Journal of Solid Mechanics Vol. 5, No. 4 (2013) pp. 391-401

Response Determination of a Beam with Moderately Large Deflection Under Transverse Dynamic Load Using First Order Shear Deformation Theory

F. Sohani, H.R. Eipakchi*

Mechanical Engineering Faculty, Shahrood University, Shahrood, Iran

Received 1 July 2013; accepted 3 September 2013

ABSTRACT

In the presented paper, the governing equations of a vibratory beam with moderately large deflection are derived using the first order shear deformation theory. The beam is homogenous, isotropic and it is subjected to the dynamic transverse and axial loads. The kinematic of the problem is according to the Von-Karman strain-displacement relations and the Hook's law is used as the constitutive equation. These equations which are a system of nonlinear partial differential equations with constant coefficients are derived by using the Hamilton's principle. The eigenfunction expansion method and the perturbation technique are applied to obtain the response. The results are compared with the finite elements method.

Keywords: First-order shear deformation theory; Response determination; Perturbation technique; Eigenfunction expansion; Moderately large deflection

1 INTRODUCTION

BEAMS with linear or nonlinear behavior have numerous applications in the mechanical structures. Lee [1] investigated the large deflection of cantilever beams made of Ludwick type material subjected to a combined loading consisting of a uniformly distributed load and one vertical concentrated load at the free end using fifth order Runge-Kutta method. Banerjee et al. [2] used the non-linear shooting and Adomian decomposition methods to determine the large deflection of a cantilever beam under arbitrary loading conditions. Chen [3] proposed the moment integral approach, which can applied to problems of complex load and varying beam properties, to solve the large deflection cantilever beam problems. Vega-Posada et al. [4] developed the large-deflection analysis and post-buckling behavior of laterally braced or unbraced slender beam- columns of symmetrical cross section subjected to end loads with elliptical integrals and Taylor expansion. Wang et al. [5] studied the large deflection problems of beam with special boundary conditions using both the elliptic integral method and the shooting-optimization technique. Li [6] presented the large deflection of the fluid-saturated poroelastic beams which is geometrically non-linear using the numerical methods. Dado [7] studied the very large deflection behavior of prismatic and non-prismatic cantilever beams subjected to various types of loadings. The formulation is based on representing the angle of rotation of the beam by a polynomial on the position variable along the deflected beam axis. The coefficients of the polynomial are obtained by minimizing the integral of the residual error of the governing differential equation and by applying the beam boundary conditions. Su and Ma [8] applied two analytical approaches, Laplace transform and normal mode methods, to investigate the dynamic transient response of a

* Corresponding author. Tel.: +98 273 3300258.
E-mail address: web2-eipakchi@shahroodut.ac.ir (E.R. Eipakchi).

cantilever Timoshenko beam subjected to impact forces. Explicit solutions for the normal mode method and the Laplace transform method were presented. The Durbin method is used to perform the Laplace inverse transformation, and numerical results based on these two approaches were compared.

In the most papers, the beam equations have been derived with the assumption of the small deflection which leads to linear equations. The papers that derived the nonlinear equations, usually have used the numerical method for solving the equations. In the presented work, the response of a nonlinear beam subjected to axial load and transverse excitation are calculated analytically using the first-order shear deformation theory (FSDT). The governing equations, which are a system of partial differential equations, are solved analytically with applying the perturbation technique and the eigenfunction expansion method. The results are compared with the FE method too.

2 GOVERNING EQUATIONS

Consider an isotropic, homogenous uniform beam with the length l, width b, thickness h, density ρ, cross-sectional area A.

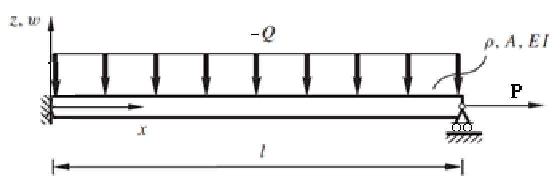

Fig. 1
Beam geometry.

The displacement field, based on the first order shear deformation theory (FSDT), is assumed as:

$$u(x,z,t)=u_0(x,t)+zu_1(x,t) \quad , \quad v(x,z,t)=0 \quad , \quad w(x,z,t)=w_0(x,t)+zw_1(x,t) \tag{1}$$

where x is along the axial direction, y is along the width, z is through the thickness, t denotes the time, $u(x,z,t)$ and $w(x,z,t)$ are the approximated axial and transverse displacements, respectively. u_0, w_0, u_1, w_1 are unknown functions. The strain components according to Von-Karman relations are [9] :

$$\begin{gathered}\varepsilon_x=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \varepsilon_y=\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \quad \varepsilon_z=\frac{\partial w}{\partial z}+\frac{1}{2}\left(\frac{\partial w}{\partial z}\right)^2, \\ \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}, \quad \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}+\frac{\partial w}{\partial z}\frac{\partial w}{\partial y}, \quad \gamma_{zx}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\end{gathered} \tag{2}$$

From Eqs. (1,2) we obtain

$$\begin{gathered}\varepsilon_x=\frac{\partial u_0}{\partial x}+z\frac{\partial u_1}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}+z\frac{\partial w_1}{\partial x}\right)^2, \quad \varepsilon_z=w_1+\frac{1}{2}w_1^2, \quad \varepsilon_y=0, \\ \gamma_{xz}=u_1+\left(1+w_1\right)\left(\frac{\partial w_0}{\partial x}+z\frac{\partial w_1}{\partial x}\right), \quad \gamma_{xy}=\gamma_{yz}=0\end{gathered} \tag{3}$$

