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A meshless technique for nonlinear Volterra-Fredholm integral equations via hybrid of radial basis functions

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Abstract

In this paper, an effective technique is proposed to determine the numerical solution of nonlinear Volterra-Fredholm integral equations (VFIEs) which is based on interpolation by the hybrid of radial basis functions (RBFs) including both inverse multiquadrics (IMQs), hyperbolic secant (Sechs) and strictly positive definite functions. Zeros of the shifted Legendre polynomial are used as the collocation points to set up the nonlinear systems. The integrals involved in the formulation of the problems are approximated based on Legendre-Gauss-Lobatto integration rule. This technique is so convenience to implement and yields very accurate results compared with the other basis.

Key words: Nonlinear Volterra-Fredholm integral equation, Strictly positive definite functions, Inverse multiquadrics, Hyperbolic secant.
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1 Introduction

In many sciences such as economics, biology, physics and engineering, a major problem is assigned to find the numerical solution of linear and nonlinear integral equations (IEs). Different basis functions have been used to estimate the solution of recent equations such as wavelets, hybrid Taylor and block-pulse functions, [12,14,1,19]. Yalinbas in [19] applied Taylor polynomial method to solve nonlinear integral equations. Razzaghi *et al.* solved these equations using Legendre wavelets method [14].

Recently, RBFs have been used to approximate the solution of a class of mixed two-dimensional nonlinear VFIEs [1]. Now in this paper, we concentrate on an idea which is based on the interpolation of hybrid of RBFs to approximate the solution of the nonlinear VFIEs. This method is included the hybrid of IMQs and sechs called briefly HISFs and the hybrid of strictly positive definite functions called briefly SPDFs. It should be noted that, IMQs and sechs have shape parameter and the stability interval of shape parameter is $[0, 2]$, therefore we show the fact that the hybrid of these functions made a good accuracy. We denote SPDFs can combine and result a good accuracy as well.

Shape parameter is the most important factor for increasing the accuracy of the numerical solution. The optimal choice of shape parameter was researched in [4,9,18,11,5] and this problem is still an open problem and under intensive investigation. We organized this paper as follows: In Section 2, we describe Bochner's theorem, details of RBFs, Legendre-Gauss-Lobatto nodes and weights. We develop the proposed method to approximate the solution of nonlinear VFIEs in Section 3. In Section 4, we establish the convergence of the proposed method. Finally, we report our numerical results which demonstrate the application of hybrid of RBFs to approximate the solution of VFIEs.

2 Strictly positive definite functions

Definition 2.1 A function ϕ on X is said to be positive definite if for any set of points $x_1, x_2, \ldots, x_N$ in X then an $N \times N$ matrix $A_{ij} = \phi(x_i - x_j)$ is nonnegative definite, i.e.

$$u^* A u = \sum_{i=1}^{N} \sum_{j=1}^{N} \bar{u}_i u_j A_{ij} \geq 0, \tag{2.1}$$

for all nonzero $u \in \mathbb{C}^N$.
If $u^* A u > 0$ whenever the points x_i are distinct and $u \neq 0$, then we say that $\phi(x)$ is an strictly positive definite functions, $[8 - 10]$. If $\phi(x)$ is a strictly positive definite function on a linear space, then the eigenvalues of A are positive and its determinant is positive.
Theorem 2.1 (Bochner's Theorem): Let a nonnegative Borel function be on R, if $0 < \int_{\mathbb{R}} f < \infty$, then $\hat{f}$ is an strictly positive definite functions, where $\hat{f}$ is the Fourier transform of function f, i.e.

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(y) e^{ixy} dy, \tag{2.2}$$

we can find many strictly positive definite functions by using this theorem. Thus for any set of distinct points $x_0, x_1, \ldots, x_N$ on $[a, b]$, the matrix $A_{ij} = f(||x_i - x_j||^2)$ is an SPDF, [5,2].

