Theory of Approximation and Applications

Vol. 12, No.2, (2018), 43-60

Application of Chebyshev polynomials for solving Abel's integral equations of the first and second kind

M. M. Shamivand [a] A. Shahsavaran [a,*]

[a] *Department of mathematics, Borujerd branch, Islamic azad university, Borujerd, Iran.*

Received 23 April 2018; accepted 19 December 2018

Abstract

A numerical implementation of an expansion method is developed for solving Abel's integral equations. The solution of such equations may demonstrate a singular behaviour in the neighbourhood of the initial point of the interval of integration. The suggested method is based on the use of Taylor series expansion to overcome the singularity which leads to approximating the unknown function and it's derivatives in terms of Chebyshev polynomials of the first kind. Some numerical examples are included to clarify the accuracy and applicability of the presented method which indicate that proposed method is computationally very attractive.

Key words: Abel's integral equations; Chebyshev polynomials; Taylor series expansion; Collocation points.

2010 AMS Mathematics Subject Classification : 45A05, 45D05, 45E10

1 Introduction

Numerical methods are widely applied for approximating the solution of weakly singular Volterra integral equations (see, [1-15]). In the present paper, we focus on weakly singular Volterra integral equations of the first and second kind given by

$$g(x) = \lambda \int_0^x \frac{f(t)}{(x-t)^\alpha} dt, \tag{1.1}$$

$$f(x) = g(x) + \lambda \int_0^x \frac{f(t)}{(x-t)^\alpha} dt, \tag{1.2}$$

where, $0 \le t < x \le 1$, and $0 < \alpha < 1$. Also, $g(x) \in L^2(R)$ on the interval $0 \le x \le 1$, α, λ and the function $g(x)$ are given and the solution $f(x)$ to be determind. For $0 < \alpha < 1$, weakly singular integral equation (1.1) and (1.2) called Abel's integral equation of the first and second kinds, respectively. We assume that (1.1) and (1.2) have a unique solution. Abel's equation is one of the integral equations derived directly from a concrete problem of mechanics or physics (without passing through a differential equation). Historically, Abel's problem is the first one to lead to the study of integral equations and have applications in different fields, e. g., stereology of spherical particles, inversion of seismic travel times, cyclic voltametry, water wave scattering by two surface-piercing barriers, percolation of water, astrophysics, theory of superfluidity, heat transfer between solids and gases under nonlinear boundary conditions, propagation of shock-waves in gas fields tubes, subsolutions of a nonlinear diffusion problem, etc [10]. Several analytical and numerical methods have been recommended for solving Abel's integral equations such as Jacobi spectral collocation scheme [1], rational basis [2], Bernstein polynomials [3], fractional calculus [4], Adomian decomposition method [5], Nonpolynomial spline collocation method [6], product integration methods [7], analytical solution [8], modified Tau method [9], operational Muntz-Galerkin approximation [10], product integration approach based on new orthogonal polynomials [11], operational Haar wavelet method [12], Block-Pulse

* Corresponding author's E-mail:a.shahsavaran@iaub.ac.ir(Ahmad Shahsavaran)

Functions [13], Hybrid Block Pulse Functions and Bernstein polynomials [14], Legendre wavelets [15] and so forth.
In this paper we use Taylor series expansion of function $f(t)$ to overcome the singularity of Abel's integral equations (1.1) and (1.2). Also we approximate the unknown solution $f(t)$ and it's derivations in terms of Chebyshev polynomials of the first kind which reduces Abel's integral equations to a set of linear algebraic equations.

2 Chebyshev polynomials of the first and second kind

Chebyshev polynomials of the first and second kind are solutions to the Chebyshev differential equations

$$(1-x^2)y^{''} - xy^{'} + n^2 y = 0, \quad |x| < 1,$$

and

$$(1-x^2)y^{''} - 3xy^{'} + n(n+2)y = 0, \quad |x| < 1,$$

respectively. The Chebyshev polynomials of the first kind is denoted by $T_n(x)$ and is given by the formula

$$T_n(x) = \cos n\theta, \quad x = \cos\theta$$

and satisfies the recursive formula

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad T_1(x) = x, \quad T_0(x) = 1,$$

where the subscript n is the degree of these polynomials and the second kind Chebyshev polynomials is denoted by $U_n(x)$ and is given by the formula

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin(\theta)}, \quad x = \cos\theta.$$

