{n\to\infty} \zeta(u_n, v_n) < 0$.

is called simulation function.

In order to illustrate that every simulation function in the original Khojasteh *et al.*'s sense (Definition 1.1) is a simulation function in sense of (Definition 1.4), but the converse is not true, they proposed the following example.

Example 1.2. *[21] The function $\zeta_k : [0,\infty) \times [0,\infty) \to \mathbb{R}$ defined by*

$$\zeta_k(u,v) = \begin{cases} 2(v-u) & \text{if } v < u \\ k\ v - u, & \text{otherwise} \end{cases}$$

where $k \in (0,1)$, verifies (ζ_1) and (ζ_2). Plus, if $\{u_n\}$, $\{v_n\}$ are sequences in $(0,\infty)$ such that

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = L > 0 \text{ and } u_n < v_n \text{ for all } n \in \mathbb{N},$$

then

$$\limsup_{n\to\infty} \zeta(u_n, v_n) = \limsup_{n\to\infty} (kv_n - u_n) = (k-1)L < 0.$$

On the other hand, considering $u_n = 2$ and $v_n = 2 - \frac{1}{n}$, we have for $n \geq 1$:

$$\zeta_k(u_n, v_n) = \zeta_k\left(2, 2 - \frac{1}{n}\right) = 2\left(2 - \frac{1}{n} - 2\right) = \frac{-2}{n}.$$

Since $\limsup\limits_{n\to\infty} \zeta_k(u_n, v_n) = 0$, we can conclude that ζ does not verify axiom (ζ_3) in Definition 1.1.

For some examples of simulation functions, see e.g.([15, 21, 4]).

Concluding, we will use in our later considerations the simulation function in the sense of the following definition:

Definition 1.4. *[20] A mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:*

(ζ_1) $\zeta(u,v) < v - u$ *for all* $u, v > 0$*;*

(ζ_2) *if* $\{u_n\}, \{v_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n > 0$*, and* $u_n < v_n$ *for all* $n \in \mathbb{N}$*, then*

$$\limsup_{n\to\infty} \zeta(u_n, v_n) < 0. \tag{1.2}$$

is called simulation function.

Definition 1.5. *[17] Let $T : \mathcal{X} \to \mathcal{X}$ be a mapping and $\alpha : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ be a function. We say that T is α-orbital admissible if*

$$\alpha(\nu, T\nu) \geq 1 \Rightarrow \alpha(T\nu, T^2\nu) \geq 1.$$

If the additional condition

$$\alpha(\nu,\omega) \geq 1 \text{ and } \alpha(\omega, T\omega) \geq 1 \Rightarrow \alpha(\nu, T\omega) \geq 1$$

is fulfilled, then the α-admissible mapping T is called triangular α-orbital admissible.

Remark 1.1. *The concept of α-orbital admissible was suggested by Popescu [17] and is a refinement of the alpha-admissible notion, defined in [22, 14].*

We can notice that each α-admissible mapping is α-orbital admissible. For more details and counter examples, see e.g. [1, 2, 3, 5, 6, 9, 17].

Definition 1.6. *A set $\mathcal{X}$ is regular with respect to mapping $\alpha : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ if $\{\nu_n\}$ is a sequence in $\mathcal{X}$ such that $\alpha(\nu_n, \nu_{n+1}) \geq 1$, for all n and $\nu_n \to \nu \in \mathcal{X}$ as $n \to \infty$, then $\alpha(\nu_n, \nu) \geq 1$ for all n.*

The notion of α-admissible $\mathcal{Z}$-contraction with respect to a given simulation function was introduced by Karapinar in [12]. Using this new type of contractive mapping he investigated the existence and uniqueness of a fixed point in standard metric space.

Definition 1.7. *[12] Let T be a self-mapping defined on a metric space $(\mathcal{X}, d)$. If there exist a function $\zeta \in \mathcal{Z}$ and $\alpha : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ such that*

$$\zeta\left(\alpha(\nu,\omega) d(T\nu, T\omega), d(\nu,\omega)\right) \geq 0 \text{ for all } \nu, \omega \in \mathcal{X}, \tag{1.3}$$

then we say that T is an α-admissible $\mathcal{Z}$-contraction with respect to ζ.

Theorem 1.1. *[12] Let $(\mathcal{X}, d)$ be a complete metric space and let $T : \mathcal{X} \to \mathcal{X}$ be an α-admissible $\mathcal{Z}$-contraction with respect to ζ. Suppose that:*

(i) T is triangular α-orbital admissible;

(ii) there exists $\nu_0 \in \mathcal{X}$ such that $\alpha(\nu_0, T\nu_0) \geq 1$;

(iii) T is continuous.

Then there exists $\nu_* \in \mathcal{X}$ *such that* $T\nu_* = \nu_*$.

Remark 1.2. *The continuity condition from Theorem 1.1 can be replaced by the "regularity"☐condition which is considered in Definition 1.6.*

Definition 1.8. *(see [11]) Let* $(\mathcal{X}, d)$ *be a metric space and* $T : \mathcal{X} \to \mathcal{X}$ *be a mapping.*

(i) T *is orbitally continuous if*

$$\lim_{i\to\infty} T^{n_i}\nu = \nu \tag{1.4}$$

implies

$$\lim_{i\to\infty} TT^{n_i}\nu = T\nu \tag{1.5}$$

for each $\nu \in \mathcal{X}$.

(ii) $(\mathcal{X}, d)$ *is orbitally complete if every Cauchy sequence of type* $\{T^{n_i}\nu\}_{i\in\mathbb{N}}$ *converges.*

Lastly, we recall the following lemma which is a standard argument to prove that a given sequence is Cauchy.

Lemma 1.1. *(See e.g. [20]) Let* $\{\nu_n\}$ *be a sequence in a metric space* $(\mathcal{X}, d)$ *such that* $\lim_{n\to\infty} d(\nu_{n-1}, \nu_n) = 0$. *If* $\{\nu_n\}$ *is not a Cauchy sequence, then there exist an* $\varepsilon > 0$ *and the sequences* $\{n_i\}$ *and* $\{m_i\}$, *with* $n_i > m_i > i$ *of positive integers such that the following sequences tend to* ε *when* $i \to \infty$*:*

$$d(\nu_{n_i}, \nu_{m_i}), d(\nu_{n_i+1}, \nu_{m_i+1}), d(\nu_{n_i-1}, \nu_{m_i}), d(\nu_{n_i}, \nu_{m_i-1}), d(\nu_{n_i-1}, \nu_{m_i-1})$$

In [13] Karapinar introduced the notion of the interpolative Hardy-Rogers type $\mathcal{Z}$*-contraction* as follows:

Definition 1.9. *[13] Let* T *be a self-mapping defined on a metric space* $(\mathcal{X}, d)$. *If there exist* $\lambda_1, \lambda_2, \lambda_3 \in (0, 1)$ *with* $\lambda_1 + \lambda_2 + \lambda_3 < 1$, *and* $\zeta \in \mathcal{Z}$ *such that*

$$\zeta(d(T\nu, T\omega), C(\nu, \omega)) \geq 0 \qquad \textit{for all } \nu, \omega \in \mathcal{X}, \tag{1.6}$$

where

$$\begin{aligned} C(\nu,\omega) \quad &:= [d(\nu,\omega)]^{\lambda_2} \cdot [d(\nu,T\nu)]^{\lambda_1} \cdot [d(\omega,T\omega)]^{\lambda_3} \cdot \\ &\cdot \left[\tfrac{1}{2}(d(\nu,T\omega) + d(\omega,T\nu))\right]^{1-\lambda_1-\lambda_2-\lambda_3} \end{aligned}$$

then we say that T *is an interpolative Hardy-Rogers type* $\mathcal{Z}$-contraction *with respect to* ζ.

