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A Simple and Systematic Approach for Implementing Boundary Conditions in the Differential Quadrature Free and Forced Vibration Analysis of Beams and Rectangular Plates

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ABSTRACT

This paper presents a simple and systematic way for imposing boundary conditions in the differential quadrature free and forced vibration analysis of beams and rectangular plates. First, the Dirichlet- and Neumann-type boundary conditions of the beam (or plate) are expressed as differential quadrature analog equations at the grid points on or near the boundaries. Then, similar to CBCGE (direct Coupling the Boundary Conditions with the discrete Governing Equations) approach, the resulting analog equations are used to replace the differential quadrature analog equations of the governing differential equations at these points in order to solve the problem. But, unlike the CBCGE approach, the grid points near the boundaries are not treated as boundary points in the proposed approach. In other words, the degrees of freedom related to Dirichlet-type boundary conditions are only eliminated from the original discrete equations. This simplifies significantly the solution procedure and its programming. A comparison of the proposed approach with other existing methodologies such as the CBCGE approach and MWCM (modifying weighting coefficient matrices) method is presented by their application to the vibration analysis of beams and rectangular plates with general boundary conditions to highlight the advantages of the new approach.

Keywords : Simple and systematic approach; Implementation of boundary conditions; Differential quadrature method; Dirichlet-type boundary conditions; Neumann-type boundary conditions; Free and forced vibration analysis; Beams; Rectangular plates.

1 INTRODUCTION

THE differential quadrature method (DQM), which was first introduced by Bellman and his associates [1, 2] in the early 1970's, is an alternative discretization approach for solving directly the governing differential equations in engineering and applied sciences. Its main idea is to approximate the derivative of a function at a discrete point with respect to a coordinate direction using a weighted linear sum of the function values at all the discrete points chosen along that direction [3, 4]. Compared to low-order methods such as the finite element and finite difference methods, the DQM can produce better accuracy by using a considerably small number of grid points. Another particular advantage of the DQM lies in its ease of use and implementation. Due to the above-

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mentioned favorable features, the DQM has been applied extensively to solve various engineering problems. The majority of these applications are concerned with static or dynamic problems [5-8]. Recently, the application of the DQM has been extended to initial value problems [9-13], where the time-derivatives are discretized using the DQM. More recently, the DQM has been successfully combined with other approximate methods such as the Ritz and finite element methods and applied to various plate problems of practical interest [14-17]. It has been found that the DQM is computationally efficient and is applicable to a large class of initial and/or boundary value problems.

In spite of above-mentioned advantages, the DQM has its own difficulty in implementation to the differential equations with multiple boundary conditions at boundary points, especially for solving fourth-order governing differential equations of classical beam and plate problems. When the DQM is employed to solve beam or plate problem, the displacement function is expressed as independent variable at any discrete point. But, two boundary conditions must be implemented at each discrete boundary point. Therefore, the number of resulting quadrature analog equations will be more than the number of the function values to be obtained. There has been a considerable effort devoted to overcoming this difficulty. Bert et al. [18] also Jang and Bert [19] proposed a δ-technique to impose the first-order derivate boundary conditions. In this technique, the Dirichlet-type boundary conditions are exactly discretized while the Neumann-type boundary conditions are approximately satisfied at a small distance δ ($\approx 10^{-5}$ in dimensionless value) adjacent to the boundary points. The technique may be applied to the solution of differential equations with more conditions at one point by choosing more successive δ-points. However, this technique may produce unexpected oscillation behavior of the DQM solutions [3, 4].

To overcome the difficulties of the δ-technique, Wang and Bert [20], Wang et al. [21], and Malik and Bert [22] proposed several approaches to impose the boundary conditions by modification of weighting coefficient matrices (say, the MWCM approach). In these approaches, the Dirichlet-type boundary conditions are exactly satisfied while the Neumann-type boundary conditions are built into the weighting coefficient matrices of the DQM. These approaches have been successfully applied to solve some beam and plate problems with very good accuracy. However, as indicated by Wang and Bert [20], there are some major limitations to the application of these techniques in implementing general boundary conditions. One limitation is in the implementation of clamped-clamped (CC) type boundary conditions. As we will show in this paper, the implementation of the FF type boundary conditions in the beam problem by this approach also leads to some wrong numerical results. Most importantly, the MWCM approach cannot be used to study the free vibration of rectangular plates involving free edges and free corners. This is because the free edge and free corner boundary conditions cannot be implemented by modification of DQM weighting coefficient matrices. For initial-value problems, Tanaka and Chen [23] have employed this technique to incorporate the given initial conditions for transient responses for elastodynamic problems.

Alternatively, Shu and Du [24, 25] proposed two approaches for implementing general boundary conditions of beams and rectangular plates. The first one, which was referred to as the SBCGE (direct Substituting the discrete Boundary Conditions into the discrete Governing Equations) approach [24], was shown to be applicable only to beams and rectangular plates with simply supported and clamped edge boundary conditions. To tackle this limitation, Shu and Du [25] proposed another approach, referred to as the CBCGE (direct Coupling the Boundary Conditions with the discrete Governing Equations) approach, for implementing general boundary conditions of the beams and rectangular plates. It was shown that the SBCGE and CBCGE approaches remove some drawbacks of the δ-technique and produce accurate solutions for beams and plates with clamped and simply supported boundary conditions. However, in these approaches, the grid points adjacent to the boundary points are treated approximately as the boundary points. Actually, these points are not boundary points but are treated as boundary points. On the other hand, the CBCGE approach was shown to have some difficulty when it is applied to the plate problems involving free corners. In this case, the results of this approach are very sensitive to the grid point distribution. To solve this difficulty, Shu and Du [25] proposed the use of stretched grid points and showed that better accuracy can be achieved by the help of this type of grid points. However, their results didn't show a uniform convergence behavior in most cases and in some cases their results showed an oscillatory behavior. Most recently, Golfam and Rezaie [26] proposed a new way of implementation of SBCGE approach to bending analysis of beams with homogeneous and non-homogeneous boundary conditions.

On the other hand, the higher-order derivative boundary conditions can be imposed exactly by modifying the trial functions to incorporate the degrees of freedom of the higher-order derivatives at the boundary [27]. This procedure, in general, is somehow similar to the conventional Ritz method where the trial functions are selected such that they satisfy at least the geometric boundary conditions of the problem. Naturally, this procedure carries the disadvantages of the conventional Ritz method. For example, the higher order trial functions tend to become unstable and, most importantly, it is not possible in general to find trial functions for the method that satisfy all the boundary conditions of the problem (both geometric and natural) [28].

In another attempt to solve the kind of differential equations which involve more than one boundary or initial conditions at one point, Wu and Liu [29-31] proposed a generalized differential quadrature rule (GDQR). They have employed the Hermite interpolation functions as the trial functions to incorporate the boundary/initial conditions into the approximate solutions. However, as pointed out by Fung [32], this approach is in fact equivalent to the conventional CBCGE approach where the differential quadrature analog equations of the boundary/initial conditions are implemented at the boundary/initial point. Alternatively, Fung [32, 33] proposed a modified differential quadrature rule to impose the higher-order initial conditions. He showed that this approach is in fact equivalent to the approach employed in Wang and Bert [20] where the derivatives initial conditions are incorporated to the solution process by modification of weighting coefficient matrices. He also showed that the accuracy and stability of the DQM is highly affected by the choice of grid point type and by the way where the initial conditions are incorporated.

In this paper a simple and systematic approach is presented to implement the Dirichlet- and Neumann-type boundary conditions in the DQM free and forced vibration analysis of beams and rectangular plates. Similar to the CBCGE approach, the two boundary conditions of the beam (or plate) are accurately satisfied at the boundary points. But, unlike the CBCGE approach, the grid points adjacent to the boundary points are not treated as the boundary points. Compared with the MWCM approach, the present formulation does not produce wrong results for free vibration of beams with clamped and/or free edges. Besides, unlike the MWCM approach, the proposed method can be easily applied to the plate problems with general boundary conditions. For instance, the case of rectangular plates with free edges and free corners can be easily handled by the proposed method. Compared with the CBCGE approach, the present formulation is superior since its implementation and programming are easier and simpler. Furthermore, the present method can produce much better accuracy than the CBCGE approach for free vibration of rectangular plates involving free corners. The obtained numerical solutions are also not very sensitive to the grid point distribution.