Now with simplicity we can write

$$
\begin{aligned}
U_n(x) - U_{n-2}(x) &= \frac{\sin(n+1)\theta - \sin(n-1)\theta}{\sin\theta} \\
&= \frac{2\cos n\theta \sin\theta}{\sin\theta} \\
&= 2\cos n\theta \\
&= 2T_n(x).
\end{aligned} \tag{2.1}
$$

Where i is odd, by taking summation of the both side of (2.1) for $n = 2, 3, ..., i(n\, odd)$ we get

$$
U_i(x) - U_1(x) = 2(T_3(x) + T_5(x) + ... + T_i(x)), \quad U_1(x) = \frac{\sin 2\theta}{\sin\theta} = 2\cos\theta = 2T_1(x),
$$

so

$$
U_i(x) = 2(T_1(x) + T_3(x) + T_5(x) + ... + T_i(x)). \tag{2.2}
$$

In a similar manner, where i is even, we get

$$
U_i(x) - U_0(x) = 2(T_3(x) + T_5(x) + ... + T_i(x)), \quad U_0(x) = \frac{\sin\theta}{\sin\theta} = 1 = T_0(x),
$$

so

$$
U_i(x) = T_0(x) + 2(T_2(x) + T_4(x) + T_6(x) + ... + T_i(x)). \tag{2.3}
$$

For evaluating $T_i'(x)$ in terms of $T_i(x)$ by using $T_i(x) = \cos i\theta,\ x = \cos\theta$ we may proceed as follows

$$
\begin{aligned}
T_i'(x) &= \frac{d(T_i(x))}{dx} \\
&= \frac{d(\cos i\theta)}{dx} \\
&= \frac{d(\cos i\theta)}{d\theta}\frac{d\theta}{dx} \\
&= \frac{-i\sin i\theta}{-\sin\theta} \\
&= \frac{i\sin i\theta}{\sin\theta} \\
&= iU_{i-1}(x).
\end{aligned} \tag{2.4}
$$

Now using (2.2)-(2.4) yields:
where i is odd

$$\begin{aligned} T_i'(x) &= i(T_0(x) + 2T_2(x) + 2T_4(x) + \dots + 2T_{i-1}(x)) \\ &= 2i(\frac{1}{2}T_0(x) + T_2(x) + T_4(x) + \dots + T_{i-1}(x)), \end{aligned} \tag{2.5}$$

and where i is even

$$\begin{aligned} T_i'(x) &= i(2T_1(x) + 2T_3(x) + 2T_5(x) + \dots + 2T_{i-1}(x)) \\ &= 2i(T_1(x) + T_3(x) + T_5(x) + \dots + T_{i-1}(x)). \end{aligned} \tag{2.6}$$

Summarily, if we expand $T_i'(x)$ in terms of Chebyshev polynomials as

$$T_i'(x) = \sum_{k=0}^{i-1} \alpha_k T_k(x)$$

we have
where i is odd:

$$\alpha_0 = i, \quad \alpha_{2p} = 2i, \quad p = 1, 2, \dots, \frac{i-1}{2}, \quad \alpha_{2p+1} = 0, \quad p = 0, 1, \dots, \frac{i-3}{2}$$

and where i is even:

$$\alpha_{2p} = 0, \quad p = 0, 1, \dots, \frac{i}{2} - 1, \quad \alpha_{2p+1} = 2i, \quad p = 0, 1, \dots, \frac{i}{2} - 1.$$

For expanding $T_i''(x)$ in terms of Chebyshev polynomials, by using (2.5)-(2.6), we may proceed as follows:

for odd k

$$
\begin{aligned}
T_i''(x) &= \sum_{k=1}^{i-1} \alpha_k T_k'(x) \\
&= \sum_{k=1}^{i-1} \alpha_k .2k(\frac{1}{2}T_0(x) + T_2(x) + ... + T_{k-1}(x)) \\
&= 2\sum_{k=1}^{i-1} k\alpha_k \sum_{i_1=0}^{\frac{k-1}{2}}{}' \, T_{2i_1}(x) \\
&= 2\sum_{k=1}^{i-1} \sum_{i_1=0}^{\frac{k-1}{2}}{}' \, k\alpha_k \, T_{2i_1}(x), \qquad (2.7)
\end{aligned}
$$

(where now, and in all future occurrences, the notation $\sum'$ denotes a sum with the first term halved)
for even k

$$
\begin{aligned}
T_i''(x) &= \sum_{k=1}^{i-1} \alpha_k T_k'(x) \\
&= \sum_{k=1}^{i-1} \alpha_k .2k(T_1(x) + T_3(x) + ... + T_{k-1}(x)) \\
&= 2\sum_{k=1}^{i-1} k\alpha_k \sum_{i_1=1}^{\frac{k}{2}} T_{2i_1-1}(x) \\
&= 2\sum_{k=1}^{i-1} \sum_{i_1=1}^{\frac{k}{2}} k\alpha_k \, T_{2i_1-1}(x). \qquad (2.8)
\end{aligned}
$$

