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Exact Implementation of Multiple Initial Conditions in the DQ Solution of Higher-Order ODEs

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ABSTRACT

The differential quadrature method (DQM) is one of the most elegant and useful approximate methods for solving initial and/or boundary value problems. It is easy to use and also straightforward to implement. However, the conventional DQM is well-known to have some difficulty in implementing multiple initial and/or boundary conditions at a given discrete point. To overcome this difficulty, this paper presents a simple and accurate differential quadrature methodology in which the higher-order initial conditions are exactly implemented. The proposed methodology is very elegant and uses a set of simple polynomials with a simple transformation to incorporate the higher-order initial conditions at the initial discrete time point. The order of accuracy of the proposed method for solving an rth order ordinary differential equation is "$m + r - 1$," where m being the number of discrete time points. This is better than the accuracy of the CBCGE (direct Coupling the Boundary/initial Conditions with the discrete Governing Equations) and MWCM (Modifying Weighting Coefficient Matrices) approaches whose order is in general "$m - 1$." Some test problems are also provided to highlight the superiority of the proposed method over the CBCGE and MWCM approaches.

Keywords : New differential quadrature methodology; Imposing multiple initial conditions; Higher-order initial-value problems; CBCGE approach; MWCM approach; Beams; Rectangular plates.

1 INTRODUCTION

THE differential quadrature method (DQM) is a high-order point discretization method for numerical solution of partial differential equations arising in engineering and applied sciences. It was first introduced by Bellman and his associates [1, 2] in the early 1970's. The central idea of the DQM is to approximate the derivative of a function with respect to a co-ordinate direction at a discrete point by a linear sum of all the function values at all the discrete points chosen along that direction [3, 4]. Compared with the finite difference and finite element methods, the DQM can produce numerical results with higher order of accuracy by using a considerably smaller number of discrete points. Another particular advantage of the DQM is its ease of use and implementation. However, in spite of its many advantages, the conventional DQM has its own difficulty in implementation to the differential equations with multiple initial/boundary conditions at a given initial/boundary point. Conventionally, the boundary conditions are expressed as differential quadrature analog equations at the discrete points on or near the boundaries. These analog equations are then used to replace the differential quadrature analog equations of the governing differential

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equations at these points in order to solve the problem. However, this procedure may become very problematic when multi-boundary conditions are involved along the boundary [3].

There has been a considerable effort to overcome this difficulty. Bert et al. [5] also Jang and Bert [6] proposed a δ-technique to impose the derivate boundary conditions. In this technique, one boundary condition is exactly discretized while the others are approximately satisfied at a small distance δ ($\approx 10^{-5}$ in dimensionless value) adjacent to the boundary points. The δ-point technique has been successfully applied to many structural problems such as the beam, plate and shell structures and has been emerged as one of the most representative steps in the development of the DQM. However, this technique may produce an unexpected oscillation behavior of the DQM solutions [3, 4]. To overcome the difficulties of the δ-technique, Wang and Bert [7], Wang et al. [8], and Malik and Bert [9] proposed several approaches to impose the boundary conditions by modification of weighting coefficient matrices (say, the MWCM approach). However, as indicated by Wang and Bert [7], Shu and Du [10, 11], and Eftekhari [12], there are some major limitations to the application of these techniques in implementing general boundary conditions. One limitation is in the implementation of CC type boundary conditions [10, 11, 12]. The implementation of the FF type boundary conditions in the beam problem by this approach was also shown to lead to some wrong numerical results [12]. For initial-value problems, Tanaka and Chen [13] have employed this technique to incorporate the given initial conditions for transient responses of elastodynamic problems.

Alternatively, Shu and Du [10, 11] proposed two approaches, referred to as the SBCGE (direct Substituting the discrete Boundary Conditions into the discrete Governing Equations) approach [10] and the CBCGE (direct Coupling the Boundary Conditions with the discrete Governing Equations) approach [11], for implementing the general boundary conditions of the beams and rectangular plates. It was shown that the SBCGE and CBCGE approaches can remove some drawbacks of the δ-technique and produce accurate solutions for beams and plates with general boundary conditions. However, in these approaches, the grid points adjacent to the boundary points are treated approximately as the boundary points. Actually, these points are not boundary points but are treated as boundary points. On the other hand, the SBCGE and CBCGE approaches may encounter some difficulties when multi-boundary conditions are involved along the boundary [11, 12]. For instance, the case of rectangular plates with free corners cannot be accurately handled using these approaches. Shu and Yao [14], Hashemi et al. [15], Malekzadeh and Rahideh [16], Civalek [17], and Civalek and Oztürk [18] have also used the CBCGE approach to impose the initial conditions in various initial-boundary-value problems. More recently, Golfam and Rezaie [19] have reformulated the SBCGE approach to implement the homogeneous and non-homogeneous boundary conditions of the beams. On the other hand, Eftekhari [12] proposed a simple and systematic scheme for imposing boundary conditions in the DQM free and forced vibration analysis of beams and rectangular plates with general boundary conditions. The method was shown to be the same order of accuracy of the CBCGE approach while it was much simpler than the CBCGE approach. Most importantly, the case of rectangular plates with free corners has been successfully handled by this approach.

In another attempt, Wu and Liu [20-22] and Wu at al. [23] proposed a generalized differential quadrature rule (GDQR), where the Hermite interpolation functions are used as the trial functions, to implement the multiple boundary/initial conditions. However, as indicated by Fung [24], this approach is in fact equivalent to the conventional CBCGE approach. Alternatively, Fung [24, 25] proposed a modified differential quadrature rule to impose the higher-order initial conditions. He showed that this approach is equivalent to the approach proposed by Wang and Bert [7] where the derivatives initial conditions are incorporated to the solution process by modification of weighting coefficient matrices.

In this paper, a new methodology in DQM application to initial-value problems that are governed by second- or higher-order ordinary differential equations in time is presented. In this methodology, a simple transformation is first employed to make the non-homogenous initial conditions homogenous. A set of simple polynomials, which satisfy the homogeneous form of the specified initial conditions, is then used as the test functions to obtain the DQM weighting coefficients. By doing so, the zero initial conditions are exactly built into the DQM weighting coefficient matrices. Besides, when this methodology is applied to higher-order initial-value problems, the non-homogenous initial conditions are automatically and exactly satisfied. The order of accuracy of the proposed method for solving an rth order ordinary differential equation is "$m + r - 1$," where m being the number of discrete time points. This is better than the accuracy of the CBCGE and MWCM approaches whose order is in general "$m - 1$." Some test problems are also provided to show the superiority of the proposed method over the CBCGE and MWCM approaches.

2 DIFFERENTIAL QUADRATURE METHOD

The DQM is based on the idea that the derivative of a function at any discrete point can be approximated by a weighted linear sum of all the functional values in the whole domain [1, 2]. Let $X(t)$ be a solution of a differential equation and $t_1, t_2, t_3, ..., t_m$ be a set of discrete points in the time direction. According to the DQM, the *r*th-order derivative of the function $X(t)$ at any discrete time point can be approximated by the following formulation [3, 26]

$$X^{(r)}(t_i) = \sum_{j=1}^{m} A_{ij}^{(r)} X(t_j) \qquad or \qquad X_i^{(r)} = \sum_{j=1}^{m} A_{ij}^{(r)} X_j \tag{1}$$

where m is the number of discrete time points, $X(t_j)$ represents the function value at a discrete time point t_j, $X^{(r)}(t_i)$ indicates the *r*th-order derivative of $X(t)$ at a discrete time point t_j, and $A_{ij}^{(r)}$ are the weighting coefficients of the *r*th-order derivative. It follows from Eq. (1) that the quadrature rules may be written collectively in matrix form as:

$$\{X^{(r)}\} = [A]^{(r)} \{X\} \tag{2}$$

where $[A]^{(r)}$ is the *r*th-order DQM weighting coefficient matrix given by

$$[A]^{(r)} = \begin{bmatrix} A_{11}^{(r)} & A_{12}^{(r)} & \cdots & A_{1m}^{(r)} \\ A_{21}^{(r)} & A_{22}^{(r)} & \cdots & A_{2m}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}^{(r)} & A_{m2}^{(r)} & \cdots & A_{mm}^{(r)} \end{bmatrix} \tag{3}$$

and

$$\{X\} = [X(t_1) \quad X(t_2) \quad . \quad . \quad . \quad X(t_m)]^T \tag{4}$$

$$\{X^{(r)}\} = [X^{(r)}(t_1) \quad X^{(r)}(t_2) \quad . \quad . \quad . \quad X^{(r)}(t_m)]^T \tag{5}$$

In order to apply the DQM to initial value problems, one needs to know the weighting coefficients a priori. The weighting coefficients can be determined by the functional approximations in the time direction. The approximation functions are known as the test (or trial) functions, and the necessary condition for the choice of the test functions is completeness in the same sense as one needs for the trial functions in the Rayleigh–Ritz analysis. The completeness requirement means that the test functions should contain all terms of the lowest order admissible, and up to the highest order desired [27]. There are many choices for the test functions such as the simple polynomials, Hermite interpolation functions, Lagrange interpolation polynomials and the trigonometric functions. Among them, the Lagrange interpolation polynomials are commonly used since it is possible to derive an explicit formula to obtain directly the DQM weighting coefficients [28, 29]. However, it is not possible to modify the language interpolation polynomials to impose the initial conditions of the problem. To solve this difficulty, this paper will introduce a set of simple polynomials, where the homogeneous form of the specified initial conditions is imposed, to derive the DQM weighting coefficients. The details will be given in Section 3.