2 DIFFERENTIAL QUADRATURE METHOD

Let $\Theta(\eta)$ be a solution of a differential equation and $\eta_1,\eta_2,\eta_3,...,\eta_n$ be a set of grid points in the η-direction. According to the DQM, the rth-order derivative of the function $\Theta(\eta)$ at any grid point $\eta=\eta_i$ can be approximated by the following formulation [3]

$$\Theta^{(r)}(\eta_i)=\sum_{j=1}^{n}A_{ij}^{(r)}\Theta(\eta_j) \quad \text{or} \quad \Theta_i^{(r)}=\sum_{j=1}^{n}A_{ij}^{(r)}\Theta_j \tag{1}$$

where n is the number of grid points in the η-direction, $W(\eta_j)$ represents the functional value at a grid point η_i, $\Theta^{(r)}(\eta_i)$ indicates the r th-order derivative of $\Theta(\eta)$ at a grid point η_i, and $A_{ij}^{(r)}$ are the weighting coefficients of the r th-order derivative. It follows from Eq. (1) that the quadrature rules may be written collectively in matrix form as:

$$\{\Theta^{(r)}\}=[A]^{(r)}\{\Theta\} \tag{2}$$

where $[A]^{(r)}$ is the r th-order DQM weighting coefficient matrix, and

$$\{W\}=[W(\eta_1)\quad W(\eta_2)\quad .\quad .\quad .\quad W(\eta_n)]^T \tag{3}$$

$$\{W^{(r)}\}=[W^{(r)}(\eta_1)\quad W^{(r)}(\eta_2)\quad .\quad .\quad .\quad W^{(r)}(\eta_n)]^T \tag{4}$$

The weighting coefficients can be determined by the functional approximations in the η coordinate direction. Using the Lagrange interpolation polynomials as the approximating functions, Quan and Chang [34] obtained the following algebraic formulations to compute the first-order weighting coefficients

$$A_{ik}^{(1)} = \begin{cases} \dfrac{M^{(1)}(\eta_i)}{(\eta_i - \eta_k) M^{(1)}(\eta_k)} & i \neq k, \quad i,k = 1,2,\dots,n \\ -\displaystyle\sum_{j=1, j\neq i}^{n} A_{ij}^{(1)} & i = k, \quad i = 1,2,\dots,n \end{cases} \tag{5}$$

where $M^{(1)}(\eta)$ is defined as:

$$M^{(1)}(\eta_i) = \prod_{j=1, j\neq i}^{n} (\eta_i - \eta_j) \tag{6}$$

The weighting coefficients of the r th-order derivative can be obtained from the following relationship [4],

$$A_{ik}^{(r)} = \begin{cases} r\left[A_{ii}^{(r-1)} A_{ik}^{(1)} - \dfrac{A_{ik}^{(r-1)}}{\eta_i - \eta_k} \right] & i \neq k, \quad i,k = 1,2,\dots,n \\ -\displaystyle\sum_{j=1, j\neq i}^{n} A_{ij}^{(r)} & i = k, \quad i = 1,2,\dots,n \end{cases} \tag{7}$$

It is also possible to compute the higher order DQM weighting coefficients matrices from the following recurrence relationships [3]:

$$[A]^{(r)} = [A]^{(1)}[A]^{(r-1)} = [A]^{(r-1)}[A]^{(1)} \tag{8}$$

It can be seen from Eq. (8) that having the matrix $[A]^{(1)}$ of the first-order derivative weighting coefficients, one can obtain the weighting coefficients of the higher-order derivatives by successive multiplications of the $[A]^{(1)}$ matrix by itself.

One of the key factors in the accuracy and rate of convergence of the DQM solutions is the choice of grid points. It is well known that the equally spaced grid points are not very desirable [3]. It has been suggested that non-uniformly spaced grid points can generate more accurate solutions. The zeros of some orthogonal polynomials are commonly adopted as the grid points. In this work, the Chebyshev-Gauss-Lobatto grid points are used for calculation of weighting coefficients. These points are given in dimensionless units of length by [3]

$$\eta_i = \frac{1}{2}\left[1 - \cos\left(\frac{(i-1)\pi}{n-1} \right) \right] \quad , \qquad i = 1,2,...,n \tag{9}$$

3 FORMULATION FOR FREE VIBRATION ANALYSIS OF BEMAS

3.1 Governing equation and boundary conditions

Consider the free vibration of a beam with length L, governed by the following non-dimensional differential equation

$$\frac{d^4 W(X)}{dX^4} = \Omega^2 W(X) \ , \qquad X = \frac{x}{L} \ , \qquad 0 \le X \le 1 \tag{10}$$

where $W(X)$ is the dimensionless mode shape function of the lateral deflection, X is dimensionless coordinate along the axis of the beam and Ω is the dimensionless frequency of the beam vibrations. The dimensionless frequency Ω is given by

$$\Omega^2 = \frac{\rho A L^4}{EI}\omega^2 \tag{11}$$

wherein ρA is the mass per unit length of the beam, EI is the bending stiffness of the beam, and ω is the dimensional frequency of the beam. The boundary conditions of the beam are:

(I) Simply-supported edge (S)

$$W = 0, \qquad \frac{d^2W}{dX^2} = 0 \tag{12}$$

(II) Clamped edge (C)

$$W = 0, \qquad \frac{dW}{dX} = 0 \tag{13}$$

(III) Free edge (F)

$$\frac{d^2W}{dX^2} = 0, \qquad \frac{d^3W}{dX^3} = 0 \tag{14}$$

The analytical solution of Eq. (10) can be expressed as [35, 36]

$$W(X) = C_1 \sin(\lambda X) + C_2 \cos(\lambda X) + C_3 \sinh(\lambda X) + C_4 \cosh(\lambda X), \qquad \lambda = \sqrt{\Omega} \tag{15}$$

where the unknown constants C_i *(i=1, 2, 3, 4)* and the value of Ω can be determined from the known boundary conditions of the beam.

3.2 Implementation of boundary conditions using conventional approaches

As we discussed in introduction, there are different approaches or ways to implement the boundary conditions of initial and/or boundary value problems. Moreover, some approaches have been shown to be equivalent to other approaches. Therefore, in this section, we only consider the CBCGE (direct Coupling the Boundary Conditions with the discrete Governing Equations) and MWCM (modifying weighting coefficient matrices) approaches which have been commonly used by many researchers. In order to explain how these approaches work, we consider the free vibration of a beam with clamped-free boundary conditions. The boundary conditions for this case are given in Eqs. (13) and (14).

3.2.1 Implementing boundary conditions by modifying the weighting coefficient matrices (MWCM)

As pointed out earlier, in this technique the Dirichlet-type boundary conditions are numerically implemented while the Neumann-type boundary conditions are applied by modifying the DQM weighting coefficients matrices. The Neumann-type boundary conditions for the clamped-free beam are:

$$\frac{dW}{dX} = 0 \qquad \text{at} \quad X = 0 \tag{16}$$

$$\frac{d^2W}{dX^2} = 0 \qquad \text{at} \quad X = 1 \tag{17}$$

$$\frac{d^3W}{dX^3} = 0 \qquad \text{at} \quad X = 1 \tag{18}$$

In order to implement the boundary condition given by Eq. (16), we define

$$[\tilde{A}]^{(1)} = \begin{bmatrix} 0 & 0 & 0 & . & . & . & 0 \\ A_{21}^{(1)} & A_{22}^{(1)} & A_{23}^{(1)} & . & . & . & A_{2n}^{(1)} \\ A_{31}^{(1)} & A_{32}^{(1)} & A_{33}^{(1)} & . & . & . & A_{3n}^{(1)} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ A_{n1}^{(1)} & A_{n2}^{(1)} & A_{n3}^{(1)} & . & . & . & A_{nn}^{(1)} \end{bmatrix} \tag{19}$$

The second-order DQM weighting coefficient matrix is then computed from the following equation

$$[\hat{A}]^{(2)} = [A]^{(1)}[\tilde{A}]^{(1)} \tag{20}$$

To implement the boundary condition given by Eq. (17), the matrix $[\hat{A}]^{(2)}$ should be modified as follows

$$[\tilde{A}]^{(2)} = \begin{bmatrix} \hat{A}_{11}^{(2)} & \hat{A}_{12}^{(2)} & \hat{A}_{13}^{(2)} & . & . & . & \hat{A}_{1n}^{(2)} \\ \hat{A}_{21}^{(2)} & \hat{A}_{22}^{(2)} & \hat{A}_{23}^{(2)} & . & . & . & \hat{A}_{2n}^{(2)} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \hat{A}_{(n-1)1}^{(2)} & \hat{A}_{(n-1)2}^{(2)} & \hat{A}_{(n-1)3}^{(2)} & . & . & . & \hat{A}_{(n-1)n}^{(2)} \\ 0 & 0 & 0 & . & . & . & 0 \end{bmatrix} \tag{21}$$

The third-order DQM weighting coefficient matrix can then be obtained from the following equation

$$[\hat{A}]^{(3)} = [A]^{(1)}[\tilde{A}]^{(2)} \tag{22}$$

To apply the boundary condition given by Eq. (18), the matrix $[\hat{A}]^{(3)}$ should be further modified as:

$$[\tilde{A}]^{(3)} = \begin{bmatrix} \hat{A}_{11}^{(3)} & \hat{A}_{12}^{(3)} & \hat{A}_{13}^{(3)} & . & . & . & \hat{A}_{1n}^{(3)} \\ \hat{A}_{21}^{(3)} & \hat{A}_{22}^{(3)} & \hat{A}_{23}^{(3)} & . & . & . & \hat{A}_{2n}^{(3)} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \hat{A}_{(n-1)1}^{(3)} & \hat{A}_{(n-1)2}^{(3)} & \hat{A}_{(n-1)3}^{(3)} & . & . & . & \hat{A}_{(n-1)n}^{(3)} \\ 0 & 0 & 0 & . & . & . & 0 \end{bmatrix} \tag{23}$$

Finally, the modified fourth-order DQM weighting coefficient matrix is obtained as:

$$[\hat{A}]^{(4)} = [A]^{(1)}[\tilde{A}]^{(3)} \tag{24}$$

It can be seen that the Neumann boundary conditions of the beam are built into the DQM weighting coefficient matrices. Similarly, one can derive the modified DQM weighting coefficient matrices for beams with other types of boundary conditions. For the present case, the quadrature analog of Eq. (10) can be written in matrix form as:

$$[\hat{A}]^{(4)}\{W\}=\Omega^2\{W\} \tag{25}$$

The Dirichlet-type boundary condition of the problem can be easily implemented by eliminating the first row and column of the matrix $[\hat{A}]^{(4)}$.