Theorem 1.2. *[13] Let* $(\mathcal{X}, d)$ *be a complete metric metric space and* T *be an interpolative Hardy-Rogers type* $\mathcal{Z}$-contraction *with respect to* ζ. *Then there exists* $\nu \in \mathcal{X}$ *such that* $T\nu_* = \nu_*$.

In [16], a generalization of the Reich-type theorem in b-metric spaces is given and in addition, the existence of non unique fixed points is ensured.

Definition 1.10. *[16] Let* $(\mathcal{X}, d, s)$, *be a* b*-metric space. A mapping* $T : \mathcal{X} \to \mathcal{X}$ *is called an* (r, a)*-weight type contraction, if there exists* $\lambda \in [0, 1)$ *such that*

$$d(Tx, Ty) \leq \lambda M^p(T, \nu, \omega, a), \tag{1.7}$$

where $p \geq 0$ *and* $a = (a_1, a_2, a_3)$, $a_i \geq 0$, $i = 1, 2, 3$ *such that* $a_1 + a_2 + a_3 = 1$ *and*

$$M^p(T,\nu,\omega,a) = \begin{cases} [a_1(d(x,y))^p + a_2(d(x,Tx))^p + a_3(d(y,Ty))^p]^{1/p}, & \text{if } p > 0 \\ d(x,y))^{a_1}(d(x,Tx))^{a_2}(d(y,Ty))^{a_3}, & \text{if } p = 0 \end{cases}$$

for all $\nu, \omega \in \mathcal{X} \setminus Fix(T)$.

Theorem 1.3. *[16] Let* $(\mathcal{X}, d, s)$ *be a complete* b*-metric space and* $T : X \to X$ *be a* (r, a)*-weight type contraction mapping. Then* T *has a fixed point* $\nu^* \in \mathcal{X}$ *and for any* $\nu_0 \in X$ *the sequence* $\{T^n \nu_0\}$ *converges to* ν^* *if one of the following conditions holds:*

(i) T *is continuous at such point* ν_**;*

(ii) $b^p a_2 < 1$*;*

(iii) $b^p a_3 < 1$.

2 Main results

Definition 2.1. *Let* $(\mathcal{X}, d)$ *be a metric space. A mapping* $T : \mathcal{X} \to \mathcal{X}$ *is called an* α*-admissible* $\mathcal{Z}$*-p-contraction with respect to* ζ *of type* K *if there is a function* $\zeta \in \mathcal{Z}$ *and* $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ *such that for* $\lambda_i > 0$, $i \in \{1, 2, 3, 4\}$ *such that* $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$ *and for all* $\nu, \omega \in \mathcal{X}$

$$\zeta(\alpha(\nu,\omega)d(T\nu,T\omega), K_p(\nu,\omega)) \geq 0, \tag{2.1}$$

where

$$\begin{aligned} K_p(\nu,\omega) = [\lambda_1 d^p(\nu,\omega) + \lambda_2 d^p(\nu,T\nu) + \lambda_3 d^p(\omega,T\omega) + \\ +\lambda_4 \left(\tfrac{d(\nu,T\omega)+d(\omega,T\nu)}{2}\right)^p\Big]^{\frac{1}{p}}, \end{aligned} \tag{2.2}$$

for $p > 0$.

Theorem 2.1. *Let* $(\mathcal{X}, d)$ *be a complete metric space and let* $T : \mathcal{X} \to \mathcal{X}$ *be a continuous* α*-admissible* $\mathcal{Z}$*-*p*-contraction with respect to* ζ *of type* K*. Suppose also that:*

(i) T *is triangular* $\alpha-$*orbital admissible;*

(ii) *there exists* $\nu_0 \in \mathcal{X}$ *such that* $\alpha(\nu_0, T\nu_0) \geq 1$*;*

Then, T *has a fixed point.*

Proof. Let $\nu_0 \in \mathcal{X}$. Starting from this initial point, we can define a sequence $\{\nu_n\} \subset \mathcal{X}$ by $\nu_{n+1} = T\nu_n = T^n \nu_0$ for all $n \in \mathbb{N}$. If for some $n_0 \in \mathbb{N}$ we have $\nu_{n_0} = \nu_{n_0+1}$ then $T\nu_{n_0} = \nu_{n_0}$, that is, ν_{n_0} is a fixed point of T,. Therefore, we will assume from now on that $\nu_{n+1} \neq \nu_n$ for all $n \in \mathbb{N}$, which means that

$$d(\nu_n, \nu_{n+1}) > 0.$$

On the other hand, due to (ii), $\alpha(\nu_0, T\nu_0) \geq 1$ and since T is $\alpha-$orbital admissible,

$$\alpha(\nu_0, T\nu_0) \geq 1 \;\Rightarrow\; \alpha(\nu_1, \nu_2) = \alpha(T\nu_0, T^2\nu_0) \geq 1$$

and recursively we get that:

$$\alpha(\nu_n, \nu_{n+1}) \geq 1, \tag{2.3}$$

for all $n \in \mathbb{N}_0$. Further, since T is triangular $\alpha-$orbital admissible, from (2.3), it is easy to conclude that

$$\alpha(\nu_n, \nu_{n+k}) \geq 1, \tag{2.4}$$

$n, k \in \mathbb{N}$.
From (2.1), by replacing $\nu = \nu_{n-1}$ and $\omega = \nu_n$ and taking into account $(\zeta 1)$ we get

$$\begin{aligned} 0 \leq \; & \zeta(\alpha(\nu_{n-1}, \nu_n) d(T\nu_{n-1}, T\nu_n), K_p(\nu_{n-1}, \nu_n)) \\ & < K_p(\nu_{n-1}, \nu_n)) - \alpha(\nu_{n-1}, \nu_n) d(T\nu_{n-1}, T\nu_n), \text{ for any } n \geq 1. \end{aligned} \tag{2.5}$$

Combining with (2.3), we have

$$\begin{aligned} d(\nu_n, \nu_{n+1}) & \leq \alpha(\nu_{n-1}, \nu_n) d(\nu_n, \nu_{n+1}) < K_p(\nu_{n-1}, \nu_n) \\ & = [\lambda_1 d^p(\nu_{n-1}, \nu_n) + \lambda_2 d^p(\nu_{n-1}, \nu_n) + \lambda_3 d^p(\nu_n, \nu_{n+1}) + \\ & \quad + \lambda_4 \left(\frac{d(\nu_{n-1}, \nu_{n+1}) + d(\nu_n, \nu_n)}{2} \right)^p \Big]^{\frac{1}{p}} \\ & \leq [\lambda_1 d^p(\nu_{n-1}, \nu_n) + \lambda_2 d^p(\nu_{n-1}, \nu_n) + \lambda_3 d^p(\nu_n, \nu_{n+1}) + \\ & \quad + \lambda_4 \left(\frac{d(\nu_{n-1}, \nu_n) + d(\nu_n, \nu_{n+1})}{2} \right)^p \Big]^{\frac{1}{p}} \end{aligned} \tag{2.6}$$

or,

$$\begin{aligned} d^p(\nu_n, \nu_{n+1}) & < \lambda_1 d^p(\nu_{n-1}, \nu_n) + \lambda_2 d^p(\nu_{n-1}, \nu_n) + \lambda_3 d^p(\nu_n, \nu_{n+1}) + \\ & \quad + \lambda_4 \left(\frac{d^p(\nu_{n-1}, \nu_n) + d^p(\nu_n, \nu_{n+1})}{2} \right), \end{aligned} \tag{2.7}$$