Similarly, for evaluating $T_i^{(3)}(x)$ in terms of Chebyshev polynomials by taking derivative of (2.7)-(2.8) and using (2.5)-(2.6) we obtain:

for odd k

$$
\begin{aligned}
T_i^{(3)}(x) &= 2\sum_{k=1}^{i-1}\sum_{i_1=0}^{\frac{k-1}{2}}{}' \, k\alpha_k T'_{2i_1}(x)\\
&= 2\sum_{k=1}^{i-1}\sum_{i_1=0}^{\frac{k-1}{2}}{}' k\alpha_k.2(2i_1)(T_1(x)+T_3(x)+...+T_{2i_1-1}(x))\\
&= 2^2\sum_{k=1}^{i-1}\sum_{i_1=0}^{\frac{k-1}{2}}{}' k\alpha_k(2i_1)\sum_{i_2=1}^{i_1} T_{2i_2-1}(x)\\
&= 2^2\sum_{k=1}^{i-1}\sum_{i_1=0}^{\frac{k-1}{2}}{}'\sum_{i_2=1}^{i_1} k\alpha_k(2i_1)\, T_{2i_2-1}(x), \qquad (2.9)
\end{aligned}
$$

and for even k

$$
\begin{aligned}
T_i^{(3)}(x) &= 2\sum_{k=1}^{i-1}\sum_{i_1=1}^{\frac{k}{2}} k\alpha_k T'_{2i_1-1}(x)\\
&= 2\sum_{k=1}^{i-1}\sum_{i_1=1}^{\frac{k}{2}} k\alpha_k.2(2i_1-1)(\frac{1}{2}T_0(x)+T_2(x)+...+T_{2i_1-2}(x))\\
&= 2^2\sum_{k=1}^{i-1}\sum_{i_1=1}^{\frac{k}{2}} k\alpha_k(2i_1-1)\sum_{i_2=0}^{i_1-1}{}' T_{2i_2}(x)\\
&= 2^2\sum_{k=1}^{i-1}\sum_{i_1=1}^{\frac{k}{2}}\sum_{i_2=0}^{i_1-1}{}' k\alpha_k(2i_1-1)\, T_{2i_2}(x). \qquad (2.10)
\end{aligned}
$$

In general, for evaluating $T_i^{(r)}(x)$ for $r=1,2,...,i$ ($T_i^{(r)}(x)=0$ for $r>i$) in terms of Chebyshev polynomials, by repeated use of (2.5)-(2.6) we obtain:
for odd k :
where r is odd:

$$
T_i^{(r)}(x) = 2^{r-1}\sum_{k=1}^{i-1}\sum_{i_1=0}^{\frac{k-1}{2}}{}' \left\{ \sum_{i_2=1}^{i_1}\sum_{i_3=0}^{i_2-1}{}' ... \sum_{i_{r-1}=1}^{i_{r-2}} k\alpha_k(\prod_{p=1}^{\frac{r-1}{2}} 2i_{2p-1})(\prod_{p=1}^{\frac{r-3}{2}}(2i_{2p}-1))T_{2i_{r-1}-1}(x), \right\} \qquad (2.11)
$$

and where r is even:

$$T_i^{(r)}(x) = 2^{r-1}\sum_{k=1}^{i-1}\sum_{i_1=0}^{\frac{k-1}{2}}{}' \left\{\sum_{i_2=1}^{i_1}\sum_{i_3=0}^{i_2-1}{}' ... \sum_{i_{r-1}=0}^{i_{r-2}-1}{}' k\alpha_k(\prod_{p=1}^{\frac{r}{2}-1} 2i_{2p-1})(\prod_{p=1}^{\frac{r}{2}-1}(2i_{2p}-1))T_{2i_{r-1}}(x).\right\} \tag{2.12}$$