3.2.2 Implementing boundary conditions by direct coupling the boundary conditions with the discrete governing equations (CBCGE)

In this technique, the grid points are first divided into two different groups, namely, domain points and boundary points. The governing differential equation is then satisfied at the domain points while the boundary conditions are applied at the boundary points. For the present problem, the domain and boundary points are as follows [3, 4, 25]

$$\{X_d\}=[X_3 \quad X_4 \quad . \quad . \quad . \quad X_{n-2}]^T \tag{26}$$

$$\{X_b\}=\begin{bmatrix}X_1 & X_2 & X_{n-1} & X_n\end{bmatrix}^T \tag{27}$$

where $\{X_d\}$ and $\{X_b\}$ indicate the vectors of domain and boundary points, respectively. It can be seen that in the CBCGE approach, the grid points X_2 and X_{n-1} are also treated as boundary points while they really belong to the domain points.

Satisfying the governing differential equation of the beam (given by Eq. (10)) at all the domain points and, applying the boundary conditions at all the boundary points yields the following eigenvalue equation:

$$\begin{bmatrix}[K_{bb}]_{4\times4} & [K_{bd}]_{4\times(n-4)} \\ [K_{db}]_{(n-4)\times4} & [K_{dd}]_{(n-4)\times(n-4)}\end{bmatrix}\begin{Bmatrix}\{W_b\}_{4\times1} \\ \{W_d\}_{(n-4)\times1}\end{Bmatrix}=\Omega^2\begin{Bmatrix}\{0\}_{4\times1} \\ \{W_d\}_{(n-4)\times1}\end{Bmatrix} \tag{28}$$

where, for the considered case of clamped-free beam,

$$[K_{bb}]=\begin{bmatrix}1 & 0 & 0 & 0 \\ A^{(1)}_{11} & A^{(1)}_{12} & A^{(1)}_{1(n-1)} & A^{(1)}_{1n} \\ A^{(2)}_{n1} & A^{(2)}_{n2} & A^{(2)}_{n(n-1)} & A^{(2)}_{nn} \\ A^{(3)}_{n1} & A^{(3)}_{n2} & A^{(3)}_{n(n-1)} & A^{(3)}_{nn}\end{bmatrix}_{4\times4} \tag{29}$$

$$[K_{bd}]=\begin{bmatrix}0 & 0 & \cdots & 0 \\ A^{(1)}_{13} & A^{(1)}_{14} & \cdots & A^{(1)}_{1(n-2)} \\ A^{(2)}_{n3} & A^{(2)}_{n4} & \cdots & A^{(2)}_{n(n-2)} \\ A^{(3)}_{n3} & A^{(3)}_{n4} & \cdots & A^{(3)}_{n(n-2)}\end{bmatrix}_{4\times(n-2)} \tag{30}$$

$$[K_{db}]=\begin{bmatrix}A^{(4)}_{31} & A^{(4)}_{32} & A^{(4)}_{3(n-1)} & A^{(4)}_{3n} \\ A^{(4)}_{41} & A^{(4)}_{42} & A^{(4)}_{4(n-1)} & A^{(4)}_{4n} \\ \vdots & \vdots & \vdots & \vdots \\ A^{(4)}_{(n-2)1} & A^{(4)}_{(n-2)2} & A^{(4)}_{(n-2)(n-1)} & A^{(4)}_{(n-2)n}\end{bmatrix}_{(n-2)\times4} \tag{31}$$

$$[K_{dd}]=\begin{bmatrix}A^{(4)}_{33} & A^{(4)}_{34} & \cdots & A^{(4)}_{3(n-2)} \\ A^{(4)}_{43} & A^{(4)}_{44} & \cdots & A^{(4)}_{4(n-2)} \\ \vdots & \vdots & \vdots & \vdots \\ A^{(4)}_{(n-2)3} & A^{(4)}_{(n-2)4} & \cdots & A^{(4)}_{(n-2)(n-2)}\end{bmatrix}_{(n-2)\times(n-2)} \tag{32}$$

$$\{W_d\} = [W(X_3) \quad W(X_4) \quad . \quad . \quad . \quad W(X_{n-2})]^T \tag{33}$$

$$\{W_b\} = \left[W(X_1) \quad W(X_2) \quad W(X_{n-1}) \quad W(X_n)\right]^T \tag{34}$$

By eliminating the vector $\{W_b\}$, Eq. (28) is reduced to the following eigenvalue problem:

$$[\boldsymbol{K}]\{W_d\} = \Omega^2\{W_d\} \tag{35}$$

where

$$[\boldsymbol{K}] = [K_{dd}] - [K_{db}][K_{bb}]^{-1}[K_{bd}] \tag{36}$$

It is noted that above procedure satisfy simultaneously the Dirichlet- and Neumann-type boundary conditions of the problem

3.3 Implementation of boundary conditions using the proposed approach

It can be seen from Sections 3.2.1 and 3.2.2 that the implementation of boundary conditions in the CBCGE approach is not as straightforward as that in the MWCM approach. Moreover, the CBCGE approach requires some mathematical manipulations to construct resulting eigenvalue problem and is somehow cumbersome and boring. The MWCM approach has also been shown not to be suitable for analysis of beams with general boundary conditions [24]. As a result, a simple procedure based on the DQM that can easily handle the beam problem with general boundary conditions is still missing. Therefore, an alternative algorithm is needed which motivates the present study to present a simple approach for implementation of general boundary conditions in the DQM free and forced vibration analysis of beams.

Using the quadrature rule, given in Eq. (2), the quadrature analog of Eq. (10) can be written in matrix form as:

$$[A]^{(4)}\{W\} = \Omega^2[I]\{W\} \tag{37}$$

where $[A]^{(4)}$ is the fourth-order DQM weighting coefficient matrix and $[I]$ is an $n \times n$ identity matrix. Eq. (37) can also be expressed as:

$$[K]\{W\} = \Omega^2[M]\{W\} \tag{38}$$

where $[K]$ and $[M]$ are stiffness and mass matrices of the beam defined as;

$$[K] = [A]^{(4)}, \quad [M] = [I] \tag{39}$$

To describe the proposed methodology, similar to previous section, we consider the free vibration problem of a beam with clamped-free boundary conditions. Applying the beam boundary conditions to Eq. (38), we obtain:

$$\begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \\ A_{11}^{(1)} & A_{12}^{(1)} & A_{13}^{(1)} & . & . & . & A_{1n}^{(1)} \\ K_{31} & K_{32} & K_{33} & . & . & . & K_{3n} \\ K_{41} & K_{42} & K_{43} & . & . & . & K_{4n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ K_{(n-2)1} & K_{(n-2)2} & K_{(n-2)3} & .. & . & . & K_{(n-2)n} \\ A_{n1}^{(2)} & A_{n2}^{(2)} & A_{n3}^{(2)} & . & . & . & A_{nn}^{(2)} \\ A_{n1}^{(3)} & A_{n2}^{(2)} & A_{n3}^{(3)} & . & . & . & A_{nn}^{(3)} \end{bmatrix} \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ . \\ . \\ . \\ W_{n-1} \\ W_n \end{Bmatrix} = \Omega^2 \begin{bmatrix} 0 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \\ M_{31} & M_{32} & M_{33} & . & . & . & M_{3n} \\ M_{41} & M_{42} & M_{43} & . & . & . & M_{4n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ M_{(n-2)1} & M_{(n-2)2} & M_{(n-2)3} & . & . & . & M_{(n-2)n} \\ 0 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \end{bmatrix} \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ . \\ . \\ . \\ W_{n-1} \\ W_n \end{Bmatrix} \tag{40}$$

It can be seen from Eq. (40) that the quadrature analogs of the boundary conditions are directly replaced into the quadrature analogs of the governing differential equation at the boundary points and their immediate adjacent points. At this step, the proposed approach is identical to the conventional CBCGE approach.

Unfortunately, the eigenvalue problem given in Eq. (40) is difficult to solve since it is highly ill-conditioned. To overcome this difficulty, some researchers [3, 25] have proposed to eliminate the degrees of freedom correspond to the boundary points and their adjacent points (i.e., W_1, W_2, W_{n-1} and W_n). As we mentioned in previous section, this approach is the CBCGE approach wherein the grid points X_2 and X_{n-1} are also treated as boundary points. However, as we will show in this section, this is not the only case where the ill-conditioning phenomenon in solving the eigenvalue problem (40) can be eliminated.

To understand the major reason for such ill-conditioning phenomenon, we considered the free vibration problem of beams with general boundary conditions and investigated the effect of elimination of quadrature analog equations of the Dirichlet- and Neumann-type boundary conditions on the accuracy and stability of numerical solutions. Our numerical experiments showed that this difficulty is highly affected by the elimination of quadrature analog equations of the Dirichlet-type boundary conditions. For example, the ill-conditioned eigenvalue problem (40) can be converted to a well-conditioned eigenvalue problem if the quadrature analog equation of the Dirichlet-type boundary condition is eliminated as follows:

$$\begin{bmatrix} A_{12}^{(1)} & A_{13}^{(1)} & \cdots & A_{1n}^{(1)} \\ K_{32} & K_{33} & \cdots & K_{3n} \\ K_{42} & K_{43} & \cdots & K_{4n} \\ \vdots & \vdots & \vdots\ \vdots\ \vdots & \vdots \\ K_{(n-2)2} & K_{(n-2)3} & \cdots & K_{(n-2)n} \\ A_{n2}^{(2)} & A_{n3}^{(2)} & \cdots & A_{nn}^{(2)} \\ A_{n2}^{(3)} & A_{n3}^{(2)} & \cdots & A_{nn}^{(3)} \end{bmatrix} \begin{Bmatrix} W_2 \\ W_3 \\ \cdot \\ \cdot \\ \cdot \\ W_{n-1} \\ W_n \end{Bmatrix} = \Omega^2 \begin{bmatrix} 0 & 0 & \cdots & 0 \\ M_{32} & M_{33} & \cdots & M_{3n} \\ M_{42} & M_{43} & \cdots & M_{4n} \\ \vdots & \vdots & \vdots\ \vdots\ \vdots & \vdots \\ M_{(n-2),2} & M_{(n-2),3} & \cdots & M_{(n-2),n} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{Bmatrix} W_2 \\ W_3 \\ \cdot \\ \cdot \\ \cdot \\ W_{n-1} \\ W_n \end{Bmatrix} \tag{41}$$

It can be seen that the mass matrix of above eigenvalue problem has three zero rows and therefore is a singular matrix. However, such an eigenvalue problem can be easily and accurately solved using the QZ algorithm [37]. Currently, some subroutines that implement QZ algorithms are included in most linear algebra-related software packages such as the MATLAB and LAPACK.