One of the key factors in the accuracy and rate of convergence of the DQ solutions is the choice of discrete time points. It is well known that the equally spaced grid points are not very desirable [3]. It has been suggested that non-uniformly spaced grid points can generate more accurate solutions. The zeros of some orthogonal polynomials are commonly adopted as the grid points. In this study, the DQM discrete time points are taken nonuniformly spaced and are given by the following equation

$$t_i = T_s / 2\left[1-\cos\left(\frac{(i-1)\pi}{m-1}\right)\right], \qquad i=1,2,...,m, \qquad 0 \le t \le T_s \tag{6}$$

where T_s is the time span.

3 PROPOSED APPROACH

3.1 Proposed transformation

As pointed out earlier in introduction, an important step of the proposed approach is to convert the non-zero initial conditions to zero ones. This can be done simply by the help of the following transformation

$$Z(t) = X(t) - X_0 - \sum_{n=1}^{r-1} \frac{(-1)^n (t_1 - t)^n X_0^{(n)}}{n!} \tag{7}$$

where $X_0 = X(0)$ and $X_0^{(n)} = X^{(n)}(0)$ are initial conditions, r is the order of the ordinary differential equation, and t_1 is the initial time point. When above transformation is used, an arbitrary initial value problem with non-zero initial conditions is converted to an equivalent initial value problem with zero initial conditions.

3.2 Weighting coefficients of the proposed approach

By using the transformation (7), an arbitrary non-homogenous initial value problem is converted to an equivalent homogenous initial value problem. This helps us to easily choose the trial functions to compute the DQM weighting coefficients and to incorporate the initial conditions of the problem. Since we now deal with a homogenous initial value problem, the following trial functions can be chosen to exactly satisfy the zero initial conditions:

$$\tau_j(t) = t^{j+r-1}, \quad j=1,2,...,m, \quad r \ge 1 \tag{8}$$

where r is the order of the ordinary differential equation, and m is the number of trial functions (also the number of discrete time points). From Eq. (8) one can easily verify that the proposed trial functions satisfy the homogeneous form of the initial value problems (for example, for second-order equations (r = 2), the proposed trial functions are $\tau_j(t) = t^{j+1}$ which satisfy the conditions $\tau_j(0) = \tau_j^{(1)}(0) = 0$).

Let $t_1, t_2, t_3, ..., t_m$ be a set of discrete points in the time direction, and the transformed function $Z(t)$, defined in Eq. (7), to be approximated by the polynomials $\tau_j(t) (j=1,2,...,m)$. Since the function value at the initial time point $t = t_1 = 0$ is zero, thus, no approximation is needed at the initial time point. Therefore, $Z(t)$ and $Z^{(r)}(t)$ (rth order derivate of function $Z(t)$ with respect to time) can be approximated as:

$$Z(t) = \sum_{j=1}^{m-1} z_j \tau_j(t) \tag{9}$$

$$Z^{(r)}(t) = \sum_{j=1}^{m-1} z_j \tau_j^{(r)}(t) \tag{10}$$

where $z_j (j=1,2,...,m-1)$ are unknown coefficients. Eqs. (9) and (10) can be written for the discrete time points $t_j (i=2,3,...,m)$ in matrix notation as:

$$\{Z\} = [\tau]\{z\} \tag{11}$$

$$\{Z^{(r)}\} = [\tau]^{(r)}\{z\} \tag{12}$$

where

$$\{Z\} = [Z(t_2) \quad Z(t_3) \quad . \quad . \quad . \quad Z(t_m)]^T \tag{13}$$

$$\{Z^{(r)}\} = [Z^{(r)}(t_2) \quad Z^{(r)}(t_3) \quad . \quad . \quad . \quad Z^{(r)}(t_m)]^T \tag{14}$$

$$\{z\} = [z_1 \quad z_2 \quad . \quad . \quad . \quad z_{m-1}]^T \tag{15}$$

$$\tau_{ij} = \tau_j(t_{i+1}) \ , \tau_{ij}^{(r)} = \tau_j^{(r)}(t_{i+1}), \ i, j = 1,2,...,m-1 \tag{16}$$

From Eqs. (11) and (12) one has

$$\{Z^{(r)}\} = [\tau]^{(r)}[\tau]^{-1}\{Z\} \tag{17}$$

which gives the *r*th order DQM weighting coefficient matrix as:

$$[A]^{(r)} = [\tau]^{(r)}[\tau]^{-1} \tag{18}$$

It can be seen that the zero initial conditions are exactly and automatically built into the DQM weighting coefficient matrices.

3.3 Accuracy of the proposed approach and comparison with that of CBCGE and MWCM approaches

Let r ($\geq$ 1) being the order of the ordinary differential equation under investigation, m be the number of discrete time points, and Δt be the time step. Let also the polynomials $T_j(t) = t^{j-1}(j = 1,2,...,m)$ are used to construct the weighting coefficients in the CBCGE and MWCM approaches. From the mathematical formulations presented in Section 3.2, one can make the following conclusions:

- The present scheme uses a polynomial of degree "$m + r - 1$" to approximate the function $X(t)$. Hence, the order of accuracy of the proposed method for approximation of $X(t)$ is "$m + r - 1$."
- In the CBCGE and MWCM approaches, an arbitrary function $X(t)$ is approximated by a polynomial of degree "$m - 1$." Therefore, the order of accuracy of these approaches for approximation of $X(t)$ is "$m - 1$."
- Since the order of an ordinary differential equation is always larger than or equal to unity (i.e., $r \geq 1$), the order of accuracy of the present scheme is always higher than that of the CBCGE and MWCM approaches.

4 A STEP-BY-STEP DQM IN TIME DOMAIN

If the whole time domain of interest is discretized simultaneously, many unknowns have to be solved simultaneously and the computational time required may increase significantly. Moreover, as the time span T_s becomes larger, more discrete time points are needed and this may lead to difficulties in obtaining the long terms solutions. Therefore, it is more convenient to apply the DQM as a step-by-step time integration scheme to advance the solutions progressively over the time domain of interest [30-34]. To apply this scheme, the time domain of interest is first divided into several time elements. For each time element, a procedure similar to a single time domain can be repeated. Similar to the conventional time step methods, the results at the end of a time element (time step) will be used as the initial conditions for the next time element.

Although this procedure (step-by-step DQM) may decrease the computational cost [33], it can increase the error of approximate solutions considerably. The reason for this is that the solutions at time element (n + 1) depend on the solutions at time element n. This means the error can grow with time as time marches. Moreover, it is well-known that the local methods produce much more computing error than the global methods. Therefore, it is desirable to

have the results at the end of the time element as accurate as possible. The present methodology is shown to be more accurate than the CBCGE and MWCM approaches and hence can tackle this difficulty somewhat.

5 NUMERICAL RESULTS

The numerical accuracy of the proposed methodology is discussed in Section 3.3 and compared with that of the CBCGE and MWCM approaches. We concluded that the proposed method is more accurate than the CBCGE and MWCM approaches. In this section, we present a number of numerical examples to support this conclusion. All of the test problems have analytical or accurate solutions. Thus, all of these examples serve well to assess the numerical accuracy of the proposed methodology. Of these examples, the first three problems are governed by second-order ordinary differential equations in time. The first and second examples are linear ordinary differential equations while the third example is nonlinear one. In the fourth and fifth examples, the governing equations of the problems are, respectively, the third- and fourth-order linear initial-value differential equations. From the exact value (or accurate solution) at a given sample time point t_i, the percent error in a quadrature solution can be obtained as:

$$Percent\ \ error\ \ in\ X\left(t_i\right)=\frac{X\left(t_i\right)|_{Exact/Accurate}-X\left(t_i\right)|_{DQM}}{X\left(t_i\right)|_{Exact/Accurate}}\times 100 \tag{19}$$

which is used in the following examples for the error and convergence analysis of the DQM solutions. In some cases in the numerical experiments, we also obtained the relative error of the solutions to the exact (or accurate solution) values using

$$Relative\ \ error\ \ in\ X\left(t_i\right)=X\left(t_i\right)|_{Exact/Accurate}-X\left(t_i\right)|_{DQM} \tag{20}$$

In all problems considered here, when a step-by-step DQM is employed, the time domain of interest is divided into n number of equal length time elements with m sample time points per each time element. For the solution of nonlinear initial-value problems, the resultant nonlinear systems of algebraic equations obtained from the application of the DQM are solved by using the Newton-Raphson method. Through the numerical experiments we found that only 3-5 iterations are sufficient to achieve accurate solutions in Newton-Raphson scheme.