2.1 Definition of RBFs

Let $\mathbb{R}^+ = \{x \in \mathbb{R},\ x \geq 0\}$ be the nonnegative half-line and let $\phi : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function with $\phi(0) \geq 0$. RBFs on $\mathbb{R}^d$ are functions of the form

$$\phi(||x - x_i||), \tag{2.3}$$

in which $x, x_i \in \mathbb{R}^d$, and $||.||$ denotes the Euclidean distance between x and x_i's. If one chooses N points $\{x_i\}_{i=1}^N$ in $\mathbb{R}^d$ then by custom

$$y(x) \simeq \sum_{i=1}^{N} \lambda_i \phi(||X - X_i||), \quad \lambda_i \in \mathbb{R}, \tag{2.4}$$

is called the RBFs, [3,10].

2.2 Function approximation by hybrid functions

Let $\mathbf{\Phi}(x)$ and $\mathbf{\Theta}(x)$ be $n-$vectors of RBFs, the expansion of $y(x)$ in terms of RBFs can be defined as follows

$$\begin{aligned} y(x) &\simeq \textstyle\sum_{i=1}^{N}[\lambda_i\phi_i(x)+\mu_i\theta_i(x)] \\ &= C^T\mathbf{\Psi}(x), \end{aligned}$$

where $\mathbf{\Psi}(x) = \begin{bmatrix} \mathbf{\Phi}(x) & \mathbf{\Theta}(x) \end{bmatrix}^T$ is a $2n-$vector and the ith element of $\mathbf{\Phi}(x)$ and $\mathbf{\Theta}(x)$ can be considered in the following cases

Case(I):

$$\begin{aligned} \phi_i(x) &= \phi(x-x_i) = e^{-|x-x_i|}, \\ \theta_i(x) &= \theta(x-x_i) = e^{-(x-x_i)^2/4}, \end{aligned}$$

Case(II):

$$\begin{aligned} \phi_i(x) &= \phi(\|x-x_i\|) = sech(c\sqrt{\|x-x_i\|}), \\ \theta_i(x) &= \theta(\|x-x_i\|) = \tfrac{1}{\sqrt{\|x-x_i\|^2+c^2}}, \end{aligned}$$

and

$$\begin{aligned} \mathbf{\Phi} &= \begin{bmatrix} \phi_1(x) & \phi_2(x) & \ldots & \phi_N(x) \end{bmatrix}^T, \\ \mathbf{\Theta} &= \begin{bmatrix} \theta_1(x) & \theta_2(x) & \ldots & \theta_N(x) \end{bmatrix}^T, \\ \Lambda &= \begin{bmatrix} \lambda_1(x) & \lambda_2(x) & \ldots & \lambda_N(x) \end{bmatrix}^T, \\ M &= \begin{bmatrix} \mu_1(x) & \mu_2(x) & \ldots & \mu_N(x) \end{bmatrix}^T, \\ C &= \begin{bmatrix} \Lambda & M \end{bmatrix}^T, \end{aligned}$$

x is input and $\{\lambda_i\}_{i=1}^N$, $\{\mu_i\}_{i=1}^N$ are the set of constant coefficients of ϕ_i's and θ_i's, respectively, which can be determined.

2.3 Legendre-Gauss-Lobatto nodes and weights

Let $\mathcal{H}_N[-1,1]$ denotes the space of algebraic polynomials of degree $\leq N$

$$(P_i, P_j) = \frac{2}{2j+1}\delta_{ij},$$

Here $(.,.)$ represents the usual $L^2[-1,1]$ inner product and $\{P_i\}_{i\geq 0}$ are the well-known Legendre polynomials of order i which are orthogonal with respect to the weight function $w(x) = 1$ on the interval $[-1,1]$, and satisfy the following formula

$$\begin{aligned} p_0(x) &= 1, \quad p_1(x) = x, \\ p_{i+1}(x) &= (\frac{2i+1}{i+1})xp_i(x) - \frac{i}{i+1}p_{i-1}(x), \end{aligned}$$

for each $i \in \{1, 2, 3, ...\}$.