In equations (2.11)-(2.12), the sigma notation into curly brackets "{ }", that is $\sum_{i_p}$, for odd p is $\sum_{i_p=0}^{i_{p-1}-1}{}'$ and for even p is $\sum_{i_p=1}^{i_{p-1}}$.
For even k :
where r is odd:

$$T_i^{(r)}(x) = 2^{r-1}\sum_{k=1}^{i-1}\sum_{i_1=1}^{\frac{k}{2}} \left\{\sum_{i_2=0}^{i_1-1}{}' \sum_{i_3=1}^{i_2} ... \sum_{i_{r-1}=0}^{i_{r-2}-1}{}' k\alpha_k(\prod_{p=1}^{\frac{r-1}{2}}(2i_{2p-1}-1))(\prod_{p=1}^{\frac{r-3}{2}} 2i_{2p})T_{2i_{r-1}}(x),\right\} \tag{2.13}$$

and where r is even:

$$T_i^{(r)}(x) = 2^{r-1}\sum_{k=1}^{i-1}\sum_{i_1=1}^{\frac{k}{2}} \left\{\sum_{i_2=0}^{i_1-1}{}' \sum_{i_3=1}^{i_2} ... \sum_{i_{r-1}=1}^{i_{r-2}} k\alpha_k(\prod_{p=1}^{\frac{r}{2}-1}(2i_{2p-1}-1))(\prod_{p=1}^{\frac{r}{2}-1} 2i_{2p})T_{2i_{r-1}-1}(x).\right\} \tag{2.14}$$

In equations (2.13)-(2.14) the sigma notation into curly brackets "{ }", that is $\sum_{i_p}$, for even p is $\sum_{i_p=0}^{i_{p-1}-1}{}'$ and for odd p is $\sum_{i_p=1}^{i_{p-1}}$.

3 Solution of Abel's integral equation of the first and second kind

In this section we numerically solve the Abel's integral equations of the form (1.1) and (1.2) using Chebyshev polynomials of the first kind. firstly, we use Taylor series expansion of the function $f(t)$ about the point $t = x$ which leads to

$$f(t) = f(x) + \sum_{r=1}^{n}\frac{f^{(r)}(x)}{r!}(t-x)^r + R_n(t,x),$$

where $R_n(t,x)$ is the reminder term and defined by

$$R_n(t,x) = \frac{f^{(n+1)}(\eta)}{(n+1)!}(t-x)^{n+1}$$

for some real number η between x and t. Furthermore, if $|f^{(n+1)}(x)| \leq M$ on the interval $[0,1]$ then, with simplicity, it can be shown that $|R_n(t,x)| \leq \frac{M}{(n+1)!}$ for arbitrary chosen points t and x on the interval $[0,1]$, thus for sufficiently large n reminder term $R_n(t,x)$ tends to zero and can be disregarded.
Therefore $f(t)$ can be represented by $n+1$ initial terms of Taylor series as

$$f(t) = f(x) + \sum_{r=1}^{n} \frac{f^{(r)}(x)}{r!}(t-x)^r, \tag{3.1}$$

which leads to

$$f(x) - f(t) = -\sum_{r=1}^{n} \frac{f^{(r)}(x)(-1)^r}{r!}(x-t)^r. \tag{3.2}$$

Now for solving Abel's integral equation of the first kind (1.1) by utilizing (3.2) we may proceed as follows

$$\begin{aligned}
g(x) &= \lambda \int_0^x \frac{f(t)}{(x-t)^\alpha} dt \\
&= -\lambda \int_0^x \frac{f(x) - f(t) - f(x)}{(x-t)^\alpha} dt \\
&= -\lambda \int_0^x \frac{f(x) - f(t)}{(x-t)^\alpha} dt + \lambda f(x) \int_0^x (x-t)^{-\alpha} dt \\
&= \lambda \sum_{r=1}^{n} \frac{f^{(r)}(x)(-1)^r}{r!} \int_0^x (x-t)^{r-\alpha} dt + \lambda f(x) \int_0^x (x-t)^{-\alpha} dt \\
&= \lambda \sum_{r=1}^{n} \frac{f^{(r)}(x)(-1)^r}{r!} \frac{x^{r-\alpha+1}}{r-\alpha+1} + \lambda f(x) \frac{x^{1-\alpha}}{1-\alpha} \\
&= \lambda \sum_{r=1}^{n} f^{(r)}(x) B_{\alpha,r}(x) + \lambda f(x) A_\alpha(x),
\end{aligned} \tag{3.3}$$

where

$$B_{\alpha,r}(x) = \frac{(-1)^r}{r!} \frac{x^{r-\alpha+1}}{r-\alpha+1}, \quad A_\alpha(x) = \frac{x^{1-\alpha}}{1-\alpha}. \tag{3.4}$$

Similarly, for solving Abel's integral equation of the second kind (1.2), by utilizing (3.2) and (3.4) we get

$$g(x) = (1 - \lambda A_\alpha(x))f(x) - \lambda \sum_{r=1}^{n} f^{(r)}(x) B_{\alpha,r}(x). \tag{3.5}$$