5.1 Example 1: Forced vibration of a mass-spring system

Consider the forced vibration of a mass-spring system governed by

$$X^{(2)}(t)+X(t)=\sin(t) \tag{21}$$

with initial conditions $x(0)=x^{(1)}(0)=1$. The exact solution of Eq. (21) is simply

$$X(t)=\left(3\sin(t)+2\cos(t)-t\cos(t)\right)/2 \tag{22}$$

Fig. 1 shows the converging trend of solutions with respect to the number of discrete time points. These results are obtained using one time element (i.e., n = 1). The obtained results are also compared with those obtained using the CBCGE and MWCM approaches. It can be observed from Fig. 1 that the present methodology produces strikingly much better accuracy than the CBCGE and MWCM approaches. It is also observed that the CBCGE approach gives much less accuracy than other two approaches. Note that the minimum percent error in present solution values at time levels T_s = 0.1 and T_s = 0.25 sec are of orders 10^{-14} and 10^{-12}, respectively, which means almost exact solution values. These values are considerably smaller than those of CBCGE and MWCM approaches.

Fig. 2 shows the converging trend of solutions with respect to the number of time elements at two different time levels (T_s = 5 and T_s = 10 sec). These results are obtained using only 3 grid points per DQM time element (i.e., m =

3). It can be clearly observed that most accurate solutions are obtained by the present methodology and the worst ones are obtained by the CBCGE approach.

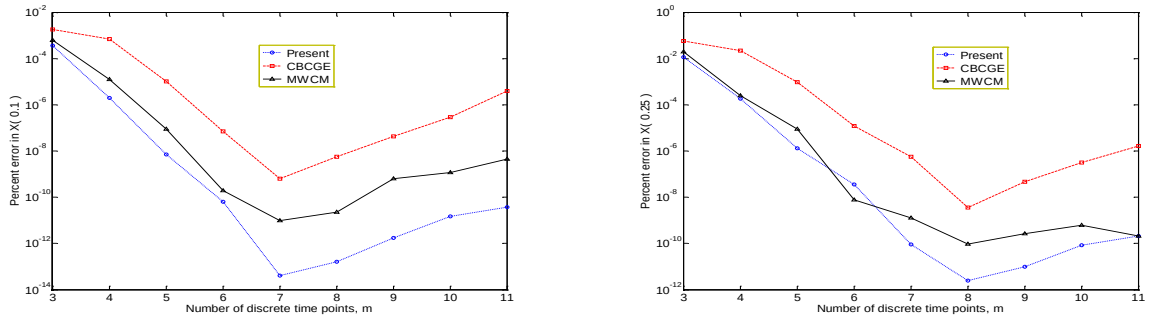

Fig.1
Convergence and comparison of results of example 1 at two different time levels (n = 1).

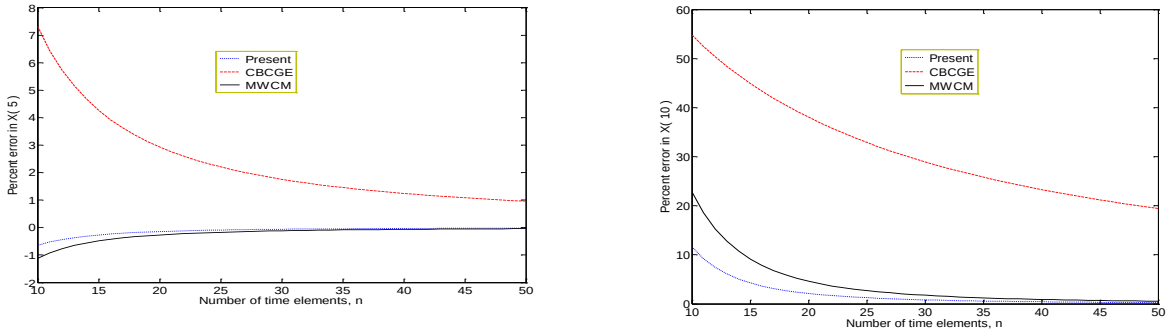

Fig.2
Convergence and comparison of results of example 1 at two different time levels (m = 3).

5.2 Example 2: Forced vibration of a mass-spring-damper system

Consider the forced vibration of a mass-spring-damper system governed by

$$X^{(2)}(t)+X^{(1)}(t)+X(t)=\cos(t) \tag{23}$$

with initial conditions $X(0)=1.2,\ X^{(1)}(0)=-1.5$ .The exact solution of Eq. (23) is simply

$$X(t)=\sin(t)+\exp(-t/2)\left(\frac{6}{5}\cos(\frac{\sqrt{3}}{2}t)-\frac{19\sqrt{3}}{5}\sin(\frac{\sqrt{3}}{2}t)\right) \tag{24}$$

In Fig. 3, the percent error in quadrature solution values at time level T_s = 55 sec is plotted against the number of DQM time elements (n) for different number of discrete time points (m). It can be seen that the results generated by the proposed method are more accurate than those obtained by the CBCGE and MWCM approaches. It can also be seen that as the number of discrete time points (m) increases, the solutions of the CBCGE and MWCM approaches approach to those of the proposed methodology. These results also illustrate the capability of the present DQ methodology in computing the long-term responses.

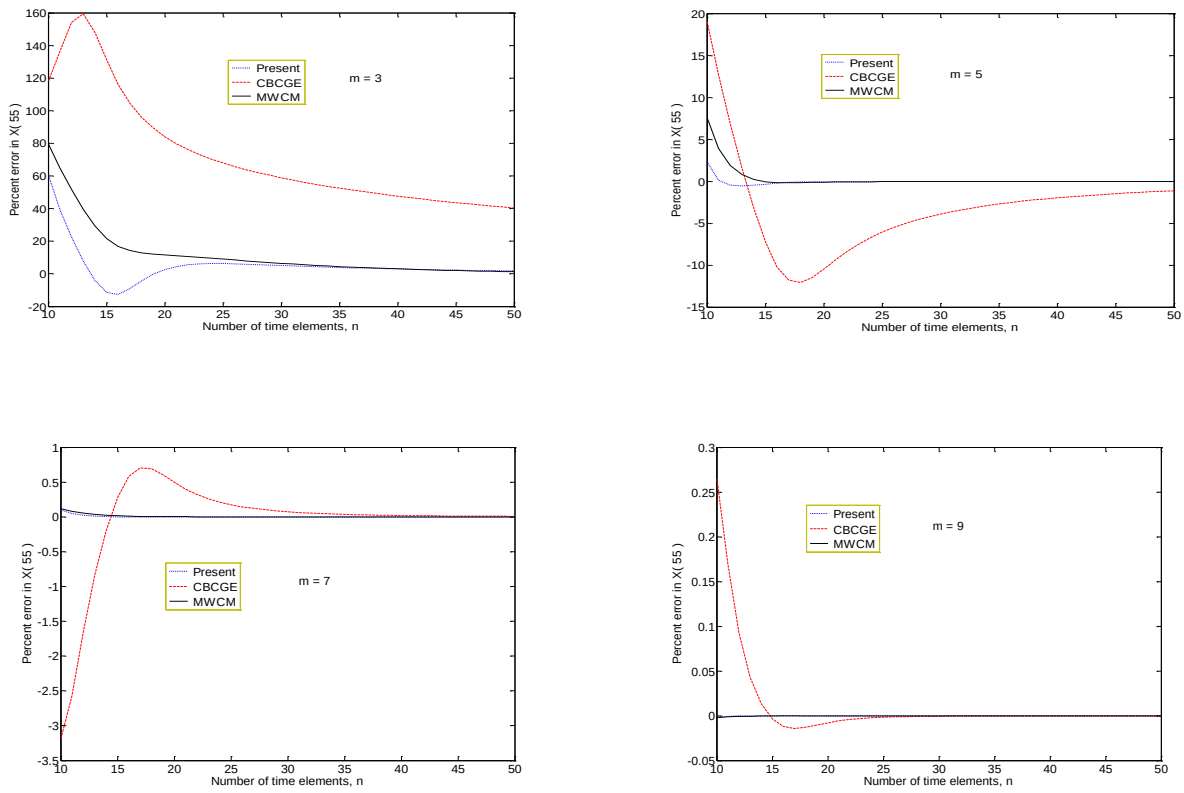

Fig.3
Convergence and accuracy of results of example 2 and comparison with the results of CBCGE and MWCM approaches.