For an isotropic, homogeneous, linear elastic material, the stress-strain relations are:

$$\begin{gathered}\sigma_x=A\varepsilon_x+\lambda\left(\varepsilon_y+\varepsilon_z\right),\tau_{xy}=\mu\gamma_{xy} \quad , \quad \sigma_y=A\varepsilon_y+\lambda\left(\varepsilon_z+\varepsilon_x\right),\tau_{yz}=\mu\gamma_{yz} \quad , \\ \sigma_z=A\varepsilon_z+\lambda\left(\varepsilon_x+\varepsilon_y\right),\tau_{xz}=\mu\gamma_{xz}\end{gathered} \tag{4}$$

where λ and μ are Lame' coefficients and $A= \lambda +2 \mu$ .The strain energy is given by:

$$U=\frac{1}{2}\iiint\left(\sigma_x\varepsilon_x+\sigma_z\varepsilon_z+\tau_{zx}\gamma_{zx}\right)dV \tag{5}$$

where dV is the shell volume element, $dV=dxdydz$, $0 \le x \le l$,$-b/2 \le y \le b/2$ and $-h/2 \le z \le h/2$. Taking the variation of Eq.(5) and using Eqs.(3,4) we have:

$$\delta U=\int\Bigg(N_x\delta\frac{\partial u_0}{\partial x}+N_{xz}\delta w_1\frac{\partial w_0}{\partial x}+N_{xz}\delta w_1\frac{\partial w_1}{\partial x}+\left(M_x\frac{\partial w_0}{\partial x}+P_x\frac{\partial w_1}{\partial x}+M_{xz}\left(1+w_1\right)\right)\delta\frac{\partial w_1}{\partial x}+\left(N_x\frac{\partial w_0}{\partial x}+M_x\frac{\partial w_1}{\partial x}+N_{xz}\left(1+w_1\right)\right)\delta\frac{\partial w_0}{\partial x}+N_{xz}\delta u_1+M_x\delta\frac{\partial u_1}{\partial x}+\left(w_1+1\right)N_z\delta w_1\Bigg)dx \tag{6}$$

where the stress resultants are:

$$\begin{aligned}&N_x=\int_{-h/2}^{h/2}\sigma_x dz, \quad M_x=\int_{-h/2}^{h/2}z\sigma_x dz, \quad P_x=\int_{-h/2}^{h/2}z^2\sigma_x dz,\\ &N_z=\int_{-h/2}^{h/2}\sigma_z dz, \quad N_{xz}=\kappa\int_{-h/2}^{h/2}\tau_{xz}dz, \quad M_{xz}=\kappa\int_{-h/2}^{h/2}z\tau_{xz}dz\end{aligned} \tag{7}$$

k is the shear correction factor which depends on geometry and material properties and is assumed $\pi^2/12$ for rectangular cross section [10]. The kinetic energy is defined as:

$$T=\frac{\rho b}{2}\int_0^l\left(h\left(\frac{\partial u_0}{\partial t}\right)^2+h\left(\frac{\partial w_0}{\partial t}\right)^2+\frac{h^3}{12}\left(\frac{\partial u_1}{\partial t}\right)^2+\frac{h^3}{12}\left(\frac{\partial w_1}{\partial t}\right)^2\right)dx \tag{8}$$

The external work due to axial force P which acts at $x=l$ is [11]:

$$dW_P=P\left(ds-dx\right) \quad , \quad ds=dx\sqrt{1+y'^2}=dx\left(1+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right) \tag{9}$$

The virtual-work is:

$$\delta W_P=\int_0^l P\frac{\partial w_0}{\partial x}\frac{\partial}{\partial x}\left(\delta w_0\right)dx=P\frac{\partial w_0}{\partial x}\delta w_0\Bigg|_0^l-\int_0^l P\frac{\partial^2 w_0}{\partial x^2}\delta w_0 dx \tag{10}$$

For transverse load Q the external work is:

$$\delta W_Q=-Q\delta\left(w_0+zw_1\right) \tag{11}$$

The motion equations and the boundary conditions can determine using the Hamilton's principle [11]:

$$\int_{t_1}^{t_2}\delta L dt+\int_{t_1}^{t_2}\delta W_{nc}dt=0 \quad , \quad L=T-U \tag{12}$$

where W_{nc} stands for non-conservative work and it is due to axial and transverse loads in this problem, i.e. $W_{nc}=W_P+W_Q$. From Eqs. (6,8,10,11,12) result:

$$\frac{\partial N_x}{\partial x}-\rho h\frac{\partial^2 u_0}{\partial t^2}=0, \qquad \frac{\partial M_x}{\partial x}+N_{xz}-\frac{\rho h^3}{12}\frac{\partial^2 u_1}{\partial t^2}=0,$$
$$\frac{\partial}{\partial x}\left(N_x\frac{\partial w_0}{\partial x}+M_x\frac{\partial w_1}{\partial x}+N_{xz}\left(1+w_1\right)\right)-\rho h\frac{\partial^2 w_0}{\partial t^2}-Q-P\frac{\partial^2 w_0}{\partial x^2}=0, \tag{13}$$
$$\frac{\partial}{\partial x}\left(M_x\frac{\partial w_0}{\partial x}+P_x\frac{\partial w_1}{\partial x}+M_{xz}\left(1+w_1\right)\right)+N_{xz}\frac{\partial w_0}{\partial x}-M_{xz}\frac{\partial w_1}{\partial x}-N_z(1+w_1)-\frac{\rho h^3}{12}\frac{\partial^2 w_1}{\partial t^2}+\frac{1}{2}Qh=0$$