Now we approximate $f(x)$ by $f_m(x)$ in terms of Chebyshev polynomials as

$$f_m(x) = \sum_{i=0}^{m}{}' f_i \, T_i(x), \tag{3.6}$$

so for $r = 1, 2, ..., m$ we have

$$f_m^{(r)}(x) = \sum_{i=0}^{m}{}' f_i \, T_i^{(r)}(x),$$

which according to (2.11)-(2.14), $f_m^{(r)}(x)$ can be expressed in terms of Chebyshev polynomials $T_i(x)$. Therefore, for finding the solution of Abel's integral equations of the first and second kind (1.1) and (1.2), we collocate (3.3) and (3.5) at the points

$$x_k = \frac{1 + \cos(\frac{k\pi}{m})}{2}, \; k = 0, 1, ..., m$$

which gives a system of $m+1$ linear algebraic equations and can be solved for unknown coefficients f_i, $i = 0, 1, ..., m$. So we get the desired approximation for $f(x)$ by (3.6).

4 Illustrative examples

To illustrate the efficiency of the proposed method in this paper we consider several examples whose exact solutions are exist.

Example 4.1 *Consider the Abel's integral equation of the second kind (see [14])*

$$f(x) = x^7(1 - \frac{4096}{6435}\sqrt{x}) + \int_0^x \frac{f(t)}{\sqrt{x-t}} dt,$$

exact solution is $f(x) = x^7$.

In this example we consider $m = n = 7$, (m is the number of Chebyshev polynomials and n is the number of terms in Taylor series expansion) so according to described method in section 2, for calculating $f_m^{(r)}(x)$ in terms of Chebyshev polynomials, we need to know the coefficients in expansion of $T_i^{(r)}(x)$ in terms of $T_i(x)$ which are listed for $r = 1, 2, ..., 7$, as below

for $r = 1$:
$T_0'(x) = 0,$
$T_1'(x) = T_0(x),$
$T_2'(x) = 4\,T_1(x),$
$T_3'(x) = 3\,T_0(x) + 6\,T_2(x),$
$T_4'(x) = 8\,T_1(x) + 8\,T_3(x),$
$T_5'(x) = 5\,T_0(x) + 10\,T_2(x) + 10\,T_4(x),$
$T_6'(x) = 12\,T_1(x) + 12\,T_3(x) + 12\,T_5(x),$
$T_7'(x) = 7\,T_0(x) + 14\,T_2(x) + 14\,T_4(x) + 14\,T_6(x),$

for $r = 2$:
$T_0''(x) = T_1''(x) = 0,$
$T_2''(x) = 4\,T_0(x),$
$T_3''(x) = 24\,T_1(x),$
$T_4''(x) = 32\,T_0(x) + 48\,T_2(x),$
$T_5''(x) = 120\,T_1(x) + 80\,T_3(x),$
$T_6''(x) = 108\,T_0(x) + 192\,T_2(x) + 120\,T_4(x),$
$T_7''(x) = 336\,T_1(x) + 280\,T_3(x) + 168\,T_5(x),$

for $r = 3$:
$T_0^{(3)}(x) = T_1^{(3)}(x) = T_2^{(3)}(x) = 0,$
$T_3^{(3)}(x) = 24\,T_0(x),$
$T_4^{(3)}(x) = 192\,T_1(x),$
$T_5^{(3)}(x) = 360\,T_0(x) + 480\,T_2(x),$
$T_6^{(3)}(x) = 1728\,T_1(x) + 960\,T_3(x),$
$T_7^{(3)}(x) = 2016\,T_0(x) + 3360\,T_2(x) + 1680\,T_4(x),$

for $r = 4$:
$T_0^{(4)}(x) = T_1^{(4)}(x) = T_2^{(4)}(x) = T_3^{(4)}(x) = 0,$
$T_4^{(4)}(x) = 192\,T_0(x),$