5.3 Example 3: Second order initial-value problem with duffing-type nonlinearity

Consider the following Duffing type nonlinear initial-value problem

$$X^{(2)}(t)+X(t)-\varepsilon X^{3}(t)=0 \ \ ,\varepsilon=1/40 \tag{25}$$

with initial conditions $X(0)=1, X^{(1)}(0)=0$. The three-term perturbed solution of Eq. (25) is given by [35]

$$X(t)=\cos(t)+\left(\cos(t)+12t\sin(t)-\cos(3t)\right)/1280+ \\ \left(23\cos(t)+96t\sin(t)-72t^{2}\cos(t)-24\cos(3t)-36t\sin(3t)+\cos(5t)\right)/1638400 \tag{26}$$

As it was pointed out in Sections 5.1 and 5.2, the results of CBCGE approach are not comparable in accuracy to those of the present methodology and MWCM approach. For this reason, the results of this approach (CBCGE approach) are no longer presented for second- and higher-order initial value problems. Fig. 4 shows the convergence of solutions with increasing number of discrete time points. Only one DQM time element is considered in the time domain. It is observed from Fig. 4 that the proposed method can produce much more accurate solutions than the MWCM approach. This demonstrates the superb converging rate and accuracy of the proposed DQ methodology. The numerical results for long-term solutions are shown in Fig. 5. It can be seen that the error grows with time, as to be expected, since the solution at time element (i + 1) depends on the solution at time element i. It can also be seen the accuracy of solutions are more affected by the number of discrete time points (m) than the number of time elements (n). Besides, the long-term solutions obtained by the present method are more accurate than those of the MWCM approach. This result also illustrates the capability of the proposed approach in the solution of nonlinear ODEs.

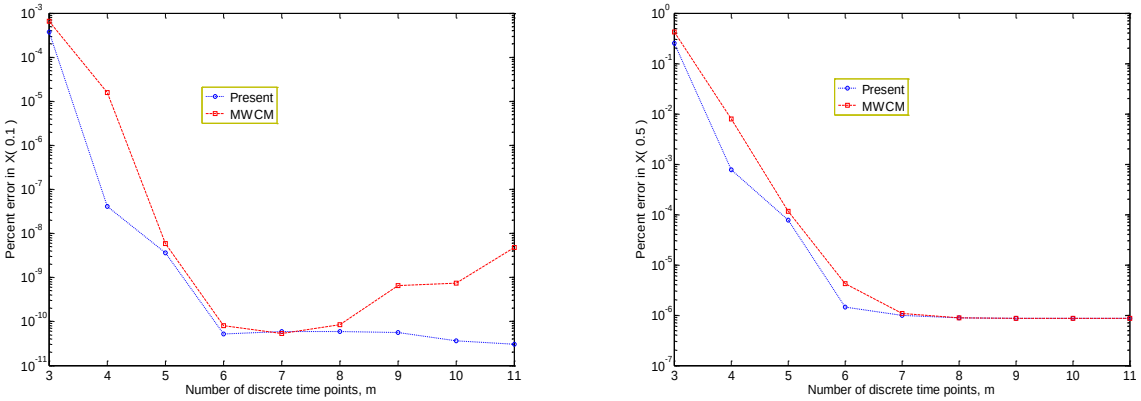

Fig.4
Convergence and comparison of results of example 3 at two different time levels (n = 1).

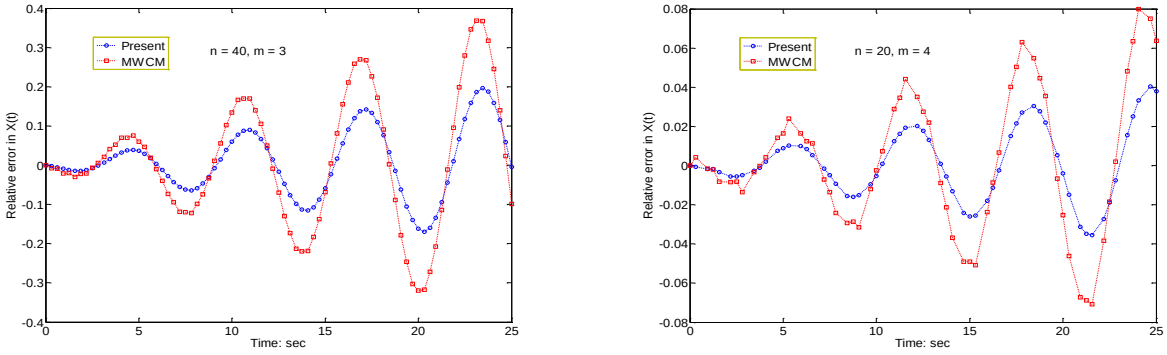

Fig.5
Convergence and comparison of results of example 3 at different time levels.

5.4 Example 4: Third-order initial-value problems

For third-order initial-value problems, we consider the following governing equation

$$X^{(3)}(t)+X^{(2)}(t)+X^{(1)}(t)+X(t)=1 \tag{27}$$

with initial conditions $X(0)=X^{(1)}(0)=X^{(2)}(0)=0$. The exact solution of Eq. (27) is simply

$$X(t)=1-\left(\sin(t)+\cos(t)+\exp(-t)\right)/2 \tag{28}$$

The results of the present test problem are shown in Fig. 6. These results are obtained using 20 DQM time elements and two different values of m. Again, one sees that the proposed methodology gives much better accuracy than the MWCM approach.

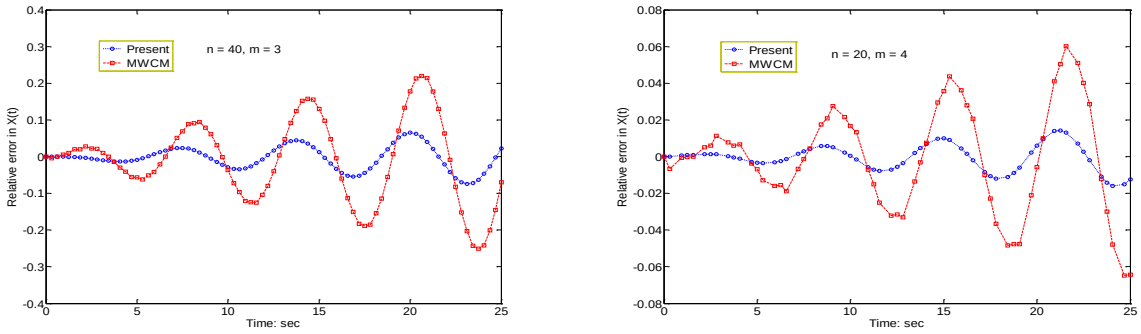

Fig.6
Convergence and comparison of results of example 4 at different time levels.

5.5 Example 5: Fourth-order initial-value problems

For fourth-order initial-value problems, we consider the following governing equation

$$X^{(4)}(t) - X(t) = t \tag{29}$$

with initial conditions $X(0) = X^{(1)}(0) = X^{(2)}(0) = X^{(3)}(0) = 0$. The exact solution of Eq. (29) is simply

$$X(t) = -t + \left(\sin(t) + \sinh(t)\right)/2 \tag{30}$$

The results of the present problem are shown in Fig. 7. By comparing the results of the present methodology with those of MWCM approach, one may conclude that the present methodology is more efficient than the MWCM approach since it can produce better accuracy by using smaller number of discrete time points.

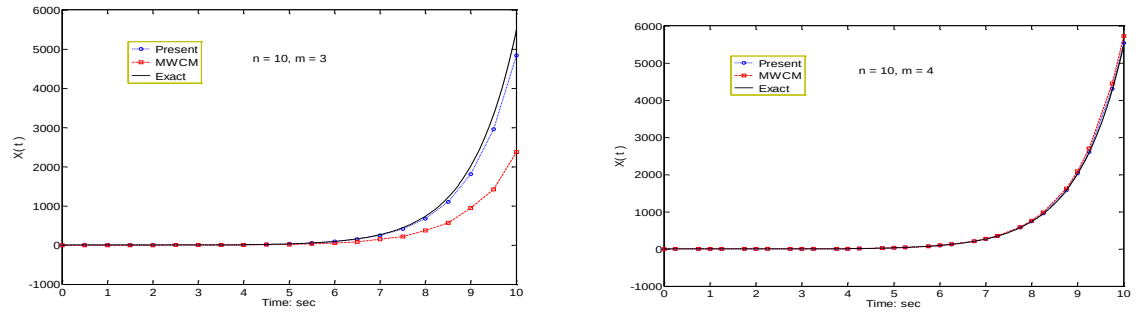

Fig.7
Convergence and comparison of results of example 5 at different time levels.