The boundary conditions are:

$$N_x\,\delta u_0\,|_0^L=0,\ M_x\,\delta u_1\,|_0^L=0, \quad \left(N_x\frac{\partial w_0}{\partial x}+M_x\frac{\partial w_1}{\partial x}+N_{xz}\left(1+w_1\right)-P\frac{\partial w_0}{\partial x}\right)\delta w_0\,|_0^L=0,$$
$$\left(M_x\frac{\partial w_0}{\partial x}+M_{xz}\left(1+w_1\right)+P_x\frac{\partial w_1}{\partial x}\right)\delta w_1\,|_0^L=0 \tag{14}$$

By substituting the stress resultants Eqs. (7) into Eqs. (13), the governing equations in terms of the displacement components are derived. They are four coupled nonlinear partial differential equations.

3 ANALYTICAL SOLUTION

In this paper, the perturbation technique is used for solving the governing equations. We start by converting the governing equations to dimensionless form, using the following parameters:

$$u_0^*=\frac{u_0}{h_0},\quad w_0^*=\frac{w_0}{h_0},\quad x^*=\frac{x}{l},\quad t^*=\frac{t}{t_0},\quad h^*=\frac{h}{h_0},\quad \varepsilon=\frac{h_0}{L},\quad \theta_1=\frac{\lambda}{A},\quad \theta_2=\frac{k\mu}{A}$$
$$P^*=\frac{P}{bAh_0},\ Q^*=\frac{Q}{bA\varepsilon},\ e=\frac{\rho h_0^2}{At_0^2},\ X=\frac{x^*}{\varepsilon} \tag{15}$$

x^* and t^* are dimensionless position and time respectively, u_0^* and w_0^* are dimensionless displacement components. h_0 and t_0 are characteristics thickness and time which are defined as $h_0=h$ and $t_0=l/c$. c is the characteristics of wave speed and it is defined as $c=\sqrt{A/\rho}$. ε is a small parameter which is considered as the perturbation parameter. By using Eq. (15), the dimensionless form of Eqs. (13) (in terms of displacement) are as the following:

$$h^*\left(\frac{\partial^2 u_0^*}{\partial X^2}+\frac{\partial w_0^*}{\partial X}\frac{\partial^2 w_0^*}{\partial X^2}+\left(\frac{1}{12}h^{*2}\frac{\partial^2 w_1^*}{\partial X^2}+\theta_1\left(1+w_1^*\right)\right)\frac{\partial w_1^*}{\partial X}\right)-eh^*\frac{\partial^2 u_0^*}{\partial t^{*2}}-f_1^*=0, \tag{16}$$

$$\frac{h^{*3}}{12}\left(\frac{\partial^2 u_1^*}{\partial X^2}+\frac{\partial}{\partial X}\left(\frac{\partial w_1^*}{\partial X}\frac{\partial w_0^*}{\partial X}\right)-e\frac{\partial^2 u_1^*}{\partial t^{*2}}\right)-\theta_2 h^*\left(u_1^*+\left(1+w_1^*\right)\frac{\partial w_0^*}{\partial X}\right)-f_2^*=0 \tag{17}$$

$$\begin{aligned}&\theta_2 h^*\left(1+w_1^*\right)^2\frac{\partial^2 w_0^*}{\partial X^2}+h^*\left(\frac{1}{4}h^{*2}\frac{\partial^2 w_1^*}{\partial X^2}+\theta_1\left(1+w_1^*\right)\right)\frac{\partial w_1^*}{\partial X}\frac{\partial w_0^*}{\partial X}+\frac{h^{*3}}{12}\left(\frac{\partial u_1^*}{\partial X}\right)\frac{\partial^2 w_1^*}{\partial X^2}\\&+h^*\left(\frac{\partial u_0^*}{\partial X}+\frac{3}{2}\left(\frac{\partial w_0^*}{\partial X}\right)^2+\frac{1}{8}h^{*2}\left(\frac{\partial w_1^*}{\partial X}\right)^2+\theta_1 w_1^*\left(1+\frac{1}{2}w_1^*\right)\right)\frac{\partial^2 w_0^*}{\partial X^2}+\frac{\partial u_1^*}{\partial X}\theta_2 h^*\left(1+w_1^*\right)+\\&h^*\frac{\partial w_0^*}{\partial X}\frac{\partial^2 u_0^*}{\partial X^2}+\theta_2 h^*\left(u_1^*+2\left(1+w_1^*\right)\frac{\partial w_0^*}{\partial X}\right)\frac{\partial w_1^*}{\partial X}+\frac{h^{*3}}{12}\frac{\partial^2 u_1^*}{\partial X^2}\frac{\partial w_1^*}{\partial X}-eh^*\frac{\partial^2 w_0^*}{\partial t^{*2}}-f_3^*=0,\end{aligned}\tag{18}$$