Next, we let $\{x_j\}_{j=0}^N$ denote the zeros of

$$(1-x^2)\dot{P}(x),$$

with

$$-1 = x_0 < x_1 < x_2 < ... < x_N = 1,$$

where $\dot{P}(x)$ is the first derivative of $P(x)$. No explicit formula for the nodes $\{x_j\}_{j=1}^{N-1}$ is known. However, they are computed numerically using the existing subroutines [7,8].

Now, we assume $f \in \mathcal{H}_{2N-1}[-1,1]$, we have

$$\int_{-1}^{1} f(x)\, dx \simeq \sum_{j=0}^{N} w_j f(x_j), \tag{2.5}$$

where w_j's are the Legendre-Gauss-Lobatto weights given in [17,13]

$$w_j = \frac{2}{N(N+1)} \times \frac{1}{(P_N(x_j))^2}. \tag{2.6}$$

3 Numerical solution of VFIES using HISFs and SPDFs

Consider a nonlinear VFIE as follows:

$$y(x) = f(x) + \lambda_1 \int_0^x k_1(x,t)G_1(t,y(t))dt + \lambda_2 \int_0^1 k_2(x,t)G_2(t,y(t))dt, \tag{3.1}$$

where $0 \leq x \leq 1$, λ_1 and λ_2 are constants, $f(x)$ and the kernels $k_1(x,t)$ and $k_2(x,t)$ are known functions assumed to have nth derivatives on the interval $0 \leq x, t \leq 1$. Let $G_1(t, y(t)) = F_1(y(t))$ and $G_2(t, y(t)) = F_2(y(t))$, where $F_1(y(t))$ and $F_1(y(t))$ are given continuous functions which are nonlinear with respect to $y(t)$.

Let $G_1(t, y(t)) = y^p(t),\ G_2(t, y(t)) = y^q(t)$, where p and q are nonnegative integers. In this paper, we propose a meshless collocation method based on both HISFs and SPDFs. We used two cases I and II to approximate the solution of nonlinear VFIEs as follows

$$\begin{aligned} \mathbf{\Psi}^T(x)C &= f(x) + \lambda_1 \textstyle\int_0^x k_1(x,t)G_1(t, \mathbf{\Psi}^T(t)C)dt \\ &\quad + \lambda_2 \textstyle\int_0^1 k_2(x,t)G_2(t, \mathbf{\Psi}^T(t)C)dt, \end{aligned} \tag{3.2}$$

for $0 \leq x \leq 1$. Now, we collocate Eq.(3.2) at points $\{x_i\}_{i=1}^N$ which are chosen as the zeros of Legendre polynomials given in Subsection 2.3

$$\begin{aligned} \mathbf{\Psi}^T(x_i)C &= f(x_i) + \lambda_1 \textstyle\int_0^{x_i} k_1(x_i,t)G_1(t, \mathbf{\Psi}^T(t)C)dt \\ &\quad + \lambda_2 \textstyle\int_0^1 k_2(x_i,t)G_2(t, \mathbf{\Psi}^T(t)C)dt. \end{aligned} \tag{3.3}$$

We first transform the integrals over $[0, x_i],\ [0, 1]$ into the integral over $[-1, 1]$ by using the following transformations, respectively.

$$\eta_1 = \frac{2}{x_i}t - 1, \quad t \in [0, x_i],$$

$$\eta_2 = 2t - 1, \ t \in [0, 1]. \tag{3.4}$$

Let

$$P(x_i, t) = k_1(x_i, t)G_1(t, \mathbf{\Psi}^T(t)C), \tag{3.5}$$

and

$$Q(x_i, t) = k_2(x_i, t)G_2(t, \mathbf{\Psi}^T(t)C), \tag{3.6}$$

by substituting Eqs.(3.4)-(3.6) in Eq.(3.3), we get

$$\mathbf{\Psi}^T(x_i)C = f(x_i) + \lambda_1 \frac{x_i}{2} \int_{-1}^{1} P(x_i, \frac{x_i}{2}(\eta_1+1))d\eta_1 + \frac{\lambda_2}{2} \int_{-1}^{1} Q(x_i, \frac{1}{2}(\eta_2+1))d\eta_2. \tag{3.7}$$