6 APLLICATION TO STRUCTURAL DYNAMIC PROBLEMS

Many problems encountered in engineering fields are governed by the hyperbolic partial differential equations. The classical methods such as the Ritz, finite element, finite difference and boundary element methods are commonly used to discretize the spatial partial derivatives. The resulting initial value problem represented as a system of ordinary differential equations (ODEs) are generally second-order in time. For instance, the governing equations for linear structural dynamic problems can be expressed as:

$$[M]\{\ddot{u}(t)\} + [C]\{\dot{u}(t)\} + [K]\{u(t)\} = \{f(t)\} \tag{31}$$

with initial conditions

$$\{u(0)\} = \{u_0\}, \qquad \{\dot{u}(0)\} = \{\dot{u}_0\} \tag{32}$$

where $[M]$,$[C]$ and $[K]$ are the structural mass, damping, and stiffness matrices, respectively; $\{\ddot{u}(t)\}, \{\dot{u}(t)\}$, and $\{u(t)\}$ are the acceleration, velocity and displacement vectors, respectively; and $\{f(t)\}$ is the load vector.

In general, Eq. (31) can be solved numerically or analytically. However, in general, it is very cumbersome and difficult to solve this system of ODEs analytically. Therefore, many time step integration algorithms have been established and developed during the past decades to solve the semi-discretized Eq. (31) numerically. Since the time domain is not bounded, Eq. (31) can be satisfied at a number of discrete time points apart. Then, a variation of displacements, velocities, and accelerations can be assumed within each time interval. As a result, the numerical solution of Eq. (31) can be obtained using a numerical step-by-step procedure. Clearly, it is the form of the assumptions within each time interval that determines the accuracy, stability, and cost of the solution procedure [36].

Conventionally, the system of ODEs (31) is approximated in time using finite difference formulas for the time derivatives. This step allows conversion of the system of ODEs into a set of algebraic equations which can be further solved for unknowns using various direct or iterative methods. However, the traditional finite difference schemes such as the Newmark, Wilson, and Houbolt methods are low-order methods, and therefore they need to use a very small time step size to achieve accurate numerical solutions. This difficulty can be overcome by the help of higher-order methods like the DQM [30-34]. In Refs. [30-33], it has been shown that the DQM time integration scheme can produce better accuracy than the Newmark, Wilson, and Houbolt methods using a considerably larger time step sizes. The superiority of the DQM over the classical Runge–Kutta method has also been shown in Refs. [33, 34, 37]. In this study, we will show that the accuracy of the DQM time integration scheme for solving the system of ODEs (31) can be highly influenced by the way in which the initial conditions are incorporated into the solution process. Most importantly, we will show that the methodology presented in this paper can give better accuracy than the CBCGE and MWCM approaches. For completeness, the application of the proposed approach for solving Eq. (31) is briefly explained. In solving Eq. (31) we assume that the matrices $[M]$,$[C]$ and $[K]$ are of order $p \times p$ and the vectors $\{u(t)\}$ and $\{f(t)\}$ are of order $p \times 1$.

6.1 Solution of Eq. (31) via proposed approach

From Eq. (7), the non-zero initial conditions (32) can be changed to zero ones if the following transformation is used

$$\{z(t)\} = \{u(t)\} - \{u_0\} + (t_1 - t)\{\dot{u}_0\} \tag{33}$$

Introducing Eq. (33) to Eq. (31) gives

$$[M]\{\ddot{z}(t)\} + [C]\{\dot{z}(t)\} + [K]\{z(t)\} = \{f(t)\} - [C]\{\dot{u}_0\} - [K]\{u_0\} + (t_1 - t)[K]\{\dot{u}_0\} \tag{34}$$

with initial conditions

$$\{z(0)\} = \{0\}, \qquad \{\dot{z}(0)\} = \{0\} \tag{35}$$

Therefore the non-homogeneous initial value problem (31) is reduced to a homogeneous one by the help of transformation (33).

Since the vector $\{z(t)\}$ has zero value at $t = t_1 = 0$, no approximation is needed at the initial time point. Therefore Eq. (34) should only be satisfied at discrete time points $t_2, t_3, ..., t_m$. By doing so and using the quadrature rule we obtain

$$[\bar{K}]\{\bar{z}\} = \{\bar{f}\} \tag{36}$$

where the $p \times p$ sub-matrices $[\bar{K}_{ij}]$, and the $p \times 1$ sub-vectors $\{\bar{z}_i\}$ and $\{\bar{f}_i\}$ are given by

$$[\bar{K}_{ij}] = A_{ij}^{(2)}[M] + A_{ij}^{(1)}[C] + \hat{I}_{ij}[K], \qquad i, j = 1, 2, ..., m-1 \tag{37}$$

$$\{\bar{z}_i\} = \{z(t_{i+1})\} \tag{38}$$

$$\{\bar{f}_i\} = \{f(t_{i+1})\} - [C]\{\dot{u}_0\} - [K]\{u_0\} + (t_1 - t_{i+1})[K]\{\dot{u}_0\} \tag{39}$$

where $A_{ij}^{(r)}(r = 1, 2)$ are defined already in Section 3.2, and $\hat{I}_{ij}$ are elements of an identity matrix of order (m -1) × (m -1). After solving Eq. (36) for $\{\bar{z}\}$, the displacement vector $\{u(t)\}$ can be obtained from Eq. (33) as:

$$\{u(t_{i+1})\} = \{z(t_{i+1})\} + \{u_0\} - (t_1 - t_{i+1})\{\dot{u}_0\}, \qquad i = 1, 2, ..., m-1 \tag{40}$$

Similarly, the velocity vector is obtained as:

$$\{\dot{u}(t_{i+1})\} = \{\dot{z}(t_{i+1})\} + \{\dot{u}_0\}, \qquad i = 1, 2, ..., m-1 \tag{41}$$

6.2 Stability of the proposed approach and comparison with that of CBCGE and MWCM approaches

In addition to numerical accuracy, another important aspect that should be considered for the integration of ODEs is numerical stability. The stability characteristics of the DQM time integration scheme for solving second-order ODEs have been well studied by Fung [25]. Fung showed that the numerical stability of the DQM time integration scheme can be highly influenced by the choice of discrete time points and the way of implementation of initial conditions. He also showed that the unconditionally stable algorithms can only be obtained using the MWCM approach with some particular type of discrete time points. However, as pointed out by Fung [25], the domain of unconditional stability may vary with the increase of time point number, and the unconditional stability of the MWCM approach is not guaranteed if the number of discrete time points is large. This undesirable feature has also been reported already by some researchers when solving static problems (see, for instance, Ref. [3] for more details). In this section, we will compare the stability of the proposed approach with that of the CBCGE and MWCM approaches.

It is cumbersome and difficult to study the characteristics of an algorithm by applying it to Eq. (31) directly. Alternatively, Eq. (31) can be uncoupled by using the modal decomposition method. It has been rigorously established that the integration of the uncoupled equations is equivalent to the integration of the original system [25, 38, 39]. Therefore, it is more convenient and sufficient for the purpose of investigating the characteristics of a proposed algorithm by considering the equation of motion of a linear single degree of freedom system in the form

$$\ddot{u}(t) + \omega^2 u(t) = 0 \tag{42}$$

with initial conditions $u(t = 0) = u_0$ and $\dot{u}(t = 0) = \dot{u}_0$, where ω is a constant (say the natural frequency of the system). Eq. (42) can be normalized as:

$$\ddot{u}(\tau) + \omega^2 \Delta t^2 u(\tau) = 0, \qquad \tau = t / \Delta t \tag{43}$$

with initial conditions $u(\tau = 0) = u_0$ and $\dot{u}(\tau = 0) = \dot{u}_0 \Delta t$.

The stability property of the DQM time integration scheme can be evaluated by calculating the spectral radius of the numerical amplification matrix $[S]$, which relates the states at the end of the time interval to the initial states, i.e.,

$$\begin{Bmatrix} u(t = \Delta t) \\ \dot{u}(t = \Delta t) \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{Bmatrix} u(t = 0) \\ \dot{u}(t = 0) \end{Bmatrix} = [S] \begin{Bmatrix} u_0 \\ \dot{u}_0 \end{Bmatrix} \tag{44}$$

In general, the eigenvalues λ of $[S]$ can be real or complex. The largest magnitude of the eigenvalues is defined as the spectral radius [25]. For unconditionally stable algorithms, it is sufficient and necessary that the spectral radius satisfies the condition $|\lambda|_{\max} \le 1$.