$T_5^{(4)}(x) = 1920\, T_1(x),$
$T_6^{(4)}(x) = 4608\, T_0(x) + 5760\, T_2(x),$
$T_7^{(4)}(x) = 26880\, T_1(x) + 13440\, T_3(x),$

for $r = 5:$
$T_0^{(5)}(x) = T_1^{(5)}(x) = T_2^{(5)}(x) = T_3^{(5)}(x) = T_4^{(5)}(x) = 0,$
$T_5^{(5)}(x) = 1920\, T_0(x),$
$T_6^{(5)}(x) = 23040\, T_1(x),$
$T_7^{(5)}(x) = 67200\, T_0(x) + 80640\, T_2(x),$

for $r = 6:$
$T_0^{(6)}(x) = T_1^{(6)}(x) = T_2^{(6)}(x) = T_3^{(6)}(x) = T_4^{(6)}(x) = T_5^{(6)}(x) = 0,$
$T_6^{(6)}(x) = 23040\, T_0(x),$
$T_7^{(6)}(x) = 322560\, T_1(x),$

for $r = 7:$
$T_0^{(7)}(x) = T_1^{(7)}(x) = T_2^{(7)}(x) = T_3^{(7)}(x) = T_4^{(7)}(x) = T_5^{(7)}(x) = T_6^{(7)}(x) = 0,$
$T_7^{(7)}(x) = 322560\, T_0(x).$

Example 4.2 *In this example we consider the second kind Volterra integral equation of Abel's type (see [4,12])*

$$f(x) = \frac{1}{x+1} + \frac{2\arcsin h(\sqrt{x})}{\sqrt{x+1}} - \int_0^x \frac{f(t)}{\sqrt{x-t}} dt,$$

exact solution is $f(x) = \frac{1}{x+1}$.

Example 4.3 *Consider the Abel's integral equation with linear singularity*

$$f(x) = x^2(1-x)^2 - \frac{729}{15400}x^{\frac{14}{3}} + \frac{243}{2200}x^{\frac{11}{3}} - \frac{27}{400}x^{\frac{8}{3}} + \frac{1}{10}\int_0^x \frac{f(t)}{(x-t)^{\frac{1}{3}}} dt,$$

exact solution is $f(x) = x^4 - 2x^3 + x^2$.

Example 4.4 *Consider the Abel's integral equation of the second kind* *(see [13])*

$$f(x) = \frac{1}{\sqrt{x+1}} + \frac{\pi}{8} - \frac{1}{4}\arcsin(\frac{1-x}{1+x}) - \frac{1}{4}\int_0^x \frac{f(t)}{\sqrt{x-t}}dt,$$

with exact solution $f(x) = \frac{1}{\sqrt{x+1}}$.

The absolute error $|f(x) - f_m(x)|$ for examples 1-4 are included in Table 1 for $m = n = 7$.

Example 4.5 *Consider the Abel's integral equation of the second kind (see [4])*

$$f(x) = x + \frac{4}{3}x^{\frac{3}{2}} - \int_0^x \frac{f(t)}{\sqrt{x-t}}dt,$$

with exact solution $f(x) = x$.

Example 4.6 *We consider the second kind Volterra integral equation of Abel's type (see [4,13])*

$$f(x) = x^2 + \frac{16}{15}x^{\frac{5}{2}} - \int_0^x \frac{f(t)}{\sqrt{x-t}}dt,$$

with exact solution $f(x) = x^2$.

The absolute error $|f(x) - f_m(x)|$ for examples 5-6 are presented in Table 1 for $m = 3$ and $n = 2$.

Table 1: Absolute error for the examples 1-6

	m=n=7				m=3,n=2	
t	example 1	example 2	example 3	example 4	example 5	example 6
0.0	0.0	3.00×10^{-10}	3.71×10^{-10}	0.0	0.0	0.0
0.1	5.40×10^{-10}	1.42×10^{-6}	3.00×10^{-11}	3.62×10^{-7}	1.00×10^{-11}	5.00×10^{-11}
0.2	2.70×10^{-10}	7.46×10^{-7}	2.00×10^{-11}	1.23×10^{-7}	0.0	5.00×10^{-11}
0.3	9.20×10^{-10}	1.81×10^{-6}	1.00×10^{-11}	4.68×10^{-7}	0.0	3.40×10^{-10}
0.4	2.93×10^{-10}	8.27×10^{-7}	2.00×10^{-10}	1.19×10^{-7}	0.0	7.00×10^{-10}
0.5	1.63×10^{-9}	2.03×10^{-6}	5.00×10^{-10}	4.86×10^{-7}	0.0	9.00×10^{-10}
0.6	1.50×10^{-9}	1.92×10^{-7}	9.00×10^{-10}	3.68×10^{-8}	1.00×10^{-10}	1.10×10^{-9}
0.7	1.07×10^{-9}	1.85×10^{-6}	8.00×10^{-10}	4.01×10^{-7}	1.00×10^{-10}	1.20×10^{-9}
0.8	2.00×10^{-9}	7.30×10^{-8}	1.00×10^{-10}	4.63×10^{-8}	1.00×10^{-10}	1.30×10^{-9}
0.9	3.10×10^{-9}	1.10×10^{-6}	2.40×10^{-9}	2.39×10^{-7}	1.00×10^{-10}	9.00×10^{-10}
1.0	2.00×10^{-9}	1.59×10^{-8}	0.0	8.00×10^{-10}	1.00×10^{-10}	4.00×10^{-10}