(we used here: $\left(\frac{a+b}{2}\right)^p \leq \frac{a^p + b^p}{2}$). Since $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$ we have

$$d^p(\nu_n, \nu_{n+1}) < \frac{2\lambda_1 + 2\lambda_2 + \lambda_4}{2 - 2\lambda_3 - \lambda_4} d^p(\nu_{n-1}, \nu_n) = d^p(\nu_{n-1}, \nu_n), \tag{2.8}$$

which shows that the sequence of non-negative real numbers $\{d(\nu_{n-1}, \nu_n)\}$ is decreasing and so, there exists $\delta \geq 0$ such that $\lim\limits_{n\to\infty} d(\nu_{n-1}, \nu_n) = \delta$. Furthermore,

$$\lim_{n\to\infty} K_p(\nu_{n-1}, \nu_n) = [(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) \cdot \delta^p]^{1/p} = \delta.$$

Now, taking into account (2.3),

$$d(\nu_n, \nu_{n+1}) \leq \alpha(\nu_{n-1}, \nu_n) d(T\nu_{n-1}, T\nu_n) < K_p(\nu_{n-1}, \nu_n) \tag{2.9}$$

and when $n \to \infty$ in (2.9) we get

$$\begin{aligned} \delta & \leq \lim_{n\to\infty} \alpha(\nu_{n-1}, \nu_n) d(T\nu_{n-1}, T\nu_n) \\ & < \lim_{n\to\infty} K_p(\nu_{n-1}, \nu_n) = \delta. \end{aligned}$$

Thus, $\lim\limits_{n\to\infty} \alpha(\nu_{n-1}, \nu_n) d(T\nu_{n-1}, T\nu_n) = \delta$. If we suppose that $\delta > 0$ and taking $u_n = \alpha(\nu_{n-1}, \nu_n) d(T\nu_{n-1}, T\nu_n)$

respectively $v_n = K_p(\nu_{n-1}, \nu_n)$, from (ζ_3) we get

$$0 \leq \limsup_{n\to\infty} \zeta(u_n, v_n) < 0. \tag{2.10}$$

This is a contradiction. Hence,

$$\lim_{n\to\infty} d(\nu_{n-1}, \nu_n) = 0. \tag{2.11}$$

In the following, we shall prove that the sequence $\{d(\nu_{n-1}, \nu_n)\}$ is Cauchy. Assuming the contrary, from Lemma (1.1), we can find $\varepsilon > 0$ and two sequences $\{n_i\}$,$\{m_i\}$ of positive integers, with $n_i > m_i > i$ such that

$$\begin{aligned} \lim_{i\to\infty} d(\nu_{n_i}, \nu_{m_i}) &= \lim_{i\to\infty} d(\nu_{n_i-1}, \nu_{m_i-1}) = \lim_{i\to\infty} d(\nu_{n_i-1}, \nu_{m_i}) \\ &= \lim_{i\to\infty} d(\nu_{n_i}, \nu_{m_i-1}) = \varepsilon. \end{aligned} \tag{2.12}$$

On the other hand, by (2.11) and (2.12)

$$\begin{aligned} \lim_{i\to\infty} K_p(\nu_{n_i-1}, \nu_{m_i-1}) &= \lim_{i\to\infty} [\lambda_1 d^p(\nu_{n_i-1}, \nu_{m_i-1}) + \lambda_2 d^p(\nu_{n_i-1}, \nu_{n_i}) + \\ &+ \lambda_3 d^p(\nu_{m_i-1}, \nu_{m_i}) + \lambda_4 \left(\tfrac{d(\nu_{n_i-1}, \nu_{m_i}) + d(\nu_{m_i-1}, \nu_{n_i})}{2}\right)^p \Big]^{\frac{1}{p}} \\ &= (\lambda_1 + \lambda_4)^{1/p} \varepsilon. \end{aligned}$$

Again, applying (2.1), we have

$$\begin{aligned} 0 &\leq \zeta(\alpha(\nu_{n_i-1}, \nu_{m_i-1}) d(T\nu_{n_i-1}, T\nu_{m_i-1}), K_p(\nu_{n_i-1}, \nu_{m_i-1})) \\ &< K_p(\nu_{n_i-1}, \nu_{m_i-1})) - \alpha(\nu_{n_i-1}, \nu_{m_i-1}) d(T\nu_{n_i-1}, T\nu_{m_i-1}), \end{aligned}$$

and together with (2.4)

$$\begin{aligned} d(\nu_{n_i}, \nu_{m_i}) &= d(T\nu_{n_i-1}, T\nu_{m_i-1}) \leq \alpha(\nu_{n_i-1}, \nu_{m_i-1}) d(T\nu_{n_i-1}, T\nu_{m_i-1}) \\ &< K_p(\nu_{n_i-1}, \nu_{m_i-1}). \end{aligned}$$

Furthermore, letting $i \to \infty$ in the previous inequality we get

$$\varepsilon < (\lambda_1 + \lambda_4)^{1/p} \varepsilon \leq \varepsilon \tag{2.13}$$

This is a contradiction and for this reason we conclude that $\varepsilon = 0$ and the sequence $\{\nu_n\}$ is Cauchy. Since the space $(\mathcal{X}, d)$ is complete, there is $\nu_* \in \mathcal{X}$ such that

$$\lim_{n\to\infty} \nu_n = \nu_*. \tag{2.14}$$

The mapping T is supposed to be continuous. Hence T is continuous at a point ν_*, which means that

$$\nu_* = \lim_{n\to\infty} \nu_{n+1} = \lim_{n\to\infty} T\nu_n = T(\lim_{n\to\infty} \nu_n) = T\nu_*$$

that is, ν_* is a fixed point of T.

□

Theorem 2.2. *Let $(\mathcal{X}, d)$ be a complete metric space and let $T : \mathcal{X} \to \mathcal{X}$ be an α-admissible $\mathcal{Z}$-p-contraction with respect to ζ of type K. Suppose also that:*

(i) T is triangular $\alpha-$orbital admissible;

(ii) there exists $\nu_0 \in \mathcal{X}$ such that $\alpha(\nu_0, T\nu_0) \geq 1$;

(iii) $\mathcal{X}$ is regular with respect to mapping α.

Then, T has a fixed point.