In Figs. 8-10 the stability properties of various DQM time integration schemes (present, CBCGE approach, and MWCM approach) for solving second-order ODEs are investigated. As it can be seen, in general, all the approaches are conditionally stable. In other words, to achieve numerical stability, $\omega\Delta t$ should be smaller than a critical value. This critical value depends on the order of algorithms and its magnitude increases as the order of algorithm increases.

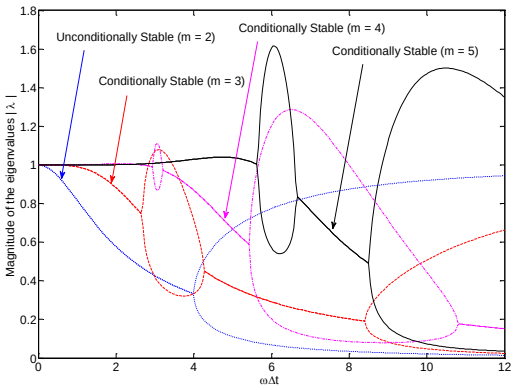

Fig.8
Stability of the proposed DQM time integration scheme for the solution of second-order ODEs for different number of discrete time points: m = 2 (dotted line), m = 3 (dashed line), m = 4 (dashed - dotted line), m = 5 (solid line).

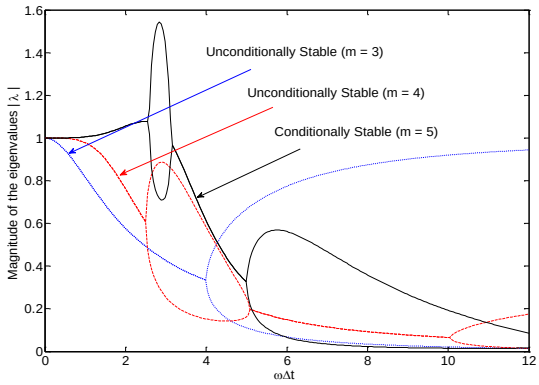

Fig.9
Stability of the CBCGE approach for the solution of second-order ODEs for different number of discrete time points: m = 3 (dotted line), m = 4 (dashed line), m = 5 (solid line).

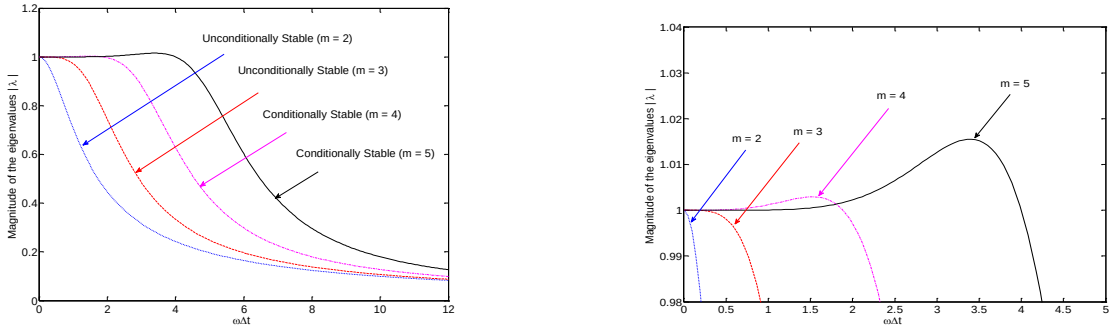

Fig.10
Stability of the MWCM approach for the solution of second-order ODEs for different number of discrete time points: m = 2 (dotted line), m = 3 (dashed line), m = 4 (dashed - dotted line), m = 5 (solid line).

6.3 Numerical results

To demonstrate the accuracy and efficiency of the proposed methodology, two numerical examples related to forced vibration of beams and rectangular plates subjected to a travelling load are now presented. These problems have many applications in engineering systems, most notably in the design and analysis of highway and railway bridges [40]. In solving these problems, the Ritz method is used for the spatial discretization, while the DQEM is used for the temporal discretization. In the test examples, the performances of various approaches for imposing the initial conditions are also studied.

6.3.1 Vibration of a simply-supported beam due to a moving point load

Consider an Euler-Bernoulli beam with length L, mass per unit length ρA , and flexural stiffness EI subjected to a concentrated load F moving at a constant velocity v. The governing differential equation of motion of the beam is given by [40-42]

$$\rho A \frac{\partial^2 w(x,t)}{\partial t^2} + EI \frac{\partial^4 w(x,t)}{\partial x^4} = -F\delta(x - vt) \tag{45}$$

where $w(x, t)$ is the lateral deflection of the beam and $\delta(x)$ is the Dirac-delta function. In the Ritz method, the solution to Eq. (45) is assumed to be in the following form

$$w(x,t) = \sum_{j=1}^{N} d_j(t)\phi_j(x) \tag{46}$$

where $d_j\,(j = 1,2,...,N)$ are Ritz parameters, $\phi_j(x)$ are Ritz approximation/trial functions that satisfy the geometric boundary conditions of the beam, and N is the number of solution terms.

Substituting Eq. (46) into Eq. (45), multiplying both sides of resulting equations by $\varphi_i(x)$, and performing the integration over the length of the beam ($0 \le x \le L$), we obtain

$$[M]\{\ddot{d}(t)\} + [K]\{d(t)\} = \{f(t)\} \tag{47}$$

where

$$M_{ij} = \rho A \int_0^L \varphi_i(x)\varphi_j(x)dx, \qquad i, j = 1,2,...,N \tag{48}$$

$$K_{ij} = EI \int_0^L \varphi_{i,xx}(x)\varphi_{j,xx}(x)dx \tag{49}$$

$$f_i(t) = -F\phi_i(vt) \tag{50}$$

$$\{d\} = \{d_1 \quad d_2 \quad \dots \quad d_N\}^T, \qquad \{\ddot{d}\} = \frac{d^2}{dt^2}\{d\} \tag{51}$$

Eq. (47) can now be solved by the help of various DQEM approaches. For the numerical experiment, we considered a simply supported beam with a span L of 10.16 *cm*, width b = 0.635 cm, thickness h = 0.635 *cm*, modulus of elasticity $E = 2.068 \times 10^{11}$ *Pa*, mass density ρ = 10686.9 *kg/m*^{3}, moment of area I = 135.4832 *mm*^{4} and fundamental period $T_f = 8.149\times10^{-4}$ *s*, subjected to a F = 4.45 *N* moving force [32, 33]. The dynamic responses at the beam center, w_{cd}, are calculated for different values of v (velocity of moving load) and normalized by the static

deflection $w_{cs} = FL^3/(48EI)$ of the beam under a point load F at mid-span. Fig. 11 demonstrates the convergence behavior of solutions for different values of n (number of time elements), and m (number of time points in each time element). Note that a polynomial interpolation scheme is used in the time direction to find a continuous representation of the approximate solutions, as it can be seen from the results shown in Fig. 11. In the results shown in Fig. 8 the parameter T is the time required for the load to cross the beam.