Similarly, one can easily formulate the present approach for beams with other type of boundary conditions. From Eqs. (40) and (41), it can be seen that the proposed approach is very simple and straightforward and, thus, its programming is very easy. Besides, it consists of only two simple steps:

1. Similar to CBCGE approach, substitute the quadrature analog equations of the boundary conditions into the quadrature analog equations of the governing differential equation at the boundary points and their immediate adjacent points.
2. Eliminate the quadrature analog equations of the Dirichlet-type boundary conditions.

3.4 Numerical results

To demonstrate the stability and accuracy of the proposed approach, natural frequencies of beams with general boundary conditions are evaluated and the results are tabulated in Tables 1-2. In Table 1. , the results of proposed approach are compared with the solutions obtained by the CBCGE and MWCM approaches. The exact frequency parameters are also shown and bolded in the table. The numerical results are obtained using 19 grid points, and the coordinates of grid points are selected as those given in Eq. (9). From Table 1. , it can be seen that the results of proposed approach are identical to the results of the CBCGE approach. This is a reasonable result, because both the proposed approach and the CBCGE approach use a similar approach to impose the boundary conditions of the beam at its boundary points. In other words, only the way of implementation of boundary conditions in these approaches (proposed approach and the CBCGE approach) is different. It can also be seen from Table 1 that, except for CC and FF beams, the MWCM approach produces better accuracy than the present approach and the CBCGE approach. This is also a reasonable result, because the Neumann-type boundary conditions are exactly satisfied in the MWCM approach while they are approximately satisfied in the present approach and the CBCGE approach. However, the MWCM approach is found to produce spurious modes with zero frequency for beams with CC and FF boundary conditions. The frequency parameters correspond to spurious modes are also bolded in the Table 1.

Table 1
Comparison of natural frequencies of a beam with different boundary conditions (n = 19).

Beam	Method	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
SS	Present	9.8696	39.4784	88.8264	157.9141	246.7488
	CBCGE	9.8696	39.4784	88.8264	157.9141	246.7488
	MWCM	9.8696	39.4784	88.8264	157.9137	246.7401
	Exact	9.8696	39.4784	88.8264	157.9137	246.7401
CC	Present	22.3733	61.6728	120.9034	199.8597	298.5597
	CBCGE	22.3733	61.6728	120.9034	199.8597	298.5597
	MWCM	00.0000	22.3733	61.6728	120.9034	199.8594
	Exact	22.3733	61.6728	120.9034	199.8594	298.5555
SC	Present	15.4182	49.9649	104.2477	178.2704	272.0350
	CBCGE	15.4182	49.9649	104.2477	178.2704	272.0350
	MWCM	15.4182	49.9649	104.2477	178.2697	272.0307
	Exact	15.4182	49.9649	104.2477	178.2697	272.0310
SF[a]	Present	15.4182	49.9649	104.2478	178.2638	271.8390
	CBCGE	15.4182	49.9649	104.2478	178.2638	271.8390
	MWCM	15.4182	49.9649	104.2477	178.2697	272.0307
	Exact	15.4182	49.9649	104.2477	178.2697	272.0310
CF	Present	3.5160	22.0345	61.6972	120.9021	199.8387
	CBCGE	3.5160	22.0345	61.6972	120.9021	199.8387
	MWCM	3.5160	22.0345	61.6972	120.9019	199.8595
	Exact	3.5160	22.0345	61.6972	120.9019	199.8595
FF[b]	Present	22.3733	61.6728	120.9037	199.8177	298.0423
	CBCGE	22.3733	61.6728	120.9037	199.8177	298.0423
	MWCM	00.0009	22.3733	61.6728	120.9034	199.8594
	Exact	22.3733	61.6728	120.9034	199.8594	298.5555

[a]In a mathematical sense the first mode has a zero frequency, and corresponds to rigid body rotation about the simply-supported end.
[b]In a mathematical sense the first two modes have zero frequencies and correspond to rigid body translation in the transverse direction and rigid body rotation about the beam center.

The DQM solution results shown in Table 1 were calculated using Chebyshev-Gauss-Lobatto grid points (see Eq. (9)). It is also possible to improve their accuracy by using the following type of grid points:

$$\begin{aligned} &\eta_1 = 0, \eta_2 = \delta, \eta_{n-1} = 1-\delta, \eta_n = 1 \\ &\eta_i = 1/2\left[1-\cos\left(\frac{(i-2)\pi}{n-3}\right)\right], \qquad i = 3,4,...,n-2, \qquad 0 \le \eta \le 1, \qquad \eta = X \end{aligned} \tag{42}$$

where δ is a parameter that determines the closeness between the boundary points (η_1 and η_n) and their immediate adjacent points (η_2 and η_{n-1}). In practice, the magnitude of δ should be as small as possible ($\le 10^{-3}$). In this study, the magnitude of parameter δ is assumed to be $\delta = 10^{-3}$.

when the above type of grid points is used, Table 2. shows the convergence and accuracy of solutions obtained by the proposed approach. Comparing these results with those in Table 1. , it can be seen that the use of δ -points improves significantly the accuracy of solutions.

Table 2
Convergence and comparison of natural frequencies of a beam with different boundary conditions, when adjacent δ-points are introduced on grid points.

Beam	Method	n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
SS	Present	19	9.8696	39.4784	88.8264	157.9137	246.7400
		21	9.8696	39.4784	88.8264	157.9137	246.7401
	Exact		9.8696	39.4784	88.8264	157.9137	246.7401
CC	Present	19	22.3733	61.6728	120.9034	199.8595	298.5555
		21	22.3733	61.6728	120.9034	199.8594	298.5555
	Exact		22.3733	61.6728	120.9034	199.8594	298.5555
SC	Present	19	15.4182	49.9649	104.2477	178.2697	272.0309
		21	15.4182	49.9649	104.2477	178.2697	272.0310
	Exact		15.4182	49.9649	104.2477	178.2697	272.0310
SF	Present	19	15.4182	49.9649	104.2477	178.2696	272.0263
		21	15.4182	49.9649	104.2477	178.2697	272.0312
		23	15.4182	49.9649	104.2477	178.2697	272.0310
	Exact		15.4182	49.9649	104.2477	178.2697	272.0310
CF	Present	19	3.5160	22.0345	61.6972	120.9019	199.8590
		21	3.5160	22.0345	61.6972	120.9019	199.8596
		23	3.5160	22.0345	61.6972	120.9019	199.8595
	Exact		3.5160	22.0345	61.6972	120.9019	199.8595
FF	Present	19	22.3733	61.6728	120.9034	199.8585	298.5430
		21	22.3733	61.6728	120.9034	199.8595	298.5563
		23	22.3733	61.6728	120.9034	199.8594	298.5555
	Exact		22.3733	61.6728	120.9034	199.8594	298.5555

4 FORMULATION FOR FORCED VIBRATION ANALYSIS OF BEMAS

4.1 Governing equation

Consider the forced vibration of a beam with length L, mass per unit length ρA, bending stiffness EI; subjected to a dynamic transverse load $f(x, t)$. The governing differential equation of motion of the beam is given by

$$\rho A\frac{\partial^2 w(x,t)}{\partial t^2}+EI\frac{\partial^4 w(x,t)}{\partial x^4}=f(x,t), \qquad 0\le x\le L \tag{43}$$

where $w(x, t)$ is the lateral deflection of the beam, and t is the time. Using the normal mode approach (modal analysis), the analytical solution of Eq. (43) can be expressed as [36]

$$w(x,t)=\sum_{i=1}^{\infty}\left[A_i\cos\omega_i t+B_i\sin\omega_i t+\frac{1}{\omega_i}\int_0^L Q_i(\tau)\sin\omega_i(t-\tau)d\tau\right]W_i(x) \tag{44}$$

where ω_i is the ith natural frequency and $W_i(x)$ is the corresponding natural mode. Moreover, $Q_i(\tau)$ is the generalized force corresponding to the ith mode given by

$$Q_i(\tau)=\int_0^L W_i(x)f(x,\tau)dx, \qquad i=1,2,3,... \tag{45}$$

It is noted that the first two terms inside the brackets of Eq. (44) denote the free vibration response while the third term indicates the forced vibration response of the beam. The constants A_i and B_i can be evaluated using the initial conditions of the beam.

4.2 Implementation of boundary conditions using conventional approaches

Similar to Section 3.2, in this section, a beam with clamped-free end conditions is considered and the CBCGE and MWCM approaches are briefly described.