Proof. Following the same steps as in the demonstration of the Theorem 2.1, we know that for any $p > 0$, the sequence $\{\nu_n\}$ is Cauchy, and due to the completeness of the metric space $(\mathcal{X}, d)$, there exists ν_* such that $\lim\limits_{n\to\infty} \nu_n = \nu_*$. Supposing that $T\nu_* \neq \nu_*$, using the triangle inequality we get

$$0 < d(\nu_*, T\nu_*) \leq d(\nu_*, T\nu_{n-1}) + d(T\nu_{n-1}, T\nu_*). \tag{2.15}$$

Replacing ν by ν_{n-1} and ω by ν_* in (2.1) and using (ζ_1) we get

$$\begin{aligned} 0 &\leq \zeta\left(\alpha(\nu_{n-1}, \nu_*)d(T\nu_{n-1}, T\nu_*), K_p(\nu_{n-1}, \nu_*)\right) \\ &< K_p(\nu_{n-1}, \nu_*) - \alpha(\nu_{n-1}, \nu_*)d(T\nu_{n-1}, T\nu_*). \end{aligned}$$

Since from the hypothesis (iii), the space $(\mathcal{X})$ is regular, so for $n \in \mathbb{N}$ we have $\alpha(\nu_{n-1}, \nu_*) \geq 1$ and

$$\begin{aligned} d(T\nu_{n-1}, T\nu_*) &\leq \alpha(\nu_{n-1}, \nu_*)d(T\nu_{n-1}, T\nu_*) < K_p(\nu_{n-1}, \nu_*) \\ &= [\lambda_1 d^p(\nu_{n-1}, \nu_*) + \lambda_2 d^p(\nu_{n-1}, \nu_n) + \lambda_3 d^p(\nu_*, T\nu_*) + \\ &\qquad +\lambda_4 \left(\tfrac{d(\nu_{n-1}, T\nu_*) + d(\nu_*, \nu_n)}{2}\right)^p\Big]^{\frac{1}{p}} \\ &= [\lambda_1 d^p(\nu_{n-1}, \nu_*) + \lambda_2 d^p(\nu_{n-1}, \nu_n) + \lambda_3 d^p(\nu_*, T\nu_*) + \\ &\qquad +\lambda_4 \tfrac{d^p(\nu_{n-1}, T\nu_*) + d^p(\nu_*, \nu_n)}{2}\Big]^{\frac{1}{p}} \end{aligned}$$

Hence, returning in (2.15) we have

$$\begin{aligned} 0 &< d(T\nu_*, \nu_*) < d(T\nu_{n-1}, \nu_*) + K_p(\nu_{n-1}, \nu_*) \\ &= d(T\nu_{n-1}, \nu_*) + [\lambda_1 d^p(\nu_{n-1}, \nu_*) + \lambda_2 d^p(\nu_{n-1}, \nu_n) + \lambda_3 d^p(\nu_*, T\nu_*) + \\ &\qquad +\lambda_4 \tfrac{d^p(\nu_{n-1}, T\nu_*) + d^p(\nu_*, \nu_n)}{2}\Big]^{\frac{1}{p}} \end{aligned} \tag{2.16}$$

Letting $n \to \infty$ in the inequality (2.16) we obtain

$$\begin{aligned} 0 &< d^p(T\nu_*, \nu_*) < \left(\lambda_3 d^p(\nu_*, T\nu_*) + \lambda_4 \tfrac{d^p(\nu_*, T\nu_*)}{2}\right) \\ &= (\lambda_3 + \tfrac{\lambda_4}{2})d^p(T\nu_*, \nu_*) \leq d^p(T\nu_*, \nu_*) \end{aligned}$$

which is a contradiction and shows that $d(T\nu_*, \nu_*) = 0$. Therefore, $T\nu_* = \nu_*$.

□

Adding an additional presumption ensures the uniqueness of the fixed point.

Theorem 2.3. *If in Theorems 2.1 and 2.2, we assume additionally that*

$$\alpha(\nu,\omega) \geq 1 \quad \text{for any} \quad \nu,\omega \in Fix(T),$$

then the fixed point of T is unique.

Proof. Let ν_* be a fixed point of T. If there exists another point, ω_* different from ν_* such that $T\omega_* = \omega_*$, then

$$0 \leq \zeta(\alpha(\nu_*,\omega_*)d(T\nu_*,T\omega_*), K_p(\nu_*,\omega_*)) < K_p(\nu_*,\omega_*) - \alpha(\nu_*,\omega_*)d(T\nu_*,T\omega_*).$$

Hence,

$$0 < d(\nu_*,\omega_*) \leq \alpha(\nu_*,\omega_*)d(T\nu_*,T\omega_*) < K_p(\nu_*,\omega_*) = \left[\lambda_1 d^p(\nu_*,\omega_*) + \lambda_4 d^p(\nu_*,\omega_*)\right]^{\frac{1}{p}}.$$

This implies that

$$0 < d^p(\nu_*,\omega_*) < (\lambda_1+\lambda_4)d^p(\nu_*,\omega_*) \leq d^p(\nu_*,\omega_*)$$

which is a contradiction. Therefore $d^p(\nu_*,\omega_*) = 0$ and hence, $\nu_* = \omega_*$, that is the fixed point of T is unique.

□

A similar result can be easily obtained, following the proof from [13], if we take for the case $p = 0$ $K_p(\nu,\omega) = C(\nu,\omega)$.

Theorem 2.4. *Let $(\mathcal{X}, d)$ be a complete metric space and let T be a self-mapping on $\mathcal{X}$, such that there exist $\zeta \in \mathcal{Z}$ and $\alpha : \mathcal{X}\times\mathcal{X} \to [0,\infty)$ such that for $\lambda_i > 0$, $i \in \{1,2,3,4\}$ with $\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 1$ and for all $\nu,\omega \in \mathcal{X}\setminus Fix(T)$*

$$\zeta(\alpha(\nu,\omega)d(T\nu,T\omega), C(\nu,\omega)) \geq 0, \tag{2.17}$$

Suppose also that:

- (i) *T is triangular α−orbital admissible;*
- (ii) *there exists $\nu_0 \in \mathcal{X}$ such that $\alpha(\nu_0, T\nu_0) \geq 1$;*
- (iii) *either, T is continuous, or*
- (iv) *$(\mathcal{X}, d)$ is regular.*

Then, T has a fixed point.

Definition 2.2. *Let $(\mathcal{X}, d)$ be a metric space. A mapping $T : \mathcal{X} \to \mathcal{X}$ is called an α-admissible $\mathcal{Z}$-p-contraction with respect to ζ of type J if there exist a function $\zeta \in \mathcal{Z}$ and $\alpha : \mathcal{X}\times\mathcal{X} \to [0,\infty)$ such that for $\lambda_1,\lambda_2 > 0$, with $\lambda_1+\lambda_2 = 1$*

$$\zeta(\alpha(\nu,\omega)d(T\nu,T\omega), J_p(\nu,\omega)) \geq 0, \tag{2.18}$$

where

$$J_p(\nu,\omega) = \begin{cases} \left[\lambda_1 d^p(\nu,\omega) + \lambda_2\left(\frac{d(\omega,T\omega)(1+d(\nu,T\nu))}{1+d(\nu,\omega)}\right)^p\right]^{\frac{1}{p}}, & \text{for } p>0 \\ [d(\nu,\omega)]^{\lambda_1}\cdot\left[\frac{d(\omega,T\omega)(1+d(\nu,T\nu))}{1+d(\nu,\omega)}\right]^{\lambda_2}, & \text{for } p=0 \end{cases} \tag{2.19}$$

for all $\nu,\omega \in \mathcal{X}\setminus Fix(T)$.