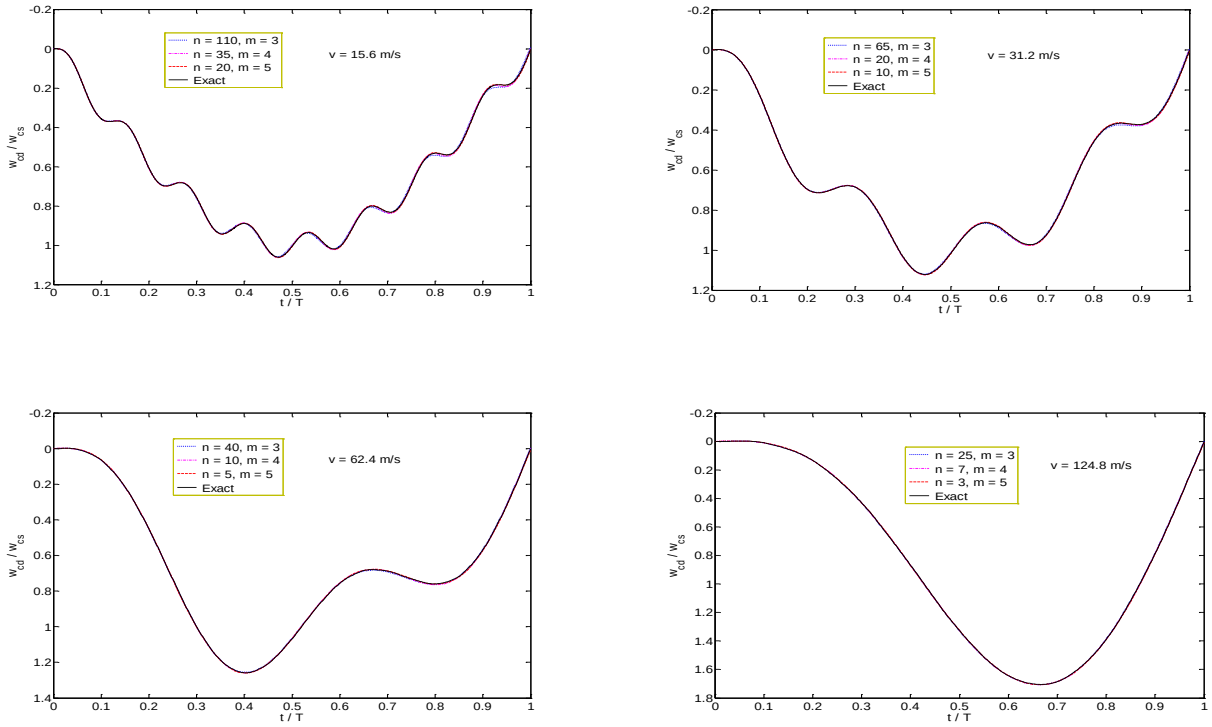

Fig.11
Convergence and accuracy of the central displacement of a simply supported beam subjected to a moving force with respect to n, and m.

An excellent rate of convergence can also be observed from the numerical results presented in Fig. 11. Besides, the convergence rate and accuracy of numerical results are improved by increasing the number of time elements (n) or by increasing the number of discrete time points (m). It is clear that when a lower-order scheme is used, a larger number of time elements are required to achieve accurate results. This implies that the accuracy and computational efficiency of the method is more influenced by the order of algorithm rather than the number of time elements. It can also be seen from Fig. 11 that a smaller values of n and m can be used in the method for larger values of v (moving speed). This is because the shape of dynamic responses of the beam becomes smoother as the velocity of moving load increases. Since the employed DQM is a polynomial-based DQM, the convergence of results is attained with smaller values of n and m for larger values of v.

In Figs. 12-14 the results of proposed method are compared with those obtained using the CBCGE and MWCM approaches for different values of load speed. These results are obtained using two different values of m (m = 3 and m = 4). It can be clearly observed from Figs. 12-14 that the proposed method can produce much better accuracy than the CBCGE and MWCM approaches. It is also observed that the CBCGE approach gives much less accuracy than other two approaches. Besides, as the number of discrete time points (m) increases, the solutions of the MWCM approach approaches to those of the proposed methodology. These results also illustrate the capability of the present DQ methodology for solving systems of second-order ODEs.

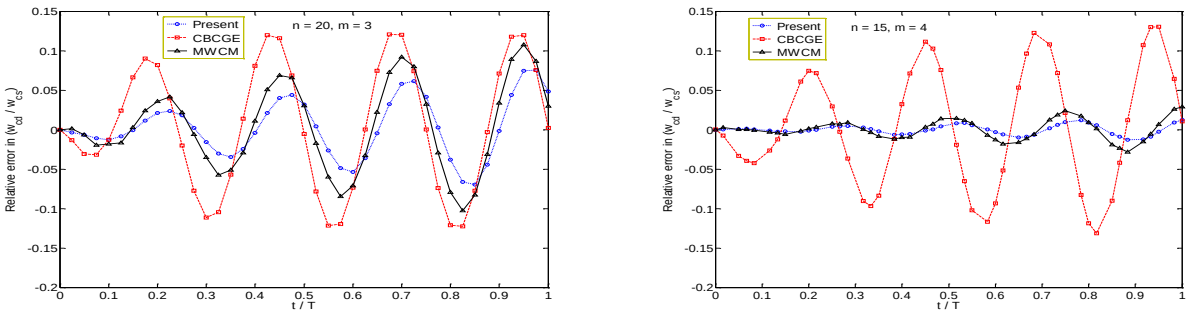

Fig.12
Convergence and accuracy of the results for central displacement of a simply supported beam subjected to a moving force (v = 31.2 m/s)**.**

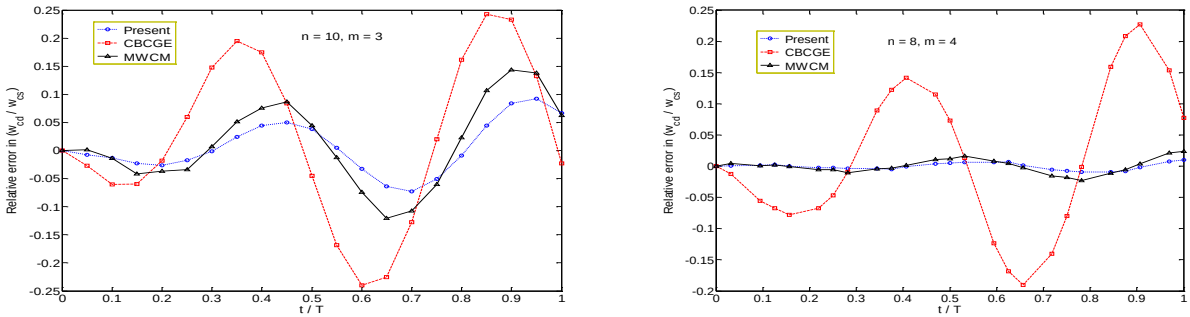

Fig.13
Convergence and accuracy of the results for central displacement of a simply supported beam subjected to a moving force (v = 62.4 m/s).

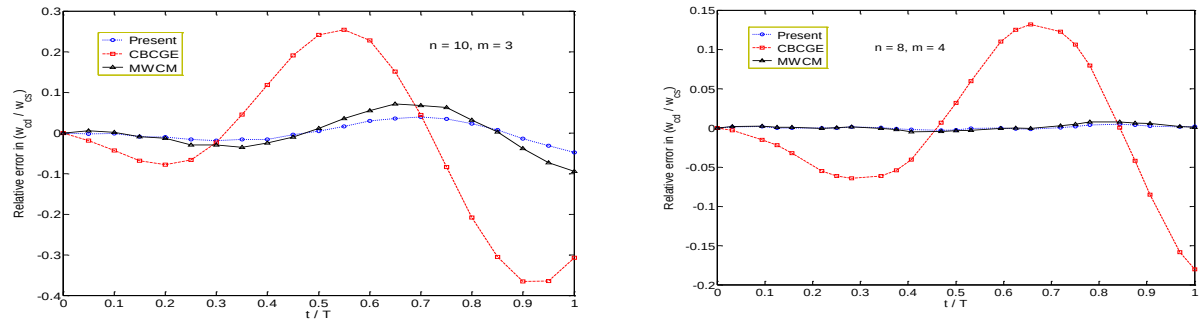

Fig.14
Convergence and accuracy of the results for central displacement of a simply supported beam subjected to a moving force (v = 93.6 m/s).

6.3.2 Vibration of a simply-supported rectangular plate due to a moving point load

Consider an isotropic thin rectangular plate with length a, width b, mass per unit area ρh , and flexural rigidity D subjected to concentrated load F moving parallel to x-axis along the path line $y = y_0$ with a constant speed v. The governing differential equation of motion of the rectangular plate is given by [40-42]

$$\rho h \frac{\partial^2 w}{\partial t^2} + D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = -F\delta(x - vt)\delta(y - y_0) \tag{52}$$

where $w(x, y, t)$ is the lateral deflection of the plate.