Example 4.7 *We consider the first kind Volterra integral equation of Abel's type (see [4])*

$$\frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3) = \int_0^x \frac{f(t)}{\sqrt{x-t}}dt,$$

with exact solution $f(x) = x^3 - x^2 + 1.$

For showing the numerical results for the example 7, the exact and approximate solutions are compared in Figure 1 with $m = n = 7$. Also, the absolute error function $|f(x) - f_m(x)|$ for this example, is plotted in Figure 2 with $m = n = 7$.

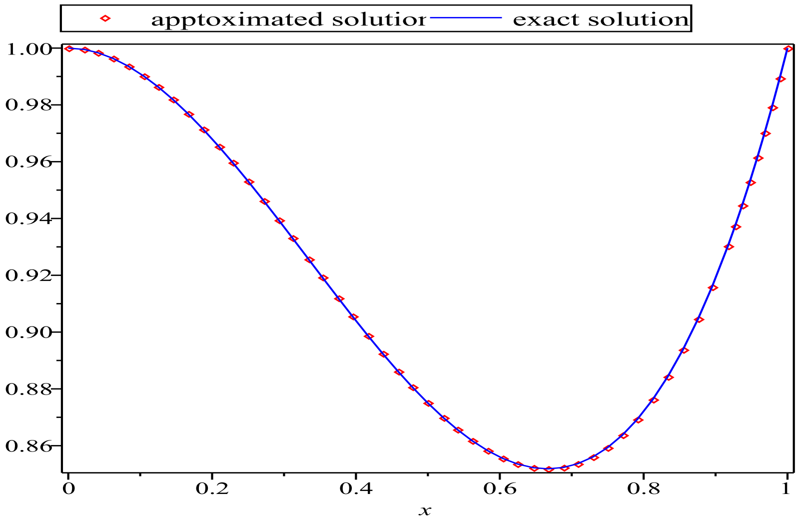

Fig. 1. Comparison of the exact and approximate solution for example 7 with m=n=7

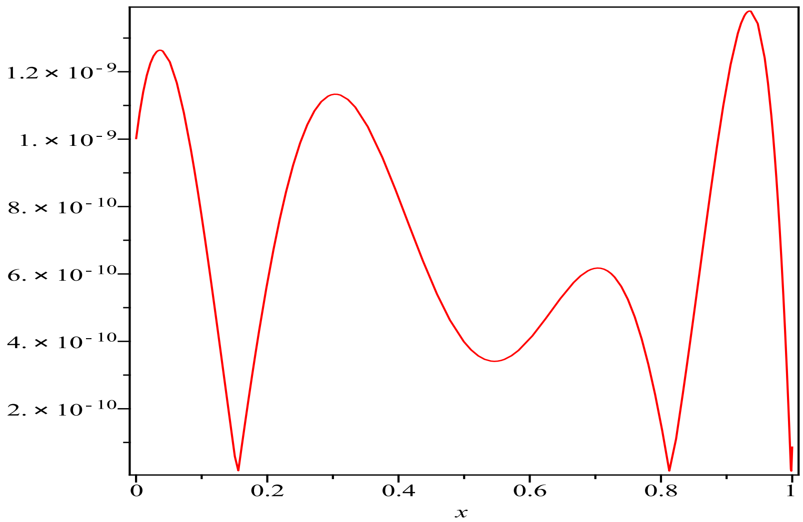

Fig. 2. Plot of the absolute error for example 7 with m=n=7

Example 4.8 *Consider the first kind Abel's integral equation (see [4])*

$$e^x - 1 = \int_0^x \frac{f(t)}{\sqrt{x-t}}dt,$$

with exact solution $f(x) = \frac{1}{\sqrt{\pi}} e^x erf(\sqrt{x})$, *where* $erf(x)$ *is the error function and defined by*

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

In Figure 3, the absolute error function $|f(x) - f_m(x)|$ for example 8, is plotted with $m = n = 7$.