Theorem 2.5. *Let $(\mathcal{X}, d)$ be a complete metric space and let T be an α-admissible $\mathcal{Z}$-p-contraction with respect to ζ of type J Suppose also that:*

(i) *T is triangular $\alpha-$orbital admissible;*

(ii) *there exists $\nu_0 \in \mathcal{X}$ such that $\alpha(\nu_0, T\nu_0) \geq 1$;*

(iii) *either, T is continuous, or*

(iv) *$(\mathcal{X}, d)$ is regular.*

Then, T has a fixed point.

Proof. Starting from an arbitrary point ν_0 in $\mathcal{X}$ we build a sequence $\{\nu_n\}$, as $\nu_n = T^n \nu_0$ for all $n \in \mathbb{N}$. If there exists some $m \in \mathbb{N}$ such that $T\nu_m = \nu_{m+1} = \nu_m$, then ν_m is a fixed point of T and the proof is finished. For this reason, we can assume from now on that $\nu_n \neq \nu_{n-1}$ for any $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned} 0 &\leq \zeta(\alpha(\nu_{n-1}, \nu_n) d(T\nu_{n-1}, T\nu_n), J_p(\nu_{n-1}, \nu_n)) \\ &< J_p(\nu_{n-1}, \nu_n) - \alpha(\nu_{n-1}, \nu_n) d(T\nu_{n-1}, T\nu_n). \end{aligned} \tag{2.20}$$

Since T is triangular $\alpha-$orbital admissible, (2.3) holds and the above inequality becomes

$$d(\nu_n, \nu_{n+1}) \leq \alpha(\nu_{n-1}, \nu_n) d(T\nu_{n-1}, T\nu_n) < J_p(\nu_{n-1}, \nu_n). \tag{2.21}$$

(1.) For the case $p > 0$

$$\begin{aligned} J_p(\nu_{n-1}, \nu_n) &= \left[\lambda_1 d^p(\nu_{n-1}, \nu_n) + \lambda_2 \left(\tfrac{d(\nu_n, T\nu_n)(1 + d(\nu_{n-1}, T\nu_{n-1}))}{1 + d(\nu_{n-1}, \nu_n)}\right)^p\right]^{\frac{1}{p}} \\ &= \left[\lambda_1 d^p(\nu_{n-1}, \nu_n) + \lambda_2 d^p(\nu_n, \nu_{n+1})\right]^{\frac{1}{p}} \end{aligned}$$

and replacing in (2.21) we get

$$d(\nu_n, \nu_{n+1}) < \left[\lambda_1 d^p(\nu_{n-1}, \nu_n) + \lambda_2 d^p(\nu_n, \nu_{n+1})\right]^{\frac{1}{p}}$$

which is equivalent with the following

$$d^p(\nu_n, \nu_{n+1}) < \frac{\lambda_1}{1 - \lambda_2} d^p(\nu_{n-1}, \nu_n) = d^p(\nu_{n-1}, \nu_n)$$

It follows then that $\{d(\nu_{n-1}, \nu_n)\}$ is a non-increasing sequence of positive real numbers and consequently, there is $\delta \geq 0$ such that $\lim_{n \to \infty} d(\nu_{n-1}, \nu_n) = \delta$. Since it can be easily seen that $\lim_{n \to \infty} O_p(\nu_{n-1}, \nu_n) = \delta$, if we suppose that $\delta > 0$ then passing the limit when $n \to \infty$ in (2.20) we get

$$0 \leq \limsup_{n \to \infty} \zeta(\alpha(\nu_{n-1}, \nu_n) d(T\nu_{n-1}, T\nu_n), J_p(\nu_{n-1}, \nu_n)) < 0$$

and hence $\delta = 0$ which contradicts our assumption. Furthermore,

$$\lim_{n \to \infty} d(\nu_{n-1}, \nu_n) = 0. \tag{2.22}$$

We shall prove that $\{\nu_n\}$ is a Cauchy sequence. If we suppose, by contradiction, than $\{\nu_n\}$ is not a Cauchy sequence then following the proof of Theorem 2.1, by Lemma 1.1 there exits $\varepsilon > 0$ such that

$$\begin{aligned} \lim_{i\to\infty} d(\nu_{n_i}, \nu_{m_i}) &= \lim_{i\to\infty} d(\nu_{n_i-1}, \nu_{m_i-1}) = \lim_{i\to\infty} d(\nu_{n_i-1}, \nu_{m_i}) \\ &= \lim_{i\to\infty} d(\nu_{n_i}, \nu_{m_i-1}) = \varepsilon. \end{aligned} \tag{2.23}$$

Replacing in (2.18)

$$\begin{aligned} 0 &\le \zeta(\alpha(\nu_{n_i-1}, \nu_{m_i-1}) d(T\nu_{n_i-1}, T\nu_{m_i-1}), J_p(\nu_{n_i-1}, \nu_{m_i-1})) \\ &< J_p(\nu_{n_i-1}, \nu_{m_i-1}) - \alpha(\nu_{n_i-1}, \nu_{m_i-1}) d(T\nu_{n_i-1}, T\nu_{m_i-1}) \end{aligned}$$

or, together with (2.4)

$$\begin{aligned} d(\nu_{n_i}, \nu_{m_i}) &\le \alpha(\nu_{n_i-1}, \nu_{m_i-1}) d(T\nu_{n_i-1}, T\nu_{m_i-1}) < J_p(\nu_{n_i-1}, \nu_{m_i-1}) \\ &= \left[\lambda_1 d^p(\nu_{n_i-1}, \nu_{m_i-1}) + \lambda_2 \left(\frac{d(\nu_{m_i-1}, \nu_{m_i})[1 + d(\nu_{n_i-1}, \nu_{n_i})]}{1 + d(\nu_{n_i-1}, \nu_{m_i-1})}\right)^p\right]^{\frac{1}{p}}. \end{aligned}$$

Letting $i \to \infty$ in the above inequality we get that

$$0 < \varepsilon < \lambda_1^{1/p} \varepsilon < \varepsilon,$$

which is a contradiction. Hence, we conclude that $\{\nu_n\}$ is a Cauchy sequence in a complete metric space $(\mathcal{X}, d)$ and there exists ν_* such that

$$\nu_n \to \nu_* \text{ as } n \to \infty. \tag{2.24}$$

If T is continuous

$$\lim_{n\to\infty} d(\nu_{n+1}, T\nu_*) = \lim_{n\to\infty} d(T\nu_n, T\nu_*) = 0,$$

and combined with the uniqueness of the limit, we get that $T\nu_* = \nu_*$, that is, ν_* forms a fixed point of T.
In the case of the alternative hypothesis, we suppose that $T\nu_* \neq \nu_*$. From (2.18)

$$0 \le \zeta\left(\alpha\left(\nu_{n(k)}, \nu_*\right) d(T\nu_{n(k)}, T\nu_*), J_p(\nu_{n(k)}, \nu_*)\right)$$

and since $(\mathcal{X}, d)$ is regular, there exists a subsequence $\left\{\nu_{n(k)}\right\}$ of $\{\nu_n\}$ such that $\alpha(\nu_{n(k)}, \nu_*) \le 1$ for any $k \in \mathbb{N}$

$$\begin{aligned} d(\nu_{n(k)+1}, T\nu_*) &\le \alpha\left(\nu_{n(k)}, \nu_*\right) d(T\nu_{n(k)}, T\nu_*) < J_p(\nu_{n(k)}, \nu_*) \\ &= \left[\lambda_1 d^p(\nu_{n(k)}, \nu_*) + \lambda_2 \left(\frac{d(\nu_*, T\nu_*)(1 + d(\nu_{n(k)}, \nu_{n(k)+1}))}{d^p(\nu_{n(k)}, \nu_*)}\right)^p\right]^{\frac{1}{p}} \end{aligned}$$

Letting $n \to \infty$ and keeping in mind (2.24) and (2.22), we have

$$0 < d(\nu_*, T\nu_*) < [\lambda_1 d^p(\nu_*, T\nu_*) + \lambda_2 d^p(\nu_*, T\nu_*)]^{\frac{1}{p}}$$

which is equivalent with

$$0 < d^p(\nu_*, T\nu_*) < (\lambda_1 + \lambda_2)\, d^p(\nu_*, T\nu_*) = d^p(\nu_*, T\nu_*).$$

This is a contradiction. Thus, $d^p(\nu_*, T\nu_*) = 0$, that is, ν_* is a fixed point of T.