The approximate analytical solution to Eq. (52) will be obtained by the help of new variational formulation recently proposed by the present author and his co-author [43, 44]. The solution to Eq. (52) is first assumed to be in the form:

$$w(x, y, t) = \sum_{j=1}^{N} \Psi_j(y, t)\Phi_j(x) \tag{53}$$

where $\Psi_j(y, t)$ are undetermined parameters, $\Phi_j(x)$ are Ritz trial functions, and N is the number of solution terms used for the approximation of the plate deflection in the x-direction. Substituting Eq. (53) into Eq. (52), multiplying both sides of the resulting equation by $\Phi_i(x)$, and performing the integration over the length of the plate ($0 \le x \le a$), we obtain

$$[A]\{\Psi\} + 2[C]\{\Psi_{,yy}\} + [B]\{\Psi_{,yyyy}\} + \frac{\rho h}{D}[B]\{\Psi_{,tt}\} = \frac{1}{D}\{f^x\} \tag{54}$$

where ($i, j = 1, 2, ..., N$)

$$A_{ij} = \int_0^a \Phi_{i,xx}\Phi_{j,xx}\,\mathrm{d}x, \qquad B_{ij} = \int_0^a \Phi_i \Phi_j\,\mathrm{d}x, \qquad C_{ij} = -\int_0^a \Phi_{i,x}\Phi_{j,x}\,\mathrm{d}x, \qquad f_i^x = -F\Phi_i(vt)\delta(y - y_0) \tag{55}$$

$$\{\Psi\} = \{\Psi_1 \quad \Psi_2 \quad \dots \quad \Psi_N\}^T, \qquad \{\Psi_{,yy}\} = \frac{\partial^2}{\partial y^2}\{\Psi\}, \qquad \{\Psi_{,yyyy}\} = \frac{\partial^2}{\partial y^4}\{\Psi\}, \qquad \{\Psi_{,tt}\} = \frac{\partial^2}{\partial t^2}\{\Psi\} \tag{56}$$

The solution to Eq. (54) is then assumed as:

$$\Psi_i(y, t) = \sum_{j=1}^{M} \beta_j^i(t)\psi_j(y), \qquad i = 1, 2, ..., N \tag{57}$$

where β_j^i ($i = 1, 2, ..., N$, $j = 1, 2, ..., M$) are Ritz coefficients, $\psi_j(y)$ are Ritz trial functions, and M is the number of Ritz terms used for approximation of solution in the y-direction.

Substituting Eq. (57) into Eq. (54), multiplying both sides of the resulting equation by $\psi_i(y)$, and performing the integration over the width of the plate ($0 \le y \le b$), we obtain

$$[\tilde{M}]\{\ddot{\tilde{\beta}}\} + [\tilde{K}]\{\tilde{\beta}\} = \{\tilde{f}\} \tag{58}$$

where the $M \times M$ sub-matrices $[\tilde{M}_{ij}]$ and $[\tilde{K}_{ij}]$, and the $M \times 1$ $\{\tilde{f}_i\}$ are given by

$$[\tilde{M}_{ij}] = \frac{\rho h}{D} B_{ij}[\bar{B}], \qquad i, j = 1, 2, ..., N \tag{59}$$

$$[\tilde{K}_{ij}] = A_{ij}[\bar{B}] + 2C_{ij}[\bar{C}] + B_{ij}[\bar{A}] \tag{60}$$

$$\{\tilde{f}_i\} = \frac{1}{D} f_i^x \{f^y\} \tag{61}$$

where ($i, j = 1,2,...,M$)

$$\bar{A}_{ij} = \int_0^b \psi_{i,yy} \psi_{j,yy} \, dy, \qquad \bar{B}_{ij} = \int_0^b \psi_i \psi_j \, dy, \qquad \bar{C}_{ij} = -\int_0^b \psi_{i,y} \psi_{j,y} \, dy, \qquad f_i^y = \psi_i(y_0) \tag{62}$$

Furthermore

$$\{\ddot{\tilde{\beta}}\} = \frac{d^2}{dt^2}\{\tilde{\beta}\} \quad \{\beta^2\}^T \quad \dots \quad \{\beta^N\}^T]^T, \qquad \{\ddot{\tilde{\beta}}\} = \frac{d^2}{dt^2}\{\tilde{\beta}\}, \qquad \{\beta^k\} = [\beta_1^k \quad \beta_2^k \quad \dots \quad \beta_M^k]^T, \quad k = 1,2,...,N \tag{63}$$

Various DQEM approaches can now be used to solve Eq. (58). For the numerical experiment, we considered the forced vibration problem of a simply supported square plate subjected to a moving point load. The parameters used in this numerical example are as follows:

$$\rho h / D = 1, F / D = 1, a = b = 1, y_0 = \text{½} \tag{64}$$

In Sections 5 and 6.3.1 we showed that the results obtained by the CBCGE approach are not comparable in term of accuracy with those generated by the proposed approach and the MWCM approach. Therefore, in this section, we only compare the results of proposed approach with those of the MWCM approach. Figs. 15-17 illustrate the results for different values of moving speed. Again, one sees that the proposed approach can produce much better accuracy than the MWCM approach.

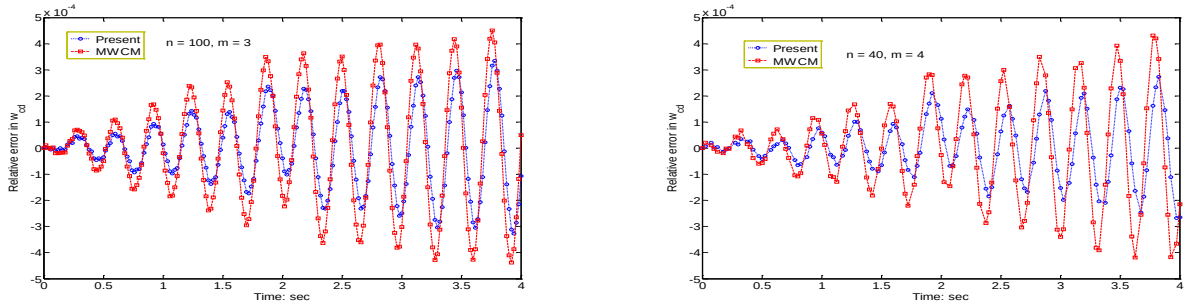

Fig.15
Convergence and accuracy of the results for central displacement of a simply supported square plate subjected to a moving force (v = 0.25 m/s).

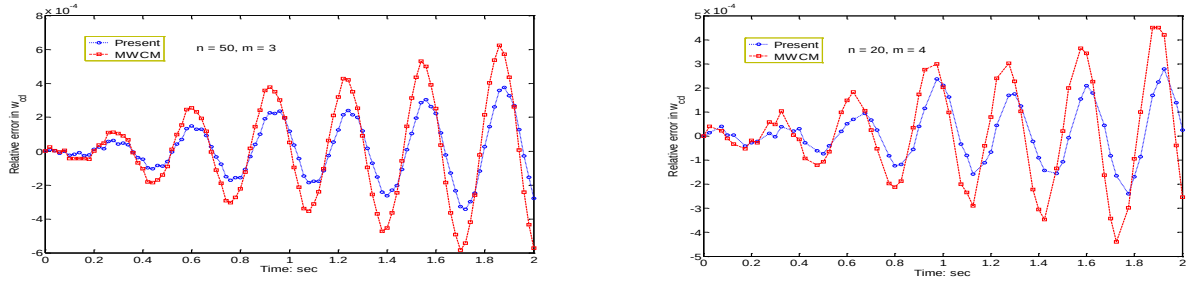

Fig.16
Convergence and accuracy of the results for central displacement of a simply supported square plate subjected to a moving force (v = 0.5 m/s).

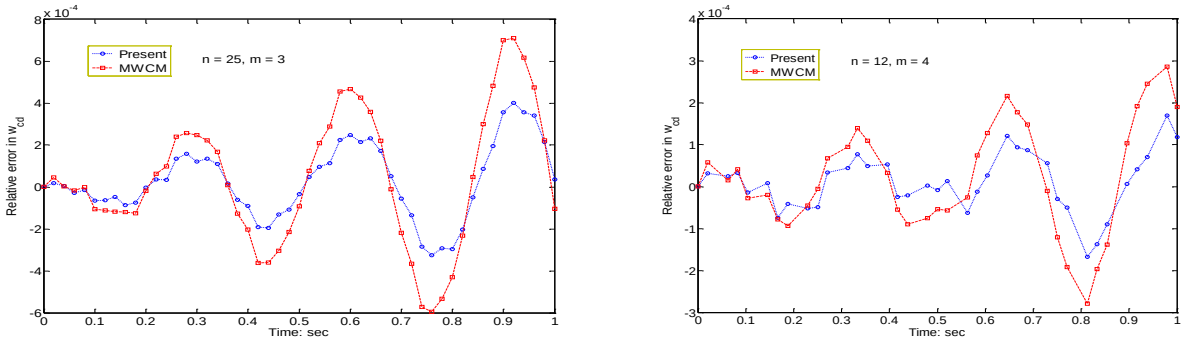

Fig.17
Convergence and accuracy of the results for central displacement of a simply supported square plate subjected to a moving force ($v = 1$ m/s).

7 CONCLUSIONS

An efficient DQ methodology is proposed to exactly implement the multiple initial conditions in the DQ solution of higher order ODEs. In general, for solving an ODE with order r, the order of accuracy of the proposed methodology is "$r + m - 1$," where m is the number discrete time points. This is better than the accuracy of the CBCGE and MWCM approaches whose order is in general "$m - 1$." A number of test problems are also presented to support this conclusion. Numerical results reveal that the proposed technique can produce much better accuracy than the CBCGE and MWCM approaches for solving higher-order initial value problems.

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