$$\begin{aligned}&\frac{h^*}{12}\left(h^*\frac{\partial^2 u_1^*}{\partial X^2}+3h^*\frac{\partial w_1^*}{\partial X}\frac{\partial^2 w_0^*}{\partial X^2}+\frac{3}{2}h^{*2}\frac{\partial w_0^*}{\partial X}\frac{\partial^2 w_1^*}{\partial X^2}\right)\frac{\partial w_0^*}{\partial X}+\frac{h^{*2}}{12}\frac{\partial u_1^*}{\partial X}\frac{\partial^2 w_0^*}{\partial X^2}\\&+\frac{h^{*2}}{12}\frac{\partial^2 u_0^*}{\partial X^2}\frac{\partial w_1^*}{\partial X}+\left(1+w_1^*\right)\left(-\left(w_1^*+\frac{1}{2}w_1^{*2}\right)-\theta_1\frac{\partial u_0^*}{\partial X}-\frac{1}{2}\theta_1\left(\frac{\partial w_0^*}{\partial X}\right)^2+\frac{h^{*2}}{24}\theta_1\left(\frac{\partial w_1^*}{\partial X}\right)^2\right.\\&\left.+\frac{h^{*2}}{12}\theta_2\left(\frac{\partial w_1^*}{\partial X}\right)^2+\frac{h^{*2}}{12}\theta_2\left(1+w_1^*\right)\frac{\partial^2 w_1^*}{\partial X^2}\right)+\theta_2\left(u_1^*+\left(-1+w_1^*\right)\frac{\partial w_0^*}{\partial X}\right)\frac{\partial w_0^*}{\partial X}\\&+h^{*2}\varepsilon^3\left(\frac{1}{12}\frac{\partial u_0^*}{\partial X}+\frac{h^{*2}}{160}\left(\frac{\partial w_1^*}{\partial X}\right)^2+\frac{\theta_1}{12}\left(w_1^*+\frac{1}{2}w_1^{*2}\right)\right)\frac{\partial^2 w_1^*}{\partial X^2}-\frac{eh^{*2}}{12}\frac{\partial^2 w_1^*}{\partial t^{*2}}-f_4^*=0\end{aligned}\tag{19}$$

where $f_1^*, f_2^*, f_3^*, f_4^*$ are:

$$f_1^*=f_2^*=0,\quad f_3^*=Q^*\varepsilon-P^*\varepsilon^2\frac{\partial^2 w_0^*}{\partial x^{*2}},\quad f_4^*=\frac{1}{2}Q^*\varepsilon \tag{20}$$

We use the multiple scale method . By defining the new scales $T_0=t^*$ and $T_1=\varepsilon t^*$, so the derivative operator is:

$$\frac{\partial^2}{\partial t^{*2}}=\frac{\partial^2}{\partial T_0^2}+2\varepsilon\frac{\partial^2}{\partial T_0\partial T_1} \tag{21}$$

We apply Eq. (21) into Eqs (16) to (19) and we consider the solution with a uniform expansion as:

$$\begin{aligned}u_0\left(X,t^*;\varepsilon\right)&=\varepsilon\left(u_0\left(X,T_0,T_1\right)+\varepsilon u_1\left(X,T_0,T_1\right)\right)\\u_1\left(X,t^*;\varepsilon\right)&=\varepsilon\left(u_2\left(X,T_0,T_1\right)+\varepsilon u_3\left(X,T_0,T_1\right)\right)\\w_0\left(X,t^*;\varepsilon\right)&=\varepsilon\left(w_0\left(X,T_0,T_1\right)+\varepsilon w_1\left(X,T_0,T_1\right)\right)\\w_1\left(X,t^*;\varepsilon\right)&=\varepsilon\left(w_2\left(X,T_0,T_1\right)+\varepsilon w_3\left(X,T_0,T_1\right)\right)\end{aligned}\tag{22}$$

By substituting Eqs (22) into Eqs (16) to (19), and separating the equations with the same order of ε, results:For the first-order equations :

$$h^{*2}\frac{\partial^2 u_0^*}{\partial X^2}+\theta_1 h^*\frac{\partial w_2^*}{\partial X}-2eh^{*2}\frac{\partial^2 u_0^*}{\partial T_0^2}=0,$$
$$\frac{1}{12}h^{*2}\frac{\partial^2 u_2^*}{\partial X^2}-\theta_2\left(u_2^*+h^*\frac{\partial w_0^*}{\partial X}\right)-\frac{eh^{*2}}{6}\frac{\partial^2 u_2^*}{\partial T_0^2}=0,$$
$$\theta_2\left(h^*\frac{\partial u_2^*}{\partial X}+h^{*2}\frac{\partial^2 w_0^*}{\partial X^2}\right)-2eh^{*2}\frac{\partial^2 w_0^*}{\partial T_0^2}-P^*\frac{\partial^2 w_0^*}{\partial X^2}-Q^*=0, \tag{23}$$
$$-\theta_1 h^*\frac{\partial u_0^*}{\partial X}+\frac{\theta_2 h^{*2}}{12}\frac{\partial^2 w_2^*}{\partial X^2}-w_2^*-\frac{eh^{*2}}{6}\frac{\partial^2 w_2^*}{\partial T_0^2}-\frac{1}{2}Q^*=0$$

The resulted equations in Eqs.(23) are two independent systems of equations. According to the eigenfunction expansion method, the solution of each system of equations for the simply supported beam is considered as the following:

$$u_2\left(X,T_0,T_1\right)=\sum_{m=1}^{\infty}A_{1m}\left(T_0,T_1\right)cos\left(m\pi X/a\right) \qquad w_0\left(X,T_0,T_1\right)=\sum_{m=1}^{\infty}A_{2m}\left(T_0,T_1\right)sin\left(m\pi X/a\right) \tag{24}$$