(2.) For the case $p=0$ we have

$$\begin{aligned} J_p(\nu_{n-1},\nu_n) &= [d(\nu_{n-1},\nu_n)]^{\lambda_1}\cdot\left[\frac{d(\nu_n,T\nu_n)(1+d(\nu_{n-1},T\nu_{n-1}))}{1+d(\nu_{n-1},\nu_n)}\right]^{\lambda_2}\\ &= [d(\nu_{n-1},\nu_n)]^{\lambda_1}\cdot\left[\frac{d(\nu_n,\nu_{n+1})(1+d(\nu_{n-1},\nu_n))}{1+d(\nu_{n-1},\nu_n)}\right]^{1-\lambda_1}\\ &= [d(\nu_{n-1},\nu_n)]^{\lambda_1}\cdot[d(\nu_n,\nu_{n+1})]^{1-\lambda_1} \end{aligned}$$

and the inequality (2.21) implies that

$$[d(\nu_n,\nu_{n+1})]^{\lambda_1} < [d(\nu_{n-1},\nu_n)]^{\lambda_1}.$$

Consequently, we derive that the sequence of non-negative real numbers $\{d(\nu_{n-1},\nu_n)\}$ is decreasing. Then, there exists $\delta\geq 0$ such that $\lim_{n\to\infty} d(\nu_{n-1},\nu_n)=\delta$. On the other hand, it is easy to see that

$$\lim_{n\to\infty} J_p(\nu_{n-1},\nu_n)=\delta.$$

Assuming that $\delta>0$, since T is an α-admissible $\mathcal{Z}$-p-contraction with respect to ζ of type J, we obtain

$$0\leq \limsup \zeta(\alpha(\nu_{n-1},\nu_n)d(\nu_{n-1},\nu_n), J_p(\nu_{n-1},\nu_n))<0$$

which is a contradiction. Therefore, $\delta=0$, which means

$$\lim_{n\infty} d(\nu_{n-1},\nu_n)=0. \tag{2.25}$$

By employing the same tools as in the case $p=1$ and taking into account (2.25) we have

$$\begin{aligned} \lim_{n\infty} J_p(\nu_{n_i-1},\nu_{m_i-1}) &= \lim_{n\to\infty}[d(\nu_{n_i-1},\nu_{m_i-1})]^{\lambda_1}\cdot\left[\frac{d(\nu_{m_i-1},\nu_{m_i})(1+d(\nu_{n_i-1},\nu_{n_i}))}{1+d(\nu_{n_i-1},\nu_{m_i-1})}\right]^{1-\lambda_1}\\ &=0, \end{aligned}$$

we shall easily obtain that $\{x_n\}$ forms a Cauchy sequence in a complete metric space. Thus, there is ν_* such that $\lim_{n\to\infty}\nu_n=\nu_*$. As a last step in our proof, we shall show that ν_* is a fixed point of T.
Sure, under the presumption that T is continuous we have

$$\lim_{n\to\infty} d(\nu_{n+1},T\nu_*)=\lim_{n\to\infty} d(T\nu_n,T\nu_*)=0,$$

and combined with the uniqueness of limit, $T\nu_*=\nu_*$, that is, ν^* forms a fixed point of T.
Under the alternative presumption, namely, the regularity of the space $\mathcal{X}$, we have from (2.18)

$$0\leq\zeta\left(\alpha\left(\nu_{n(k)},\nu_*\right)d(T\nu_{n(k)},T\nu_*), J_p(\nu_{n(k)},\nu_*)\right)$$

or,

$$\begin{aligned} d(\nu_{n(k)+1},T\nu_*) &= d(T\nu_{n(k)},T\nu_*) < J_p(\nu_{n(k)},\nu_*)\\ &= [d(\nu_{n(k)},\nu_*)]^{\lambda_1}\cdot\left[\frac{d(\nu_*,T\nu_*)(1+d(\nu_{n(k)},\nu_{n(k)+1}))}{1+d(\nu_{n(k)},\nu_*)}\right]^{1-\lambda_1}. \end{aligned}$$

Letting $n\to\infty$ in the above inequality we get $d(\nu_*,T\nu_*)=0$, that is $T\nu_*=\nu_*$.

□

Example 2.1. *On set $\mathcal{X}$, endowed with metric $d(\nu,\omega) = |\nu - \omega|$ we consider the mapping $O : \mathcal{X} \to \mathcal{X}$ given as follows:*

$$O(1) = O(5) = O(7) = 7,\ O(2) = 5.$$

Let the function $\zeta \in \mathcal{Z}$, where for any ν,ω, $\zeta(u,v) = \frac{v(v+1)}{v+2} - u$ and also, $\alpha : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ be defined by:

$$\alpha(\nu,\omega) = \begin{cases} 0, & \text{if } (\nu,\omega) \in \{(1,2),(2,5)\} \\ 1, & \text{if } (\nu,\omega) \in \{(2,1),(5,2)\} \\ 3, & \text{otherwise} \end{cases}$$

By elementary calculations, we can reach that O is triangular $\alpha-$orbital admissible and the space $\mathcal{X}$ is regular. The inequality (2.18)

$$\zeta\left(\alpha(\nu,\omega)d(O\nu,O\omega), J_p(\nu,\omega)\right) \geq 0$$

becomes in this case, for any $\nu,\omega \in \mathcal{X} \setminus\ Fix(T)$

$$\frac{J_p(\nu,\omega)(J_p(\nu,\omega)+1)}{J_p(\nu,\omega)+2} \geq \alpha(\nu,\omega)d(O\nu,O\omega), \tag{2.26}$$

where for $p = 0$ and $\lambda_1 = \lambda_2 = \frac{1}{2}$ we have $J_p(\nu,\omega) = \sqrt{\frac{d(\nu,\omega)d(\omega,O\omega)(1+d(\nu,O\nu))}{1+d(\nu,\omega)}}$. Since $O1 = O5 = 7$, we have $d(O1,O5) = d(7,7) = 0$ from (2.26) we have

$$\frac{J_p(\nu,\omega)(J_p(\nu,\omega)+1)}{J_p(\nu,\omega)+2} \geq 0.$$

Also, due to the way the mapping α was defined it is clear that the interesting cases are the following:
***(a)** $\nu = 2, \omega = 1$. In this case, (2.26) becomes*

$$\frac{J_p(2,1)(J_p(2,1)+1)}{J_p(2,1)+2} \geq \alpha(2,1)d(O2,O1),$$

or, since $J_p(2,1) = \sqrt{\frac{d(2,1)d(1,O1)(1+d(2,O2))}{1+d(2,1)}} = \sqrt{\frac{1\cdot 6\cdot 4}{1+1}} = \sqrt{12}$,

$$\frac{12+\sqrt{12}}{\sqrt{12}+2} \geq 2 \ \Leftrightarrow\ 8 \leq \sqrt{12}.$$

***(b)** $\nu = 5, \omega = 2$. Similarly, we have $J_p(5,2) = \sqrt{\frac{d(5,2)d(2,O2)(1+d(5,O5))}{1+d(5,2)}} = \sqrt{\frac{3\cdot 3\cdot 3}{4}} = \sqrt{\frac{27}{4}}$ and then*

$$\frac{\frac{27}{4}+\sqrt{\frac{27}{4}}}{\sqrt{\frac{27}{4}}+2} \geq 2 \ \Leftrightarrow\ \frac{19}{2} \leq \sqrt{27}.$$

So, we checked that all the presumptions of Theorem 2.5 are fulfilled and therefore $\nu = 7$ is a fixed point for O.