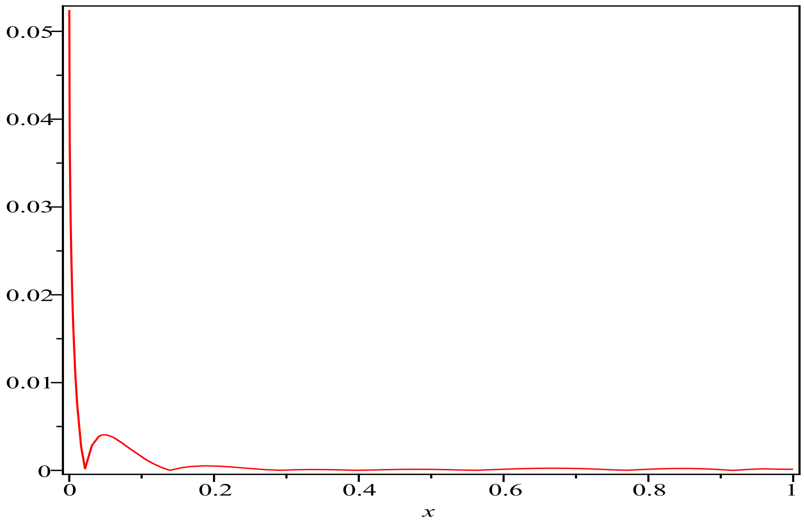

Fig. 3. Plot of the absolute error for example 8 with m=n=7

5 Conclusion

In this paper, a numerical technique is constructed to introduce an approximate solution for Abel's integral equations of the first and second kind. The proposed method is consisting of reducing Abel's integral equations to a system of linear algebraic equations by expanding the approximate solution in terms of Chebyshev polynomials with unknown coefficients. The numerical results demonstrate that good accuracy of the proposed method is obtaind by taking only a small number of Chebyshev polynomials.

Acknowledgments

The authors have greatly benefited from valuable suggestions made by the reviewers.

References

[1] M. A. Abdelkawy, S. S. Ezz-Eldien and A. Z. Amin, A Jacobi spectral collocation scheme for solving Abel's integral equations, Progress in Fractional Differentiation and Applications, 1 (2015) 1-14.

[2] S. Abelman and D. Eyre, A rational basis for second-kind Abel integral equations, J. Comput. Appl. Math., 34 (1991) 281–290.

[3] M. Alipour and D. Rostamy, Bernstein polynomials for solving Abel's integral equation, The J. Math. Comp. Sci., 3 (2011) 403-412.

[4] Z. Avazzadeh, B. Shafiee and G. B. Loghmani, Fractional calculus for solving Abel's integral equations using Chebyshev polynomials, Appl. Math. Sci., 5 (2011) 2207-2216.

[5] L. Bougoffa, R. C. Rach and A. Mennouni, A convenient technique for solving linear and nonlinear Abel integral equations by the Adomian decomposition method, Appl. Math. Comput., 218 (2011) 1785–1793.

[6] H. Brunner, Nonpolynomial spline collocation method for Volterra equations with weakly singular kernels, SIAM J. Numer. Anal., 20 (1983) 1106–1119.

[7] R. F. Cameron, Product integration methods for second kind Abel integral equations, J. Comput. Appl. Math., 11 (1984) 1–10.

[8] S. Kumar, A. Kumar, D. Kumar, J. Singh and A. Singh, Analytical solution of Abel integral equation arising in astrophysics via Laplace transform, J. Egypt. Math. Soc., 23 (2015) 102–107.

[9] P. Mokhtary, High-order modified Tau method for non-smooth solutions of Abel integral equations, Electron. Trans. Numer. Anal., 44 (2015) 462–471.

[10] P. Mokhtary, Operational Muntz-Galerkin approximation for Abel-Hammerstein integral equations of the second kind, Electron. Trans. Numer. Anal., 45 (2016) 183–200.

[11] M. Rasty, "A product integration approach based on new orthogonal polynomials for nonlinear weakly singular integral equations", Acta. Appl. Math., 109 (2010) 861-873.

[12] H. Saeedi, N. Mollahasani, M. M. Moghadam and G. N. Chuev, An operational Haar wavelet method for solving fractional Volterra integral equatiobs, Int. J. Appl. Math. Comput. Sci., 21 (2011) 535–547.

[13] A. Shahsavaran, Numerical approach to solve second kind Volterra integral equation of Abel Type using Block-Pulse Functions and Taylor expansion by collocation method, Appl. Math. Sci., 5 (2011) 685-696.

[14] M. Tavassoli Kajani, Numerical solution of Volterra integral equations by using Hybrid Block Pulse Functions and Bernstein polynomials, Int. J. Pure and Appl. Math., 107 (2016) 331-341.

[15] S. A. Yousefi, "Numerical solution of Abel's integral equation by using Legendre wavelets", Appl. Math. Comput., 175 (2006) 574-580.