Substituting Eqs. (24) into the second and third equations of Eqs. (23) yields:

$$\sum_{m=1}P_{1m}\,cos\left(m\pi X/a\right)=F_1,\ \sum_{m=1}P_{2m}\,sin\left(m\pi X/a\right)=F_2 \tag{25}$$

where

$$P_1=-\frac{1}{12}h^{*2}A_1\pi^2\varepsilon^2-\theta_2\left(A_1+A_2\pi\varepsilon\right)-\frac{1}{12}eh^{*2}\frac{d^2A_1}{dT_0^2},$$
$$P_2=\theta_2\left(h^*A_1\pi\varepsilon+h^{*2}A_2\pi^2\varepsilon^2\right)+eh^{*2}\frac{d^2A_2}{dT_0^2}-P^*A_2\pi^2\varepsilon^2,\qquad F_1=0,\qquad F_2=Q^* \tag{26}$$

The index "m" has been removed for simplicity.
Based on the Fourier half-range expansion for Eqs (25), P_1 and P_2 are obtained as:

$$P_1=\frac{2}{a}\int_0^a F_1\,cos\left(m\pi X/a\right)dX \qquad ,P_2=\frac{2}{a}\int_0^a F_2\,sin\left(m\pi X/a\right)dX \qquad ,a=1/\varepsilon \tag{27}$$

Eqs. (27) are two coupled differential equations whose solution are:

$$A_1\left(T_0,T_1\right)=a_1\left(T_1\right)e^{i\omega_2T_0}+a_2\left(T_1\right)e^{i\omega_3T_0}+cc \qquad A_2\left(T_0,T_1\right)=a_3\left(T_1\right)e^{i\omega_2T_0}+a_4\left(T_1\right)e^{i\omega_3T_0}+cc \tag{28}$$

where *cc* stands for the complex conjugate terms, ω_2 and ω_3 are natural frequencies for the first systems of equations and $a_3(T_1)$, $a_4(T_1)$ are dependent to $a_1(T_1)$, $a_2(T_1)$. The solution procedure for the second systems of equations is similar where ω_1 and ω_4 are its natural frequencies. The numerical results shows that $m=1$ is sufficient for convergence of Eqs. (24) so we used just one term in deriving the second order equations.

The resulted equations in the second-order of ε are:

$$\frac{\partial^2 u_1^*}{\partial X^2}+\theta_1\frac{\partial w_3^*}{\partial X}-e\frac{\partial^2 u_1^*}{\partial T_0^2}-2e\frac{\partial^2 u_1^*}{\partial T_0\partial T_0}+\theta_1 w_2^*\frac{\partial w_2^*}{\partial X}+\frac{\partial^2 w_0^*}{\partial X^2}\frac{\partial w_0^*}{\partial X}+\frac{1}{12}h^{*2}\frac{\partial^2 w_2^*}{\partial X^2}\frac{\partial w_2^*}{\partial X}=0, \tag{29}$$

$$\begin{aligned}&h^{*2}\frac{\partial^2 w_2^*}{\partial X^2}\frac{\partial w_0^*}{\partial X}+h^{*2}\frac{\partial^2 u_3^*}{\partial X^2}-eh^{*2}\frac{\partial^2 u_3^*}{\partial T_0^2}-2eh^{*2}\frac{\partial^2 u_3^*}{\partial T_0\partial T_1}+h^{*2}\frac{\partial^2 w_0^*}{\partial X^2}\frac{\partial w_2^*}{\partial X}\\&-12\theta_2 w_2^*\frac{\partial w_0^*}{\partial X}-12\theta_2\frac{\partial w_1^*}{\partial X}-12\theta_2 u_3^*=0\end{aligned} \tag{30}$$

$$\begin{aligned}&h^{*2}\frac{\partial^2 u_2^*}{\partial X^2}\frac{\partial w_2^*}{\partial X}+h^{*2}\frac{\partial^2 w_2^*}{\partial X^2}\frac{\partial u_2^*}{\partial X}+12\theta_2\frac{\partial u_3^*}{\partial X}+12\frac{\partial^2 u_0^*}{\partial X^2}\frac{\partial w_0^*}{\partial X}+12\theta_2\frac{\partial^2 w_1^*}{\partial X^2}\\&+12\theta_1\frac{\partial w_2^*}{\partial X}\frac{\partial w_0^*}{\partial X}+12\theta_2 u_2^*\frac{\partial w_2^*}{\partial X}+12\theta_1 w_2^*\frac{\partial^2 w_0^*}{\partial X^2}-12e\frac{\partial^2 w_1^*}{\partial T_0^2}-24e\frac{\partial^2 w_1^*}{\partial T_0\partial T_1}\\&+24\theta_2\frac{\partial w_2^*}{\partial X}\frac{\partial w_0^*}{\partial X}+12\frac{\partial^2 w_0^*}{\partial X^2}\frac{\partial u_0^*}{\partial X}+12\theta_2 w_2^*\frac{\partial u_2^*}{\partial X}-P^*\frac{\partial^2 w_1^*}{\partial X^2}=0\end{aligned} \tag{31}$$