By using the Legendre-Gauss-Lobatto integration rule, we can rewrite Eq.(3.7) as follows

$$\mathbf{\Psi}^T(x_i)C = f(x_i) + \lambda_1 \frac{x_i}{2} \sum_{j=0}^{r_1} w_{1j} P(x_i, \frac{x_i}{2}(\eta_{1j}+1)) + \frac{\lambda_2}{2} \sum_{j=0}^{r_2} w_{2j} Q(x_i, \frac{1}{2}(\eta_{2j}+1)), \tag{3.8}$$

for $i = 1, \ldots, N$. Eq.(3.8) generates a system of $2N$ equations and $2N$ unknowns which can be solved by MATLAB software for the constant coefficients $\{\lambda_i\}_{i=1}^N$ and $\{\mu_i\}_{i=1}^N$, respectively.

4 Convergence analysis

Assume $(C[J], ||.||)$ the Banach space of all continuous functions on $J = [0, 1]$ with norm $||f(s)|| = \max_{\forall s \in J} |f(s)|$. Let $k_1(x, t) \leq M_1$ and $k_2(x, t) \leq M_2$, $\forall\ 0 \leq x, t \leq 1$. Suppose the nonlinear term $G(u)$ and $F(u)$ are satisfied in lipschitz condition such that

$$|F(u) - F(v)| \leq L_1 |u - v|,$$

$$|G(u) - G(v)| \leq L_2 |u - v|.$$

Theorem 4.1 The solution of nonlinear VFIE by using both HISFs and SPDFs approximation converges if $0 < \alpha < 1$.
Proof.

$$\begin{aligned}\|y_N - \bar{y}\| &= \max_{\forall x \in J} |y_N(x) - \bar{y}(x)| \\ &\leq \max_{\forall x \in J} (|\lambda_1| \int_0^x |k_1(x,t)|\ |F(y_N) - F(\bar{y})| dt \\ &+ |\lambda_2| \int_0^1 |k_2(x,t)|\ |G(y_N) - G(\bar{y})| dt) \\ &\leq (\lambda_1 M_1 L_1 x + \lambda_2 M_2 L_2) \max_{\forall x \in J} |y_N(x) - \bar{y}(x)| \\ &\leq \alpha \max_{\forall x \in J} |y_N(x) - \bar{y}(x)|, \end{aligned} \tag{4.1}$$

where $\alpha = \lambda_1 M_1 L_1 + \lambda_2 M_2 L_2$, it implies that $(1-\alpha)||y_N - \bar{y}|| \leq 0$ and choose $0 < \alpha < 1$, by increasing n it implies $||y_N - \bar{y}|| \to 0$ as $n \to \infty$ and this completes the proof. □
It should be noted that the conditions which were described in Theorem 4.1, are satisfied in the following examples.

5 Numerical examples

In order to illustrate the performance of the proposed method to obtain the numerical solution of VFIEs and justify the accuracy and efficiency of the presented method, we consider the absolute error between the exact solution and the numerical solution defined as

$$e(x) = |y(x) - \bar{y}(x)|, \quad x \in [0,1],$$

where $y(x)$ and $\bar{y}(x)$ are the exact and approximate solutions, respectively.

The errors and numerical results for examples 5.1-5.4 are shown in Tables 1-8 for different values of shape parameter, using both HISFs and SPDFs. x_i's are chosen the zeros of the shifted Legendre polynomials. The numerical results for examples 5.1-5.4 are plotted in Figures 1-6 as well. The computations associated with the examples were performed using **MATLAB** on a **PC**.

Example 5.1 Consider a nonlinear Fredholm integral equation given by [15]

$$y(x) = \int_0^1 xty^2(t)dt - \frac{5}{12}x + 1,$$

where $0 \leq x \leq 1$ such that the exact solution is $y(x) = 1 + \frac{1}{3}x$. Errors are listed in Tables 1-4 and Figures 1 and 2.