4.2.1 Implementing boundary conditions by modifying the weighting coefficient matrices (MWCM)

The procedure is exactly identical to that described in Section 3.2.1. In this case, the quadrature analog of Eq. (43) can be written in matrix form as:

$$\rho A[I]\{\ddot{w}\}+EI[\hat{A}]^{(4)}\{w\}=\{f\} \tag{46}$$

where the matrices $[\hat{A}]^{(4)}$ and $[I]$ are defined in Sections 3.2.1 and 3.2.2, respectively, and

$$\{w\}=[w(x_1,t)\quad w(x_2,t)\quad .\ .\ .\quad w(x_n,t)]^T \tag{47}$$

$$\{\ddot{w}\}=[\ddot{w}(x_1,t)\quad \ddot{w}(x_2,t)\quad .\ .\ .\quad \ddot{w}(x_n,t)]^T \tag{48}$$

$$\{f\}=[f(x_1,t)\quad f(x_2,t)\quad .\ .\ .\quad f(x_n,t)]^T \tag{49}$$

Eq. (46) is a system of linear ordinary differential equations of second-order in time with constant coefficients and can be solved using various time integration schemes (after applying the Dirichlet-type boundary conditions). In this study, the Newmark method [38] is used for the time integration of the DQM equations of motion given in Eq. (46).

4.2.2 Implementing boundary conditions by direct coupling the boundary conditions with the discrete governing equations (CBCGE)

The procedure is identical to that described in Section 3.2.2. Following the procedure explained in Section 3.2.2, the quadrature analog of Eq. (43) can be written in matrix form as:

$$\begin{aligned}&\rho A\begin{bmatrix}[M_{bb}]_{4\times 4} & [M_{bd}]_{4\times(n-4)}\\ [M_{db}]_{(n-4)\times 4} & [M_{dd}]_{(n-4)\times(n-4)}\end{bmatrix}\begin{Bmatrix}\{\ddot{w}_b\}_{4\times 1}\\ \{\ddot{w}_d\}_{(n-4)\times 1}\end{Bmatrix}+\\ &EI\begin{bmatrix}[K_{bb}]_{4\times 4} & [K_{bd}]_{4\times(n-4)}\\ [K_{db}]_{(n-4)\times 4} & [K_{dd}]_{(n-4)\times(n-4)}\end{bmatrix}\begin{Bmatrix}\{w_b\}_{4\times 1}\\ \{w_d\}_{(n-4)\times 1}\end{Bmatrix}=\begin{Bmatrix}\{0\}_{4\times 1}\\ \{f_d\}_{(n-4)\times 1}\end{Bmatrix}\end{aligned} \tag{50}$$

where $[M_{bb}]$, $[M_{bd}]$ and $[M_{db}]$ are zero matrices; $[M_{dd}]$ is an identity matrix of size $(n - 4) \times (n - 4)$; and

$$\{w_d\}=[w(x_3,t)\quad w(x_4,t)\quad .\ .\ .\quad w(x_{n-2},t)]^T,\qquad \{\ddot{w}_d\}=\frac{d^2}{dt^2}\{w_d\} \tag{51}$$

$$\{w_b\}=\left[w(x_1,t)\quad w(x_2,t)\quad w(x_{n-1},t)\quad w(x_n,t)\right]^T,\qquad \{\ddot{w}_b\}=\frac{d^2}{dt^2}\{w_b\} \tag{52}$$

$$\{f_d\}=[f(x_3,t)\quad f(x_4,t)\quad .\ .\ .\quad f(x_{n-2},t)]^T \tag{53}$$

Note that the matrices $[K_{bb}]$, $[K_{bd}]$, $[K_{db}]$ and $[K_{dd}]$ are defined already in Section 3.2.2. By eliminating the vector $\{w_b\}$, Eq. (50) is reduced to the following system of ordinary differential equations:

$$[\boldsymbol{M}]\{\ddot{w}_d\}+[\boldsymbol{K}]\{w_d\}=\{f_d\} \tag{54}$$

where

$$[\boldsymbol{M}]=\rho A[M_{dd}], \qquad [\boldsymbol{K}]=EI\left([K_{dd}]-[K_{db}][K_{bb}]^{-1}[K_{bd}]\right) \tag{55}$$

4.3 Implementation of boundary conditions using the proposed approach

The procedure is identical to that described in Section 3.3. Following the procedure explained in Section 3.3, the quadrature analog of Eq. (43) can be expressed in matrix form as:

$$\rho A\begin{bmatrix} 0 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \\ M_{31} & M_{32} & M_{33} & . & . & . & M_{3n} \\ M_{41} & M_{42} & M_{43} & . & . & . & M_{4n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ M_{(n-2)1} & M_{(n-2)2} & M_{(n-2)3} & . & . & . & M_{(n-2)n} \\ 0 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \end{bmatrix}\begin{Bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \\ \ddot{w}_3 \\ . \\ . \\ . \\ \ddot{w}_{n-1} \\ \ddot{w}_n \end{Bmatrix}+ EI\begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \\ A_{11}^{(1)} & A_{12}^{(1)} & A_{13}^{(1)} & . & . & . & A_{1n}^{(1)} \\ K_{31} & K_{32} & K_{33} & . & . & . & K_{3n} \\ K_{41} & K_{42} & K_{43} & . & . & . & K_{4n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ K_{(n-2)1} & K_{(n-2)2} & K_{(n-2)3} & .. & . & . & K_{(n-2)n} \\ A_{n1}^{(2)} & A_{n2}^{(2)} & A_{n3}^{(2)} & . & . & . & A_{nn}^{(2)} \\ A_{n1}^{(3)} & A_{n2}^{(2)} & A_{n3}^{(3)} & . & . & . & A_{nn}^{(3)} \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \\ . \\ . \\ . \\ w_{n-1} \\ w_n \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ f_3 \\ f_4 \\ \vdots \\ f_{n-2} \\ 0 \\ 0 \end{Bmatrix} \tag{56}$$

Eq. (56) can also be expresses as:

$$[\boldsymbol{M}]\{\ddot{w}\}+[\boldsymbol{K}]\{w\}=\{\boldsymbol{f}\} \tag{57}$$

Our numerical experiments showed that above system of ordinary differential equations is a well-conditioned system of ordinary differential equations. Therefore, it can be easily and directly solved for unknowns using various time integration schemes.

4.4 Numerical results

To demonstrate the applicability of the proposed approach for the forced vibration analysis of beams, numerical experiments are carried out for the dynamic analysis of a simply supported beam subjected to a harmonically varying load in the from:

$$f(x,t)= f_0\sin\frac{m\pi x}{L}\sin\omega_f t \tag{58}$$

where f_0 and ω_f are the amplitude and frequency of variation of the applied load, respectively, and m is a constant. Assuming that the beam is initially at rest, an analytical solution for the present problem can be obtained from Eq. (44) as:

$$w(x,t)=\frac{f_0L^4}{EI(m\pi)^4\left[1-(\omega_f/\omega_m)^2\right]}\sin\frac{m\pi x}{L}\left(\sin\omega_f t-\frac{\omega_f}{\omega_m}\sin\omega_m t\right),\omega_m=m^2\pi^2\sqrt{\frac{EI}{\rho AL^4}} \tag{59}$$

Thus, the performance of various DQM algorithms can be easily tested by comparing their solutions with above analytic solution. For the numerical experiment, the following data are used:

$$\frac{\rho A}{EI}=1,\qquad \frac{f_0}{EI}=1,\qquad \omega_f=1,\qquad L=1,\qquad m=1 \tag{60}$$

The dynamic response of the simply supported beam subjected to harmonically varying load (58) is computed using the proposed approach and using the CBCGE and MWCM approaches. The Newmark method with a time step of Δt = 0.001s is used to solve the resulting dynamic equations. Table 3. shows the convergence of solutions with respect to the number of DQM grid points. Note that the Lagrange interpolation scheme in the *x*-direction is used to obtain the solution values which were not obtainable directly from the quadrature solutions. From Table 3. , it may be seen that the results of proposed approach are identical to those obtained by the CBCGE approach. Again, one sees that the MWCM approach gives better accuracy than the present approach and the CBCGE approach.

As we mentioned earlier in Section 3.4, a way for improving the accuracy of proposed approach is to use the Chebyshev grid points with adjacent δ -points. To demonstrate this for the case of forced vibration, the results of the present problem obtained using different DQM approaches are presented in Table 4. From Table 4. it can be seen that, for a small number of grid points, the MWCM approach provides better accuracy than other approaches. But, by increasing the number of grid points, the accuracy of solutions obtained by the proposed approach and the CBCGE approach becomes comparable to that of the MWCM approach.