Theorem 2.6. *Let T be an orbitally continuous self-map on the T-orbitally complete metric space $(\mathcal{X}, d)$ and a map $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$. Suppose that there exist $\zeta \in \mathcal{Z}$ such that for each $\nu, \omega \in \mathcal{X}$*

$$\zeta(\alpha(\nu,\omega)d(\nu,\omega), L_p(\nu,\omega)) \geq 0, \tag{2.27}$$

where for $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 + \lambda_2 = 1$,

$$L_p(\nu,\omega) = \begin{cases} \left[\lambda_1 \left[d(\nu,\omega)\right]^p + \lambda_2 \left[\frac{d(\nu,T^2\nu)}{2}\right]^p\right]^{\frac{1}{p}}, & \text{for } p > 0 \\ \left[d(\nu,\omega)\right]^{\lambda_1} \cdot \left[\frac{d(\nu,T^2\nu)}{2}\right]^{\lambda_2}, & \text{for } p = 0 \end{cases}$$

for all $\nu, \omega \in \mathcal{X} \setminus Fix(T)$. Suppose also that:

(i) T is orbital α-admissible;

(ii) there exists $\nu_0 \in \mathcal{X}$ such that $\alpha(\nu_0, T\nu_0) \geq 1$;

Then T has a fixed point.

Proof. As in the corresponding lines in the proof of previous theorems, starting by ν_0, we built-up a recursive sequence $\{\nu_n\}$ as:

$$\nu_0 := \nu \text{ and } \nu_n = T\nu_{n-1} \text{ for all } n \in \mathbb{N}. \tag{2.28}$$

Without loss of generality, we assume that

$$x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}. \tag{2.29}$$

Indeed, if for some $m \in \mathbb{N}$ we have the equality $\nu_m = T\nu_{m-1} = \nu_{m-1}$, then the proof is completed.

On the account of (ii), $\alpha(\nu_0, T\nu_0) \geq 1$. Due to α-admissibility of T, we derive that

$$\alpha(\nu_n, \nu_{n+1}) \geq 1 \qquad \text{for all } n \in \mathbb{N}_0. \tag{2.30}$$

For $\nu = \nu_{n-1}$ and $\omega = \nu_n$ in (2.27) and regarding the inequality (2.30), we derive that

$$\begin{aligned} 0 &\leq \zeta(\alpha(\nu_{n-1}, \nu_n)d(\nu_{Tn-1}, T\nu_n), L_p(\nu_{n-1}, \nu_n)) \\ &< L_p(\nu_{n-1}, \nu_n) - \alpha(\nu_{n-1}, \nu_n)d(\nu_{n-1}, \nu_n) \end{aligned} \tag{2.31}$$

which yields

$$d(\nu_n, \nu_{n+1}) = d(T\nu_{n-1}, T\nu_n) \leq \alpha(\nu_{n-1}, \nu_n)d(\nu_{Tn-1}, T\nu_n) < L_p(\nu_{n-1}, \nu_n). \tag{2.32}$$

(1.) For the case $p > 0$, due to (2.28), the statement (2.32) turns into

$$d^p(\nu_n, \nu_{n+1}) < \lambda_1[d(\nu_{n-1}, \nu_n)]^p + \lambda_2 \left[\frac{d(\nu_{n-1}, \nu_{n+1})}{2}\right]^p. \tag{2.33}$$

By using the triangle inequality, one can get

$$d^p(\nu_n, \nu_{n+1}) < \lambda_1 d^p(\nu_{n-1}, \nu_n) + \lambda_2 \left[\frac{d^p(\nu_{n-1}, \nu_n) + d^p(\nu_n, \nu_{n+1})}{2}\right] \tag{2.34}$$

which implies, since $\lambda_1 + \lambda_2 = 1$, that

$$d(\nu_n, \nu_{n+1}) < d(\nu_{n-1}, \nu_n) \tag{2.35}$$

Thus, $\{d(\nu_n, \nu_{n+1})\}$ is a decreasing sequence of positive real numbers and there is $\delta \geq 0$ such that $\lim_{n\to\infty} d(\nu_n, \nu_{n+1}) = \delta$. Then, also

$$\lim_{n\to\infty} L_p(\nu_{n-1}, \nu_n) = \delta.$$

We presume that $\delta > 0$. Considering in (2.27) $u_n = \alpha(\nu_{n-1}, \nu_n)d(T\nu_{n-1}, T\nu_n)$, $v_n = L_p(\nu_{n-1}, \nu_n)$ and keeping in mind the presumption (ζ_3) it follows that

$$0 \leq \limsup_{n\to\infty} \zeta\left(\alpha(\nu_{n-1}, \nu_n)d(T\nu_{n-1}, T\nu_n), L_p(\nu_{n-1}, \nu_n)\right) < 0$$

But since this is a contradiction we have $\lim_{n\to\infty} d(\nu_n, \nu_{n+1}) = 0$. We shall prove that $\{\nu_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. As in the proof of the previous theorem, assuming the opposite, that the sequence $\{\nu_n\}$ is not Cauchy, by Lemma 1.1 we can find $\varepsilon > 0$ and the sequences of positive integers $\{n_i\}, \{m_i\}$ such that $n_i > m_i > i$ and

$$\lim_{n\to\infty} d(\nu_{n_i-1}, \nu_{m_i-1}) = \lim_{n\to\infty} d(\nu_{n_i}, \nu_{m_i}) = \varepsilon. \tag{2.36}$$

Replacing in (2.27) ν by ν_{n_i-1} and ω by ν_{m_i-1} and taking into account (2.4) we get

$$\begin{aligned} d(\nu_{n_i}, \nu_{m_i}) &\leq \alpha(\nu_{n_i-1}, \nu_{m_i-1})d(T\nu_{n_i-1}, T\nu_{m_i-1}) < L_p(\nu_{n_i-1}, \nu_{m_i-1}) \\ &= \left[\lambda_1[d(\nu_{n_i-1}, \nu_{m_i-1})]^p + \lambda_2\left[\frac{d(\nu_{n_i-1}, \nu_{n_i+1})}{2}\right]^p\right]^{\frac{1}{p}} \\ &\leq \left[\lambda_1 d^p(\nu_{n_i-1}, \nu_{m_i-1}) + \lambda_2 \frac{d^p(\nu_{n_i-1}, \nu_{n_i}) + d^p(\nu_{n_i}, \nu_{n_i+1})}{2}\right]^{\frac{1}{p}} \end{aligned} \tag{2.37}$$

Letting $i \to \infty$ in the previous inequality and accordance with (2.36) we obtain

$$\varepsilon < \lambda_1\, \varepsilon < \varepsilon.$$

This is a contradiction. Thus, $\varepsilon = 0$ and $\{\nu_n\}$ is a Cauchy sequence. Regarding the construction $\nu_n = T^n\nu_0$ and using the fact that $(\mathcal{X}, d)$ is T-orbitally complete, there is $\nu_* \in \mathcal{X}$ such that $\nu_n \to \nu_*$. Furthermore by the orbital continuity of T, we obtain that $\nu_n \to T\nu_*$. Hence $\nu_* = T\nu_*$.