Example 5.2 Consider a linear Volterra integral equation given in [6,16]

$$y(x) = \cos(x) - \int_0^x (x-t)\cos(x-t)y(t)dt,$$

in which the exact solution is $y(x) = \frac{1}{3}(2\cos(\sqrt{3x}) + 1)$.
Errors are listed in Tables 5 and 6 and Figures 3 and 4.

Example 5.3 Consider a nonlinear Fredholm integral equation given in [2,**?**]

$$y(x) = \sinh(x) - \frac{1}{2} + \frac{1}{2}\cosh(1)\sinh(1) - \int_0^1 y^2(t)\ dt,$$

in which the exact solution is $y(x) = \sinh(x)$.

Example 5.4 Consider a nonlinear Fredholm integral equation given in [**?**]

$$y(x) = \sin(\pi x) + \frac{1}{5}\int_0^1 \cos(\pi x)\sin(\pi t)(y(t))^3\ dt,$$

in which the exact solution is $y(x) = \sin(\pi x) = \frac{20-\sqrt{391}}{3}\cos(\pi x)$.

Errors for examples 5.3 and 5.4 are listed in Tables 7 and 8 and Figures 5 and 6.

Table 1
Errors for Example 5.1 with $N = 8$ and $c = 0.1$

x	IMQ	sech	HISFs
0.013047	$9.139135E-3$	$4.243228E+14$	$4.247976E-3$
0.067468	$1.405161E-2$	$5.878156E+14$	$3.391788E-3$
0.160295	$2.055859E-2$	$8.670335E+14$	$1.992636E-3$
0.283302	$2.097754E-2$	$1.237529E+15$	$3.171492E-5$
0.425563	$1.281974E-2$	$1.666386E+15$	$3.470305E-3$
0.574437	$1.516602E-2$	$2.115158E+15$	$4.255340E-3$
0.716698	$3.082801E-2$	$2.543520E+15$	$5.445200E-4$
0.839705	$4.319414E-2$	$2.913197E+15$	$1.598424E-3$
0.932532	$6.697896E-3$	$3.191557E+15$	$1.521770E-3$
0.986953	$3.454421E-2$	$3.354451E+15$	$1.312679E-3$

Table 2
Errors for Example 5.1 with $N = 8$.

x	$e^{-\lvert x\rvert}$	$e^{\frac{-x^2}{4}}$	SPDFs
0.013047	$4.814595E-3$	$5.427293E-5$	$1.473544E-6$
0.067468	$1.698740E-3$	$1.170152E-4$	$1.952419E-7$
0.160295	$4.435115E-3$	$4.251683E-4$	$3.893693E-7$
0.283302	$6.803056E-3$	$8.710090E-4$	$1.483682E-6$
0.425563	$7.128748E-3$	$1.249202E-3$	$2.014688E-6$
0.574437	$9.068486E-3$	$1.639990E-3$	$2.326591E-6$
0.716698	$1.273383E-2$	$2.213368E-3$	$1.940971E-6$
0.839705	$1.366459E-2$	$2.868747E-3$	$1.577390E-6$
0.932532	$1.306527E-2$	$3.371800E-3$	$1.309427E-6$
0.986953	$2.380632E-2$	$3.631618E-3$	$4.625621E-6$

Table 3
Maximum errors for Example 5.1, using SPDFs method and the method in [17].

Presented method (SPDFs)			Method in [17]	
$N = 5$	$N = 8$	$N = 10$	$N = 5$	$N = 15$
1.654380E-5	4.625821E-6	6.662324E-6	4.791920E-3	3.607700E-5

Table 4
Maximum errors for Example 5.1, using HISFs method and the method in [17].