$$\begin{aligned}&\theta_2 h^{*2}\frac{\partial^2 w_3^*}{\partial X^2}+\theta_2 h^{*2}\left(\frac{\partial w_2^*}{\partial X}\right)^2-eh^{*2}\frac{\partial^2 w_3^*}{\partial T_0^2}-2eh^{*2}\frac{\partial^2 w_3^*}{\partial T_0\partial T_1}+\frac{1}{2}\theta_1 h^{*2}\left(\frac{\partial w_2^*}{\partial X}\right)^2\\&+2\theta_2 h^{*2}w_2^*\frac{\partial^2 w_2^*}{\partial X^2}+h^{*2}\frac{\partial u_0^*}{\partial X}\frac{\partial^2 w_2^*}{\partial X^2}+\theta_1 h^{*2}w_2^*\frac{\partial^2 w_2^*}{\partial X^2}+h^{*2}\frac{\partial^2 u_2^*}{\partial X^2}\frac{\partial w_0^*}{\partial X}\\&+h^{*2}\frac{\partial^2 w_0^*}{\partial X^2}\frac{\partial u_2^*}{\partial X}+h^{*2}\frac{\partial^2 u_0^*}{\partial X^2}\frac{\partial w_2^*}{\partial X}-w_3^*-\theta_1\frac{\partial u_1^*}{\partial X}-\theta_2 u_{10}^*\frac{\partial w_0^*}{\partial X}-\frac{3}{2}w_2^{*2}\\&-\theta_1 u_2^*\frac{\partial u_0^*}{\partial X}-\theta_2\left(\frac{\partial w_0^*}{\partial X}\right)^2-\frac{1}{2}\theta_1\left(\frac{\partial w_0^*}{\partial X}\right)^2=0\end{aligned} \tag{32}$$

For solving equations of the first system containing Eqs.(30,31) , the solutions are:

$$u_3\left(X,T_0,T_1\right)=\sum_{m=1}A_{3m}\left(T_0,T_1\right)cos\left(m\pi X/a\right),\qquad w_1\left(X,T_0,T_1\right)=\sum_{m=1}A_{4m}\left(T_0,T_1\right)sin\left(m\pi X/a\right) \tag{33}$$

By substituting Eqs. (33) into Eqs.(30,31), we have

$$\sum_{m=1}P_{3m}\,cos\left(m\pi X/a\right)=F_3\ ,\ \sum_{m=1}P_{4m}\,sin\left(m\pi X/a\right)=F_4 \tag{34}$$

where

$$\begin{aligned}&P_3=P_{31}=-\frac{1}{12}h^{*2}A_3\pi^2\varepsilon^2-\theta_2\left(A_3+A_4\pi\varepsilon\right)-\frac{1}{12}eh^{*2}\frac{d^2A_3}{dT_0^2}=\frac{2}{a}\int_0^a F_3\,cos\left(\pi X/a\right)dX\ ,\\&P_4=P_{41}=\theta_2\left(h^*A_3\pi\varepsilon+h^{*2}A_4\pi^2\varepsilon^2\right)+eh^{*2}\frac{d^2A_4}{dT_0^2}-P^*A_4\pi^2\varepsilon^2=\frac{2}{a}\int_0^a F_4\,sin\left(\pi X/a\right)dX\end{aligned} \tag{35}$$

so:

$$P_3 = f_1 e^{\pm i\omega_1 T_0} + f_2 e^{\pm i\omega_2 T_0} + f_3 e^{\pm i\omega_3 T_0} + f_4 e^{\pm i\omega_4 T_0} + f_5 e^{i(\omega_1 \pm \omega_2)T_0} + f_6 e^{\pm i(\omega_3 \pm \omega_4)T_0} + f_7 e^{\pm i(\omega_2 \pm \omega_4)T_0} + f_8 e^{\pm i(\omega_1 \pm \omega_3)T_0}$$
$$P_4 = g_1 e^{\pm i\omega_1 T_0} + g_2 e^{\pm i\omega_2 T_0} + g_3 e^{\pm i\omega_3 T_0} + g_4 e^{\pm i\omega_4 T_0} + g_5 e^{i(\omega_1 \pm \omega_2)T_0} + g_6 e^{\pm i(\omega_3 \pm \omega_4)T_0} + g_7 e^{\pm i(\omega_2 \pm \omega_4)T_0} + g_8 e^{\pm i(\omega_1 \pm \omega_3)T_0}$$

The non-homogenous terms of Eqs. (35) contain $e^{\pm i\omega_2 T_0}$ and $e^{\pm i\omega_3 T_0}$ which are secular for the second order Eqs. (35). To find a uniform expansion solution, one can remove the secular terms by defining the solvability conditions [12] as the follows:

$$\int_0^a \left(Eq_1 \cos\left(\pi X / a\right) + Eq_2 \sin\left(\pi X / a\right)\right) dX = \int_0^a \left(F_3 \cos\left(\pi X / a\right) + F_4 \sin\left(\pi X / a\right)\right) dX \tag{36}$$

where Eq_1 and Eq_2 are left-hand sides of Eqs. (34) i.e:.

$$Eq_1 = P_3 \cos\left(\pi X / a\right) \quad , \quad Eq_2 = P_4 \sin\left(\pi X / a\right) \tag{37}$$