(2.) For the case $p = 0$, the statement (2.32) becomes

$$\begin{aligned} d(\nu_n, \nu_{n+1}) &< [d(\nu_{n-1}, \nu_n)]^{\lambda_1} \cdot \left[\frac{d(\nu_{n-1}, \nu_{n+1})}{2}\right]^{1-\lambda_1} \\ &\leq [d(\nu_{n-1}, \nu_n)]^{\lambda_1} \cdot \left[\frac{d(\nu_{n-1}, \nu_n) + d(\nu_n, \nu_{n+1})}{2}\right]^{1-\lambda_1}. \end{aligned} \tag{2.38}$$

If we presume that there exists some $n_0 \in \mathbb{N}$ such that $d(\nu_{n-1}, \nu_n) \leq d(\nu_n, \nu_{n+1})$ for any $n \leq n_0$, then (2.38) turns into $d(\nu_n, \nu_{n+1}) < d(\nu_n, \nu_{n+1})$ which is a contradiction. Therefore, we have $d(\nu_{n-1}, \nu_n) > d(\nu_n, \nu_{n+1})$ for all $n \in \mathbb{N}$. We conclude that $\{d(\nu_{n-1}, \nu_n)\}$ is a monotonically decreasing sequence of non-negative real numbers, so that there is some $\delta \geq 0$ such that $\lim_{n\to\infty} d(\nu_{n-1}, \nu_n) = \delta$. Since $\lim_{n\to\infty} L_P(\nu_{n-1}, \nu_n) = \delta$, following the proof for the case $p > 0$ we get that $\delta = 0$. Again, following the case $p > 0$ it follows that the sequence $\{\nu_n\}$ is convergent to a point $\nu_* \in \mathcal{X}$, being a Cauchy sequence in a complete metric space and the point ν_* is a fixed point of T.

□

Remark 2.1. *Many consequences can be listed either by considering different functions or by taking different values for* $p \geq 0$.

References

[1] M.U. Ali, T. Kamram, E. Karapınar, *An approach to existence of fixed points of generalized contractive multivalued mappings of integral type via admissible mapping*, Abstr. Appl. Anal. 2014, (2014) Article ID 141489.

[2] M.U. Ali, T. Kamran, E. Karapınar, *On* (α, ψ, η)*-contractive multivalued mappings*, Fixed Point Theory Appl. (2014), 2014:7.

[3] H. Alsulami, S. Gulyaz, E. Karapınar, I.M. Erhan, *Fixed point theorems for a class of* α*-admissible contractions and applications to boundary value problem*, Abstr. Appl. Anal. 2014 (2014) Article ID 187031.

[4] H.H. Alsulami, E. Karapınar, F. Khojasteh, A.F. Roldán-López-de-Hierro, *A proposal to the study of contractions in quasi-metric spaces*, Discrete Dynamics in Nature and Society 2014, Article ID 269286, 10 pages.

[5] S. Alharbi, H.H. Alsulami, E. Karapınar, *On the Power of Simulation and Admissible Functions in Metric Fixed Point Theory*, Journal of Function Spaces, Volume 2017 (2017), Article ID 2068163, 7 pages.

[6] S. AlMezel, C.M. Chen, E. Karapınar and V. Rakočević, *Fixed point results for various* α*-admissible contractive mappings on metric-like spaces*, Abstract and Applied Analysis Volume 2014 (2014) ,Article ID 379358.

[7] H. Argoubia, B. Samet, C. Vetro, *Nonlinear contractions involving simulation functions in a metric space with a partial order*, J. Nonlinear Sci. Appl. 8 (2015), 1082–1094.

[8] H. Aydi, M. Jellali, E. Karapınar, *Common fixed points for generalized* α*-implicit contractions in partial metric spaces: Consequences and application*, RACSAM - Revista de la Real Academia de Ciencias Exactas, Fasicas y Naturales. Serie A. Matematicas, September 2015, Volume 109, Issue 2, pp 367-384

[9] M. Arshad, E. Ameer, E. Karapınar, *Generalized contractions with triangular alpha-orbital admissible mapping on Branciari metric spaces* , Journal of Inequalities and Applications 2016, 2016:63 (16 February 2016)

[10] C.M. Chen, A. Abkar, S. Ghods and E. Karapınar, *Fixed Point Theory for the* α*-Admissible Meir-KeelerType Set Contractions Having KKM* Property on Almost Convex Sets*, Appl. Math. Inf. Sci. 11, No. 1, 171-176 (2017)

[11] L.B. Ćirić, *On some maps with a non-unique fixed point*, Publ. Inst. Math., **17** (1974), 52–58.

[12] E. Karapınar, *Fixed points results via simulation functions*, Filomat, 2016, 30, 2343–2350.

[13] E. Karapınar, *Revisiting simulation functions via interpolative contractions*, in press.

[14] E. Karapınar, P. Kumam, P. Salimi, *On $\alpha - \psi$-Meir-Keeler contractive mappings*, Fixed Point Theory Appl. (2013), 2013:94 .

[15] F. Khojasteh, S. Shukla, S. Radenović, *A new approach to the study of fixed point theorems via simulation functions*, Filomat 29:6 (2015), 1189–1194.

[16] Z. Mitrović, H. Aydi, M.SM. Noorani, H. Qawaqneh, *The Weight Inequalities on Reich Type Theorem in b-Metric Spaces*, J. Math. Computer Sci., 19 (2019), 51–57.

[17] O. Popescu, *Some new fixed point theorems for $\alpha-$Geraghty-contraction type maps in metric spaces*, Fixed Point Theory Appl. (2014), 2014:190.

[18] S. Radenović, F. Vetro, and J. Vujaković, *An alternative and easy approach to fixed point results via simulation functions*, Demonstr. Math. 2017; 50:223–23.

[19] S. Radenović, Z. Kadelburg, D. Jandrlić and A. Jandrlić, *Some results on weak contraction maps*, Bulletin of the Iranian Mathematical Society Vol. 38 No. 3 (2012), pp 625–645.

[20] S. Radenovic, F. Vetro, J. Vujakovic, *An alternative and easy approach to fixed point results via simulation functions*, Demonstr. Math. 2017; 50:223–230.

[21] A.F Roldán-López-de-Hierro, E. Karapinar, C. Roldán-López-de-Hierro, J. Martinez-Moreno, *Coincidence point theorems on metric spaces via simulation functions*, Journal of Computational and Applied Mathematics 275 (2015) 345–355.

[22] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for α-ψ-contractive type mappings*, Nonlinear Anal. **75** (2012), 2154–2165.