Presented method (HISFs)			Method in [17]
$N = 5, c = 0.2$	$N = 5, c = 0.3$	$N = 8, c = 0.1$	$N = 5$
2.257346E-3	1.509883E-3	4.255340E-3	4.791920E-3

Table 5
Errors for Example 5.2 with $N = 8$ and $c = 0.4$

x	IMQ	sech	HISFs
0.013047	$9.864016E-1$	$9.867827E-1$	$1.918793E-2$
0.067468	$9.285990E-1$	$9.279337E-1$	$1.667732E-2$
0.160295	$8.126207E-1$	$8.134875E-1$	$5.901942E-2$
0.283302	$6.348577E-1$	$6.342302E-1$	$8.931348E-2$
0.425563	$3.883411E-1$	$3.884240E-1$	$1.223069E-1$
0.574437	$8.994568E-2$	$8.980631E-2$	$1.832225E-1$
0.716698	$2.314131E-1$	$2.306404E-1$	$1.965060E-1$
0.839705	$5.300552E-1$	$5.311744E-1$	$1.425429E-1$
0.932532	$7.707179E-1$	$7.699431E-1$	$8.442405E-2$
0.986953	$9.135256E-1$	$9.139772E-1$	$4.864761E-2$

Table 6
Errors for Example 5.2 with $N = 8$.

x	$e^{-\|x\|}$	$e^{\frac{-x^2}{4}}$	SPDFs
0.013047	$9.515649E-1$	$9.868128E-1$	$1.283342E-2$
0.067468	$9.230834E-1$	$9.279226E-1$	$6.178702E-2$
0.160295	$8.021322E-1$	$8.134704E-1$	$1.284306E-1$
0.283302	$6.204309E-1$	$6.342426E-1$	$1.851344E-1$
0.425563	$3.722676E-1$	$3.884391E-1$	$2.091179E-1$
0.574437	$8.021935E-2$	$8.979031E-2$	$1.928978E-1$
0.716698	$2.356439E-1$	$2.306551E-1$	$1.461502E-1$
0.839705	$5.309179E-1$	$5.311519E-1$	$8.832946E-2$
0.932532	$7.692418E-1$	$7.699274E-1$	$3.816246E-2$
0.986953	$8.832978E-1$	$9.140216E-1$	$7.426841E-3$

Table 7
Errors for Examples 5.3 and 5.4 with $N = 5$

x	Example 5.3		Example 5.4	
	HISFs ($c = 0.2$)	SPDFs	HISFs ($c = 0.4$)	SPDFs
0.013047	$8.384882E-3$	$1.126497E-5$	$1.401904E-1$	$2.789338E-4$
0.067468	$7.617687E-3$	$1.126491E-5$	$1.088373E-1$	$2.729209E-4$
0.160295	$5.367900E-3$	$1.126505E-5$	$3.866894E-2$	$2.445125E-4$
0.283302	$1.020434E-3$	$1.126496E-5$	$6.905376E-2$	$1.757075E-4$
0.425563	$5.410554E-3$	$1.126503E-5$	$1.767838E-1$	$6.469054E-5$
0.574437	$1.007245E-2$	$1.126507E-5$	$2.105485E-1$	$6.469037E-5$
0.716698	$7.803156E-3$	$1.126498E-5$	$1.629329E-1$	$1.757076E-4$
0.839705	$7.692589E-3$	$1.126489E-5$	$1.166511E-1$	$2.445126E-4$
0.932532	$7.546369E-3$	$1.126493E-5$	$9.423770E-2$	$2.729208E-4$
0.986953	$7.050005E-3$	$1.126497E-5$	$8.471860E-2$	$2.789340E-4$

Table 8
Maximum errors for Examples 5.3 and 5.4, using $e^{-|x|}$, $e^{-\frac{x^2}{4}}$ and SPDFs and the method in [10] with $N = 5$.

	the presented method			method in [10]	
	$e^{-|x|}$	$e^{-x^2/4}$	SPDFs	$e^{-|x|}$	$\frac{1}{1+x^2}$
Exa 5.3	$8.784853E-2$	$2.498559E-4$	$1.126507E-5$	$6.0E-3$	$2.0E-3$
Exa 5.4	$1.038136E-1$	$7.286141E-4$	$2.789349E-4$		

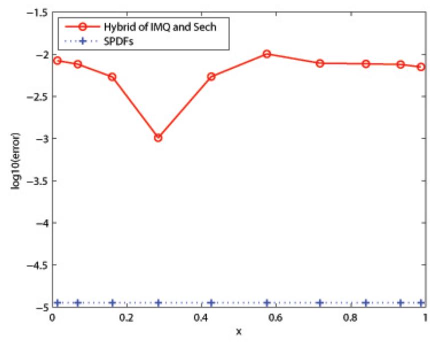

Fig. 1. Errors for Example 5.3 with $N = 5$ and $c = 0.2$ using HISFs, and SPDFs.