Table 3
Convergence and comparison of forced vibration responses ($w(x, 0.5)$) of a simply supported beam subjected to a harmonically varying load.

n	Method	x = 0.1	x = 0.25	x = 0.5	x = 0.75	x = 0.9
5	Present	2.43248e-003	5.52097e-003	7.74873e-003	5.52097e-003	2.43248e-003
	CBCGE	2.43248e-003	5.52097e-003	7.74873e-003	5.52097e-003	2.43248e-003
	MWCM	1.86551e-003	4.29099e-003	6.06706e-003	4.29099e-003	1.86551e-003
6	Present	2.14595e-003	4.87064e-003	6.83599e-003	4.87064e-003	2.14595e-003
	CBCGE	2.14595e-003	4.87064e-003	6.83599e-003	4.87064e-003	2.14595e-003
	MWCM	1.85785e-003	4.25034e-003	5.99175e-003	4.25034e-003	1.85785e-003
7	Present	1.83069e-003	4.19257e-003	5.93271e-003	4.19257e-003	1.83069e-003
	CBCGE	1.83069e-003	4.19257e-003	5.93271e-003	4.19257e-003	1.83069e-003
	MWCM	1.85328e-003	4.24099e-003	5.99789e-003	4.24099e-003	1.85328e-003
8	Present	1.84280e-003	4.21864e-003	5.96745e-003	4.21864e-003	1.84280e-003
	CBCGE	1.84280e-003	4.21864e-003	5.96745e-003	4.21864e-003	1.84280e-003
	MWCM	1.85338e-003	4.24120e-003	5.99777e-003	4.24120e-003	1.85338e-003
9	Present	1.85381e-003	4.24191e-003	5.99891e-003	4.24191e-003	1.85381e-003
	CBCGE	1.85381e-003	4.24191e-003	5.99891e-003	4.24191e-003	1.85381e-003
	MWCM	1.85345e-003	4.24115e-003	5.99790e-003	4.24115e-003	1.85345e-003
10	Present	1.85361e-003	4.24151e-003	5.99837e-003	4.24151e-003	1.85361e-003
	CBCGE	1.85361e-003	4.24151e-003	5.99837e-003	4.24151e-003	1.85361e-003
	MWCM	1.85345e-003	4.24115e-003	5.99790e-003	4.24115e-003	1.85345e-003
	Exact	1.85345e-003	4.24115e-003	5.99789e-003	4.24115e-003	1.85345e-003

Table 4
Convergence and comparison of forced vibration responses ($w(x, 0.5)$) of a simply supported beam subjected to a harmonically varying load, when adjacent δ-points are introduced on grid points.

n	Method	$x = 0.1$	$x = 0.25$	$x = 0.5$	$x = 0.75$	$x = 0.9$
7	Present	1.86404e-003	4.26647e-003	6.03408e-003	4.26647e-003	1.86404e-003
	CBCGE	1.86404e-003	4.26647e-003	6.03408e-003	4.26647e-003	1.86404e-003
	MWCM	1.85536e-003	4.24602e-003	6.00441e-003	4.24602e-003	1.85536e-003
8	Present	1.85435e-003	4.24257e-003	5.99807e-003	4.24257e-003	1.85435e-003
	CBCGE	1.85435e-003	4.24257e-003	5.99807e-003	4.24257e-003	1.85435e-003
	MWCM	1.85342e-003	4.24080e-003	5.99604e-003	4.24080e-003	1.85342e-003
9	Present	1.85342e-003	4.24111e-003	5.99786e-003	4.24111e-003	1.85342e-003
	CBCGE	1.85342e-003	4.24111e-003	5.99786e-003	4.24111e-003	1.85342e-003
	MWCM	1.85345e-003	4.24116e-003	5.99791e-003	4.24116e-003	1.85345e-003
10	Present	1.85345e-003	4.24114e-003	5.99787e-003	4.24114e-003	1.85345e-003
	CBCGE	1.85345e-003	4.24114e-003	5.99787e-003	4.24114e-003	1.85345e-003
	MWCM	1.85345e-003	4.24116e-003	5.99789e-003	4.24116e-003	1.85345e-003
	Exact	1.85345e-003	4.24115e-003	5.99789e-003	4.24115e-003	1.85345e-003

5 FORMULATION FOR FREE VIBRATION ANALYSIS OF RECTANGULAR PLATES

5.1 Governing equation and boundary conditions

Consider the free vibration of an elastic isotropic thin rectangular plate with length a and width b, governed by the following non-dimensional differential equation

$$\frac{\partial^4 W(X,Y)}{\partial X^4} + 2\lambda^2 \frac{\partial^4 W(X,Y)}{\partial X^2 \partial Y^2} + \lambda^4 \frac{\partial^4 W(X,Y)}{\partial Y^4} = \Omega^2 W(X,Y) \tag{61}$$

where $W(X,Y)$ is the dimensionless mode function of the lateral deflection; $X = x/a$ and $Y = y/b$ are dimensionless coordinates; $\lambda = a/b$ is the aspect ratio; and Ω is the dimensionless frequency. Furthermore, $\Omega = \omega a^2 \sqrt{\rho h / D}$ wherein ω, ρ, h, and D, are, respectively, the circular frequency, mass density, thickness, and flexural rigidity of the plate. The boundary conditions of the plate are:

(I) Simply-supported edge (S)

$$W = \frac{\partial^2 W}{\partial X^2} = 0 \quad \text{at} \quad X = 0 \ \text{and/or} \quad X = 1 \tag{62}$$

$$W = \frac{\partial^2 W}{\partial Y^2} = 0 \quad \text{at} \quad Y = 0 \ \text{and/or} \quad Y = 1 \tag{63}$$

(II) Clamped edge (C)

$$W = \frac{\partial W}{\partial X} = 0 \quad \text{at} \quad X = 0 \ \text{and/or} \quad X = 1 \tag{64}$$

$$W = \frac{\partial W}{\partial Y} = 0 \quad \text{at} \quad Y = 0 \ \text{and/or} \quad Y = 1 \tag{65}$$

(III) Free edge (F)

$$\frac{\partial^3 W}{\partial X^3} + (2-\mu)\lambda^2 \frac{\partial^3 W}{\partial X \partial Y^2} = \frac{\partial^2 W}{\partial X^2} + \mu\lambda^2 \frac{\partial^2 W}{\partial Y^2} = 0 \quad \text{at} \quad X = 0 \ \text{and/or} \quad X = 1 \tag{66}$$

$$\frac{\partial^3 W}{\partial Y^3}+\frac{(2-\mu)}{\lambda^2}\frac{\partial^3 W}{\partial Y \partial X^2}=\frac{\partial^2 W}{\partial Y^2}+\frac{\mu}{\lambda^2}\frac{\partial^2 W}{\partial X^2}=0 \qquad \text{at } Y=0 \text{ and/or } Y=1 \tag{67}$$

wherein μ is the Poisson's ratio. For a free corner formed by the intersection of two free edges, the additional condition

$$\frac{\partial^2 W}{\partial X\, \partial Y}=0 \tag{68}$$

Must also be satisfied at the corner [39].

5.2 Discretization of plate governing equation using the DQM

Consider n grid points with coordinates $X_1, X_2, \ldots, X_n$ in the X-direction, and m grid points with coordinates $Y_1, Y_2, \ldots, Y_m$ in the Y-direction. Satisfying Eq. (61) at any grid point $X = X_i$ and substituting the quadrature rule, given in Eq. (1), into results gives

$$[A]^{(4)}\{W(Y)\}+2\lambda^2[A]^{(2)}\frac{d^2}{dY^2}\{W(Y)\}+\lambda^4[I]\frac{d^4}{dY^4}\{W(Y)\}=\Omega^2[I]\{W(Y)\} \tag{69}$$

where $[I]$ is an identity matrix of order $n \times n$, $[A]^{(2)}$ and $[A]^{(4)}$ are the DQM weighting coefficient matrices of the second- and fourth-order X-derivatives, respectively. Furthermore,

$$\{W(Y)\}=[W(X_1,Y) \quad W(X_2,Y) \quad . \;\; . \;\; . \quad W(X_n,Y)]^T \tag{70}$$

Satisfying Eq. (69) at any grid point $Y = Y_i$ and substituting the quadrature rule, given in Eq. (1), into results gives

$$[\tilde{\boldsymbol{K}}]\{\tilde{\boldsymbol{W}}\}=\Omega^2[\tilde{\boldsymbol{M}}]\{\tilde{\boldsymbol{W}}\} \tag{71}$$

where the $n \times n$ sub-matrices $[\tilde{\boldsymbol{M}}_{ij}]$ and $[\tilde{\boldsymbol{K}}_{ij}]$ are given by

$$[\tilde{\boldsymbol{M}}_{ij}]=\bar{I}_{ij}[I], \qquad i,j=1,2,\ldots,m \tag{72}$$

$$[\tilde{\boldsymbol{K}}_{ij}]=\bar{I}_{ij}[A]^{(4)}+2\lambda^2\bar{A}_{ij}^{(2)}[A]^{(2)}+\lambda^4\bar{A}_{ij}^{(4)}[I] \tag{73}$$

where $\bar{I}_{ij}$ are the elements of an $m \times m$ identity matrix, $\bar{A}_{ij}^{(2)}$ and $\bar{A}_{ij}^{(4)}$ are the DQM weighting coefficients of the second- and fourth-order Y-derivatives, respectively. Furthermore,

$$\{\tilde{\boldsymbol{W}}\}=[\{W(Y_1)\}^T \quad \{W(Y_2)\}^T \quad . \;\; . \;\; . \quad \{W(Y_m)\}^T]^T \tag{74}$$

The general eigenvalue problem (71) can be solved for eigenvalues if the boundary conditions of the plate are also applied. The details are given in the following sections.

5.3 Discretization of plate boundary conditions using the DQM

When the CBCGE approach or proposed approach is applied, it is necessary to discretize the boundary conditions of the problem using the DQM. Therefore, the quadrature analog equations of the plate boundary conditions are derived in this section. The results are given in the following sections.

5.3.1 DQM analogs of the plate boundary conditions in the X-direction

The boundary conditions of the rectangular plate in the *X*-direction are given in Eqs. (62), (64) and (66). The corresponding quadrature analogs are given below.

(I) Simply supported edge condition at $X = X_p$ (p =1 or n)

From Eqs. (1) and (62), the quadrature analog of the boundary conditions are obtained as:

$$W(X_p, Y) = 0, \qquad \sum_{j=1}^{n} A_{pj}^{(2)} W(X_j, Y) = 0 \tag{75}$$

Above quadrature analog equations should be applied at $Y_1, Y_2, \ldots, Y_m$.