Letting the coefficients of $e^{\pm i\omega_3 T_0}$ and $e^{\pm i\omega_3 T_0}$ equal to zero, the functions $a_1(T_1)$ and $a_2(T_1)$ are obtained and finally the solution of the first-order equations are determined. The second-order equation may be solved by substituting $a_1(T_1)$ and $a_2(T_1)$ and canceling the secular terms $e^{\pm i\omega_3 T_0}$ and $e^{\pm i\omega_3 T_0}$. The particular solution of the second order equations are:

$$u_1(X, T_0, T_1) = h_1 e^{\pm i\omega_1 T_0} + h_2 e^{\pm i\omega_2 T_0} + h_3 e^{\pm i\omega_3 T_0} + h_4 e^{\pm i\omega_4 T_0} + h_5 e^{i(\omega_1 \pm \omega_2)T_0} + h_6 e^{\pm i(\omega_3 \pm \omega_4)T_0} + h_7 e^{\pm i(\omega_2 \pm \omega_4)T_0} + h_8 e^{\pm i(\omega_1 \pm \omega_3)T_0} \tag{38}$$

The total solution can be set as the homogenous first-order solution plus the particular part of the second-order solution.

$$u_0(X, t^*; \varepsilon) = \varepsilon(u_0(X, T_0, T_1) + \varepsilon u_1(X, T_0, T_1)) \tag{39}$$

The constants in determining of $a_1(T_1)$ and $a_2(T_1)$ can be calculated by applying the initial conditions.

4 NUMERICAL RESULTS

ANSYS 11 FE package is used for the response analysis of the beam with the axial and transverse loads by using element BEAM189, which has three nodes and six degrees-of-freedom in each node. The boundary conditions are considered simply supported at *x=0* and *x=l*. The characteristics of the beam have been listed in Table 1. Before the final analysis, the sensitivity to the mesh size was investigated. The optimum elements numbers was 38. Also, the optimum time step was chosen 0.02 seconds by trial and error. The transverse load is a step function as Fig.2.

Table 1
Beam properties

Length (m)	1	
Width (m)		0.02
Thickness (m)	0.002	
Poisson's ratio		0.3
Young's modulus (Pa)	2e11	
Density(kg/m^3)	7800	

Fig. 2
Transverse load- time variations.

The analytical calculations performed on Maple 13 environment. Fig. 3 shows the transverse displacement response at the mid-span of the beam due to transverse load and axial load P=10 N. The response has been calculated for three cases: analytical results (FSDT) which we formulated in this paper, the FE results by considering the large deflection effects and the FE results with small deflection option. According to this figure, the difference of the FSDT with the large deflection FE is closer than the small deflection FE. Fig. 4 shows the analytical and FE results for transverse response at x^*=0.05, which is a point near the boundary. The behavior is similar to Fig. 3 and the difference is not more than the results at x^*=0.5. In Fig. 5 the deformed shape of the beam at a special time (t=0.5) is shown with the FE and FDST method which is in accordance with the Fig. 3 and simply supported boundary conditions. For different axial load, the transverse deflection has been plotted in Fig. 6. By increasing the tensile load, the transverse deflection will decrease whereas by increasing the compressive load, the transverse deflection will increase, as expected. Fig. 7 shows the effect of number of terms Eqs. (22) on the solution. So, a two term expansion is sufficient for convergence of the transverse displacement. Fig. 8 shows the axial displacement of the upper and middle layer of the beam at a special time. It is possible to show the transverse displacement of the different layers with the FSDT too. The Euler-Bernolli and Timoshenko theories can not predict the variations of the transverse displacements across the section but it is possible for FSDT.

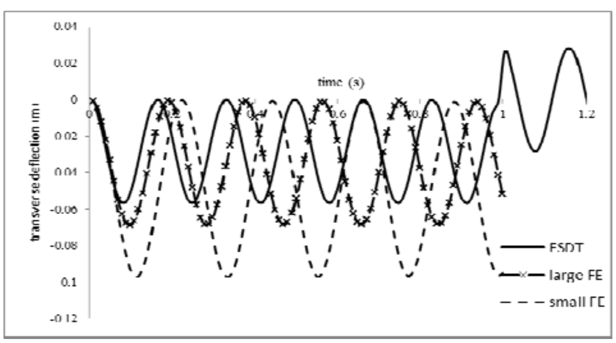

Fig. 3
Analytical and numerical transverse response at x^*=0.5.

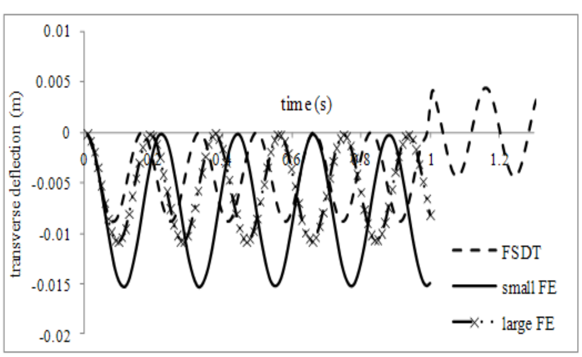

Fig. 4
Analytical and numerical transverse response at x^*=0.05.

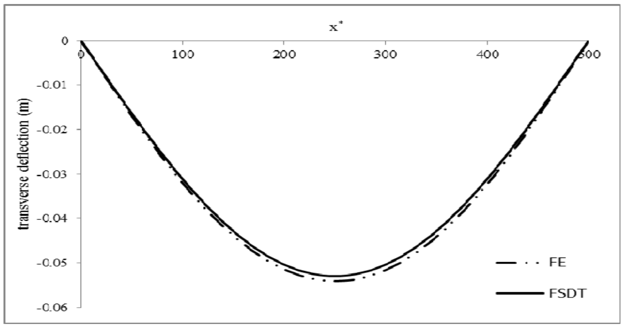

Fig. 5
Analytical and numerical transverse deflection at t=0.5.