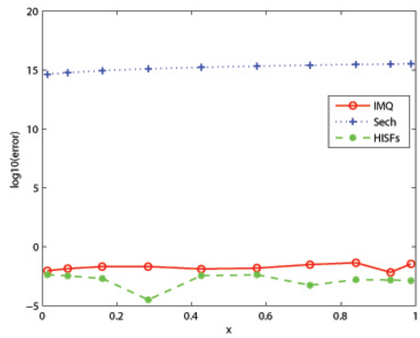

Fig. 2. Errors for Example 5.1 with $N = 8$ and $c = 0.1$ using IMQs, sechs and HISFs.

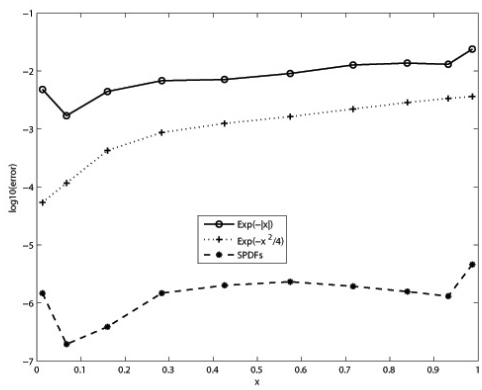

Fig. 3. Errors for Example 5.1 with $N = 8$ using $e^{-|x|}$, $e^{-\frac{x^2}{4}}$ and SPDFs.

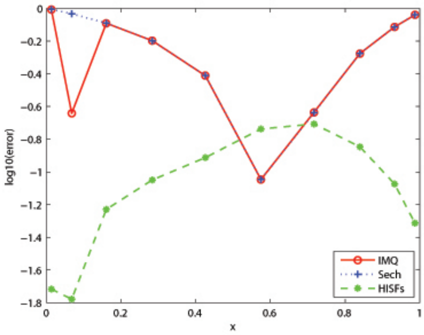

Fig. 4. Errors for Example 5.2 with $N = 8$ and $c = 0.4$ using IMQs, sechs and HISFs.

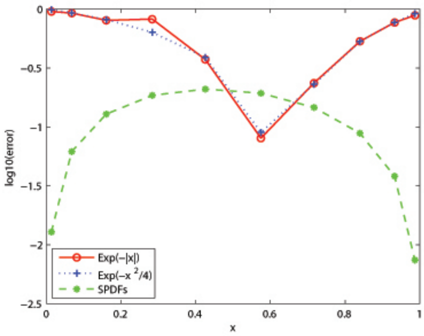

Fig. 5. Errors for Example 5.2 with $N = 5$ using $e^{-|x|}$, $e^{-\frac{x^2}{4}}$ and SPDFs

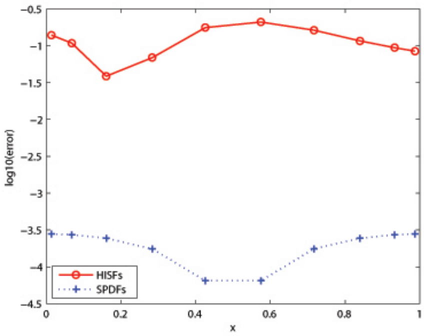

Fig. 6. Errors for Example 5.4 with $N = 5$ and $c = 0.4$ using HISFs, and SPDFs.

6 Conclusion

In this paper, we developed the application of interpolation by hybrid of RBFs including both HISFs and SPDFs for solving nonlinear VFIEs. This technique is based on the zeros of Legendre-Gauss-Lobatto as collocation points. In addition through the comparison with exact solution, we denoted that this method has good reliability and efficiency.

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