(II) Clamped edge condition at $X = X_p$ (p =1 or n)

From Eqs. (1) and (64), the quadrature analog of the boundary conditions are simply:

$$W(X_p, Y) = 0, \qquad \sum_{j=1}^{n} A_{pj}^{(1)} W(X_j, Y) = 0 \tag{76}$$

Above analog equations should be implemented at $Y_1, Y_2, \ldots, Y_m$.

(III) Free edge condition at $X = X_p$ (p =1 or n)

From Eqs. (1) and (66), the quadrature analog of the boundary conditions are written as:

$$\sum_{j=1}^{n} A_{pj}^{(3)} W(X_j, Y) + (2-\mu)\lambda^2 \sum_{j=1}^{n} A_{pj}^{(1)} \frac{d^2 W(X_j, Y)}{dY^2} = 0 \tag{77}$$

$$\sum_{j=1}^{n} A_{pj}^{(2)} W(X_j, Y) + \mu\lambda^2 \frac{d^2 W(X_p, Y)}{dY^2} = 0 \tag{78}$$

Eqs. (77) and (78) can be expressed in matrix form as:

$$[A_{p1}^{(3)} \quad A_{p2}^{(3)} \quad . \; . \; . \quad A_{pn}^{(3)}]\{W(Y)\} + (2-\mu)\lambda^2 [A_{p1}^{(1)} \quad A_{p2}^{(1)} \quad . \; . \; . \quad A_{pn}^{(1)}] \frac{d^2}{dY^2}\{W(Y)\} = 0 \tag{79}$$

$$[A_{p1}^{(2)} \quad A_{p2}^{(2)} \quad . \; . \; . \quad A_{pn}^{(2)}]\{W(Y)\} + \mu\lambda^2 [Z_1 \quad Z_2 \quad . \; . \; . \quad Z_n] \frac{d^2}{dY^2}\{W(Y)\} = 0 \tag{80}$$

where

$$Z_i = \begin{cases} 1 & i = p \\ 0 & i \neq p \end{cases} \tag{81}$$

Eqs. (79) and (80) can be rewritten as:

$$[A_p^{(3)}]_{1\times n}\{W(Y)\}_{n\times 1} + (2-\mu)\lambda^2 [A_p^{(1)}]_{1\times n} \frac{d^2}{dY^2}\{W(Y)\}_{n\times 1} = 0 \tag{82}$$

$$[A_p^{(2)}]_{1\times n}\{W(Y)\}_{n\times 1} + \mu\lambda^2 [Z]_{1\times n} \frac{d^2}{dY^2}\{W(Y)\}_{n\times 1} = 0 \tag{83}$$

where

$$[A_p^{(1)}] = [A_{p1}^{(1)} \quad A_{p2}^{(1)} \quad . \; . \; . \quad A_{pn}^{(1)}], [A_p^{(2)}] = [A_{p1}^{(2)} \quad A_{p2}^{(2)} \quad . \; . \; . \quad A_{pn}^{(2)}] \tag{84}$$

$$[A_p^{(3)}] = [A_{p1}^{(3)} \quad A_{p2}^{(3)} \quad . \quad . \quad . \quad A_{pn}^{(3)}], [Z] = [Z_1 \quad Z_2 \quad . \quad . \quad . \quad Z_n] \tag{85}$$

Eqs. (82) and (83) should be applied at Y_1, Y_2, …, Y_m. Satisfying these equations at any grid point $Y = Y_i$ and substituting the quadrature rule, given in Eq. (1), into results gives

$$[A_p^{(3)}]\{W(Y_i)\} + (2-\mu)\lambda^2 [A_p^{(1)}] \sum_{j=1}^{m} \bar{A}_{ij}^{(2)} \{W(Y_j)\} = 0, \qquad i = 1, 2, ..., m \tag{86}$$

$$[A_p^{(2)}]\{W(Y_i)\} + \mu\lambda^2 [Z] \sum_{j=1}^{m} \bar{A}_{ij}^{(2)} \{W(Y_j)\} = 0 \tag{87}$$

which can be used to implement the free-edge boundary conditions of the plate in the X-direction.

5.3.2 DQM analogs of the plate boundary conditions in the Y-direction

The boundary conditions of the rectangular plate in the Y-direction are given in Eqs. (63), (65) and (67). The corresponding quadrature analogs are given below.

(I) Simply supported edge condition at $Y = Y_q$ (q =1 or m)

From Eqs. (1) and (63), the quadrature analog of the boundary conditions are obtained as:

$$\{W(Y_q)\} = \{0\}_{n\times1}, \qquad \frac{d^2}{dy^2}\{W(Y_q)\} = \sum_{j=1}^{m} \bar{A}_{qj}^{(2)} \{W(Y_j)\} = \{0\}_{n\times1} \tag{88}$$

(II) Clamped edge condition at $Y = Y_q$ (q =1 or m)

From Eqs. (1) and (65), the quadrature analog of the boundary conditions are obtained as:

$$\{W(Y_q)\} = \{0\}_{n\times1}, \qquad \frac{d}{dy}\{W(Y_q)\} = \sum_{j=1}^{m} \bar{A}_{qj}^{(1)} \{W(Y_j)\} = \{0\}_{n\times1} \tag{89}$$

(III) Free edge condition at $Y = Y_q$ (q =1 or m)

Satisfying Eq. (67) at any grid point $X = X_i$ and substituting the quadrature rule, given in Eq. (1), into results gives

$$\frac{d^3}{dY^3}\{W(Y_q)\} + \frac{2-\mu}{\lambda^2}[A]^{(2)} \frac{d}{dY}\{W(Y_q)\} = \{0\}_{n\times1} \tag{90}$$

$$\frac{d^2}{dY^2}\{W(Y_q)\} + \frac{\mu}{\lambda^2}[A]^{(2)}\{W(Y_q)\} = \{0\}_{n\times1} \tag{91}$$

Now, using the quadrature rule, Eqs. (90) and (91) may be written as:

$$\sum_{j=1}^{m} \bar{A}_{qj}^{3}\{W(Y_j)\} + \frac{2-\mu}{\lambda^2}[A]^{(2)} \sum_{j=1}^{m} \bar{A}_{qj}^{(1)}\{W(Y_j)\} = \{0\}_{n\times1} \tag{92}$$

$$\sum_{j=1}^{m} \bar{A}_{qj}^{2}\{W(Y_j)\} + \frac{\mu}{\lambda^2}[A]^{(2)}\{W(Y_q)\} = \{0\}_{n\times1} \tag{93}$$

which can be used to implement the free-edge boundary conditions of the plate in the Y-direction.

5.3.3 DQM analogs of the free corner boundary condition at (X_p, Y_q)

Consider a rectangular plate having a free corner at (X_p, Y_q). The boundary condition of the rectangular plate, in this case, is given in Eq. (68). Using the quadrature rule, given in Eq. (1), the quadrature analog of the corner boundary condition is obtained as:

$$\frac{\partial^2 W(X_p,Y_q)}{\partial X \partial Y} = [A_p^{(1)}]\frac{d}{dY}\{W(Y_q)\} = [A_p^{(1)}]\sum_{j=1}^{m}\bar{A}_{qj}^{(1)}\{W(Y_j)\} \tag{94}$$

where $[A_p^{(1)}]$ is defined in Eq. (84). Eq. (94) can be used to implement the free-corner boundary conditions of the plate at $(X, Y) = (X_p, Y_q)$

5.4 Implementation of plate boundary conditions

Since the plate problem may be considered as a combination of the two beam problems, the approaches presented in Sections 3.2 and 3.3 can also be used to implement the plate boundary conditions. However, there are some major limitations to the application of CBCGE and MWCM approaches in implementing general boundary conditions of the rectangular plates. These limitations are:

1. It is not possible to modify the DQM weighting coefficient matrices such that the free edge and free corner boundary conditions are implemented. In other words, the application of MWCM approach is limited to the plate problems involving simply supported and clamped edges.
2. As we discussed earlier in Section 3.2, in the CBCGE approach, the quadrature analog equations of the boundary conditions should be eliminated from the discrete equations of the problem. Therefore, it is not possible to implement the free corner boundary condition in the CBCGE approach.

The above-mentioned limitations have also been addressed earlier by some researchers in the field (for instance, see Refs. [4, 22, 25]). As we will show in the numerical experiments, such limitations can be easily tackled using the proposed approach. This is because the proposed approach only eliminates the degrees of freedom correspond to the Dirichlet-type boundary conditions from the final discrete equations of the problem. Therefore, the free edge and free corner boundary conditions of the plate can also be easily implemented using the proposed approach.

5.5 Numerical results

To demonstrate the accuracy and efficiency of the proposed approach, natural frequencies of square plates with different boundary conditions are evaluated and the results are tabulated in Tables 5-8. The numerical results shown in Tables 5-7. are obtained using the Chebyshev grid points (see Eq. (9)) while those given in Table 8. are calculated using the Chebyshev grid points with adjacent δ-points (see Eq. (42)).

The first five dimensionless natural frequencies of square plates with SSSS, SCSC, and CCCC boundary conditions are shown in Table 5. and compared with the analytical solution results of Leissa [39]. The results of the SBCGE and MWCM approaches by Shu and Du [24] are also shown for comparison purposes. Similar to Ref. [24], the numerical results are calculated using 12 grid points in both the *X* and *Y* directions. It can be seen that, in all cases, the solutions of proposed method are very close to those predicted by the SBCGE approach. It can also be seen that he MWCM approach provides better accuracy than the present approach and the SBCGE approach for square plates with SSSS boundary conditions. However, this technique predicts spurious modes with non-zero frequency for square plates with SCSC and CCCC boundary conditions.