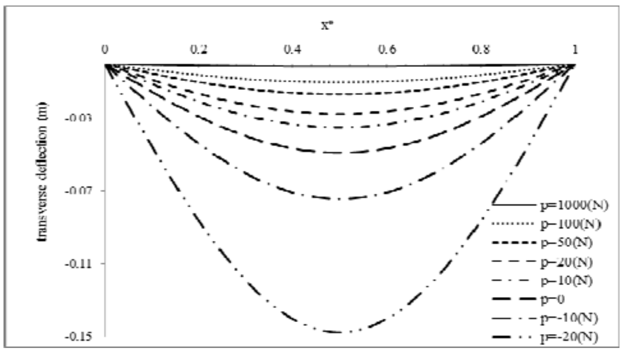

Fig. 6
Transverse deflection at t=0.5 for different axial load.

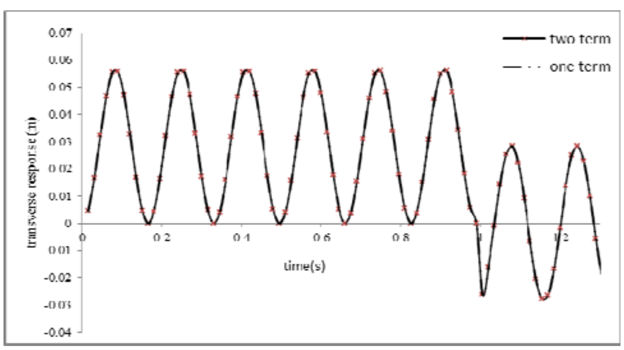

Fig. 7
Transverse response for one and two term.

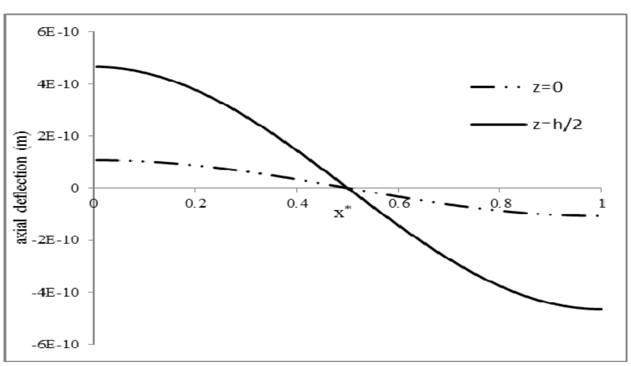

Fig. 8
Axial deflection at t=0.5 for top and mid-span of the beam.

5 CONCLUSIONS

In the presented paper, a mathematical procedure base on the shear deformation theory is proposed for response determination of a geometrical nonlinear beam subjected to transverse and axial loads. The perturbation technique and eigenfunction expansion method were employed for solution. This formulation can predict the axial response as well as transverse one for different layers of the beam. The FSDT results is closer than to the nonlinear FE results. So , the FSDT theory can be used to predict the nonlinear behavior of the beam too.

REFERENCES

[1] Lee K., 2002, Large deflections of cantilever beams of non-linear elastic material under a combined loading, *International Journal of Non-Linear Mechanics* **37**:439-443.
[2] Banerjee A., Bhattacharya B., Mallik A.K., 2008, Large deflection of cantilever beams with geometric non-linearity analytical and numerical approaches, *International Journal of Non-Linear Mechanics* **43**:366 –376.
[3] Chen L., 2010, An integral approach for large deflection cantilever beams, *International Journal of Non-Linear Mechanics* **45**:301-305.
[4] Vega-Posada C., Areiza-Hurtado M., Aristizabal-Ochoa J., 2011, Large-deflection and post-buckling behavior of slender beam-columns with non-linear end-estraints, *International Journal of Non-Linear Mechanics* **46**:79-95.
[5] Wang C. M., Lam K. Y., Hel X. Q., Chucheepsakul S., 1997, Large deflections of an end supported beam subjected to a point load, *International Journal of Non-Linear Mechanics* **32**:63-72.
[6] Li L.P., Schulgasser K., Cederbaum G., 1998, Large deflection analysis of poroelastic beams, *International Journal of Non-Linear Mechanics* **33**:1-14.
[7] Dado M., Al-Sadder S., 2005, A new technique for large deflection analysis of non-prismatic cantilever beams, *Mechanics Research Communications* **32**:692-703.
[8] Su Y., Ma C., 2012, Transient wave analysis of a cantilever Timoshenko beam subjected to impact loading by Laplace transform and normal mode methods, *International Journal of Solids and Structures* **49**:1158-1176.
[9] Amabili M., 2008, *Nonlinear Vibration and Stability of Shells and Plates*, Cambridge University Press, New York.
[10] Wang C.M., Reddy J.N., Lee K.H., 2000, Shear Deformable Beams and Plates, Relationships with Classical Solutions, Elsevier, New York.
[11] Rao S.S., 2007, *Vibration of Continuous Systems*, John Wiley & Sons, New Jersey.
[12] Nayfeh A.H., 1993, *Introduction to Perturbation Techniques*, John Wiley, New York.