As pointed out earlier in introduction, Shu and Du [25] also considered the free vibration problem of rectangular plates with general boundary conditions. They proposed a CBCGE approach for implementing the plate boundary conditions (see Section 3.2.2 for more details). They also showed that the solutions of this approach may be very sensitive to the grid point distribution. For instance, when the Chebyshev-Gauss-Lobatto grid points are used for calculation of weighting coefficients (see Eq. (9)), the CBCGE approach has shown to lead to erroneous results for natural frequencies of rectangular plates involving free corners [25]. In this section, we will show that this difficulty can be solved if the proposed approach will be used to handle the plate boundary conditions.

The numerical results, shown in Table 5., were obtained using an equal number of grid points in both the *X* and *Y* directions ($n = m$). Our numerical experiments for the present problem showed that better accuracy in the DQM solutions can be achieved if an unequal number of grid points is considered in each coordinate direction of the plate.

Table 6. presents the first five natural frequencies of square plates involving free corners. The obtained numerical results are also compared with those obtained by the CBCGE approach [25], new Ritz formulation [28], and the FE-Ritz method [40]. It is noted that in both the present and CBCGE approaches, the Chebyshev-Gauss-Lobatto grid points are used for calculation of weighting coefficients (see Eq. (9)). It is also noted that the results of Refs. [28, 40] are believed to be highly accurate since both the geometrical and natural boundary conditions of the plate are strongly satisfied in the algorithms presented in these references. It can be seen from Table 6. that the results of proposed method have closer agreement with the results of Refs. [28, 40] than the CBCGE approach [25] and, thus, are more accurate. The results of the CBCGE approach [25] are found to be in serious error. This is due to the lack of satisfaction of free corner boundary conditions in the CBCGE approach. It can also be seen from Table 6. that the accuracy and convergence rate of solutions of the proposed method can be significantly improved when *m* value is chosen to be smaller than *n* value. This numerical observation has also been reported by Bert and Malik [41] in solving the free vibration problem of plates with irregular geometries.

To better see the accuracy and efficiency of the proposed method, the percent error in quadrature solutions (defined as $|\Omega_{DQM}-\Omega_{FE\text{-}Ritz\ [40]}|/\ \Omega_{FE\text{-}Ritz\ [40]}\times 100$) is shown in Table 7. The results are also compared with those of the CBCGE approach [25]. Needless to say, the present method is more accurate than the CBCGE approach.

As we discussed earlier in Sections 3.4 and 4.4, the accuracy of numerical results of proposed method can be significantly improved if the adjacent δ-points are introduced on grid points. In this case, the coordinates of the grid points can be obtained from Eq. (42) by replacing the variable η with variables X or Y. When the adjacent δ-points are introduced on grid points, Table 8. presents the first five natural frequencies of square plates with free corners. These results are obtained using $\delta = 10^{-3}$.The results are also compared with those obtained by the FE-Ritz approach [40]. By comparing the results of Table 8. with those of Table 6., it can be seen that the DQM solution with the inclusion of δ-points produces much more accurate natural frequencies. Besides, depending on the boundary conditions of the plate, better accuracy and convergence rate can be achieved if *n* value is chosen to be smaller or larger than *m* value. This behavior is the same for plates involving simply supported edges (i.e., for plates with SSFF, SFFF and FFFF boundary conditions) or for plates involving clamped edges (i.e., for plates with CCFF, CSFF and CFFF boundary conditions).

Table 5
Comparison of natural frequencies of square plates with SSSS, SCSC, and CCCC boundary conditions ($n = m = 12$).

Plate	Method	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
SSSS	Present	19.7392	49.3494	49.3494	78.9586	98.4154
	SBCGE [24]	19.7392	49.3495	49.3495	78.9586	98.4154
	MWCM [24]	19.7392	49.3480	49.3480	78.9568	98.6956
	Exact [39]	19.7392	49.3480	49.3480	78.9568	98.6960
SCSC	Present	28.9509	54.7450	69.3293	94.5898	101.9498
	SBCGE [24]	28.951	54.745	69.329	94.589	101.950
	MWCM [24]	9.870	28.951	39.482	54.743	69.344
	Exact [39]	28.9509	54.7431	69.3270	94.5853	102.2162
CCCC	Present	35.9861	73.3988	73.3988	108.2305	131.4177
	SBCGE [24]	35.986	73.399	73.399	108.230	131.418
	MWCM [24]	0.788	22.381	22.381	35.985	61.368
	Ritz [39]	35.992	73.413	73.413	108.27	131.64

Table 6
Comparison of natural frequencies of square plates with free corners.

Plate	Method	n	m	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
SSFF	Present	15	15	3.4030	17.3164	19.3883	38.3193	51.0394
			14	3.3870	17.3166	19.3495	38.2782	51.0420
			13	3.3696	17.3165	19.3064	38.2309	51.0385
	CBCGE [25]	15	15	2.549	17.316	17.662	36.576	51.039
	New Ritz [28]			3.3670	17.316	19.293	38.211	51.035
	FE-Ritz [40]			3.3670	17.3164	19.2929	38.2112	51.0354
CSFF	Present	15	15	5.9605	19.6747	24.0289	42.7452	52.8101
			14	5.9425	19.6371	24.0352	42.7547	52.7730
			13	5.8700	19.5316	24.1248	42.8147	52.7146

	CBCGE [25]	15	15	5.780	20.703	20.926	40.296	52.255
	New Ritz [28]			5.351	19.075	24.671	43.088	52.707
	FE-Ritz [40]			5.3511	19.0752	24.6705	43.0876	52.7075
CCFF	Present	15	15	7.4435	23.9613	26.4819	47.4852	62.7220
			14	7.8116	23.9452	26.1845	47.1993	62.8072
			13	7.3887	24.0238	26.3397	47.5000	62.7017
	CBCGE [25]	15	15	7.873	23.615	23.873	44.587	62.730
	New Ritz [28]			6.919	23.904	26.585	47.651	62.706
	FE-Ritz [40]			6.9195	23.9040	26.5851	47.6519	62.7063
CFFF	Present	15	15	3.8935	9.6875	20.9346	27.7802	30.0323
			14	3.8641	9.7817	20.9937	27.5841	29.9884
			13	3.9077	9.3734	21.0125	27.5565	30.2556
	CBCGE [25]	15	15	3.898	9.459	20.206	26.150	26.500
	New Ritz [28]			3.4712	8.5074	21.2864	27.1990	30.9590
	FE-Ritz [40]			3.4711	8.5067	21.2850	27.1989	30.9563
SFFF	Present	15	15	6.7053	14.9092	25.5050	26.1287	48.4739
			14	6.7038	14.9042	25.5017	26.0413	48.4540
			13	6.6341	14.9036	25.3839	26.0324	48.4512
	CBCGE [25]	15	15	5.161	14.725	23.082	24.156	46.296
	New Ritz [28]			6.6437	14.9015	25.3757	26.0005	48.4495
	FE-Ritz [40]			6.6437	14.9015	25.3757	26.0005	48.4495
FFFF	Present	15	15	13.6686	19.5962	24.3793	35.0164	35.1960
			14	13.6676	19.5958	24.3039	34.9416	35.0069
			13	13.4671	19.5959	24.2958	34.8298	34.9197
	CBCGE [25]	15	15	10.303	19.596	22.146	30.026	30.803
	New Ritz [28]			13.4682	19.5961	24.2702	34.8009	34.8009
	FE-Ritz [40]			13.4682	19.5961	24.2702	34.8009	34.8009

Table 7
Percent error in solutions of the proposed approach and comparison with that of the CBCGE approach.

Plate	Method	n	m	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Average Error (%)
SSFF	Present	15	15	1.0692	0	0.4945	0.2829	0.0078	0.3709
			13	0.0772	0.0006	0.0700	0.0516	0.0061	0.0411
	CBCGE [25]	15	15	24.2946	0.0023	8.4534	4.2794	0.0071	7.4074
CSFF	Present	15	15	11.3883	3.1428	2.6007	0.7947	0.1947	3.6242
			13	9.6971	2.3926	2.2120	0.6334	0.0135	2.9897
	CBCGE [25]	15	15	8.0152	8.5336	15.1780	6.4789	0.8585	7.8128
CCFF	Present	15	15	7.5728	0.2397	0.3882	0.3498	0.0250	1.7151
			13	6.7808	0.5012	0.9231	0.3188	0.0073	1.7062
	CBCGE [25]	15	15	13.7799	1.2090	10.2016	6.4319	0.0378	6.3320
CFFF	Present	15	15	12.1691	13.8808	1.6462	2.1372	2.9849	6.5636
			13	12.5781	10.1884	1.2802	1.3148	2.2635	5.5250
	CBCGE [25]	15	15	12.2987	11.1947	5.0693	3.8564	14.3955	9.3629
SFFF	Present	15	15	0.9272	0.0517	0.5095	0.4931	0.0504	0.4064
			13	0.1445	0.0141	0.0323	0.1227	0.0035	0.0634
	CBCGE [25]	15	15	22.3174	1.1844	9.0390	7.0941	4.4448	8.8159
FFFF	Present	15	15	1.4879	0.0005	0.4495	0.6192	1.1353	0.7385
			13	0.0082	0.0010	0.1055	0.0830	0.3414	0.1078
	CBCGE [25]	15	15	23.5013	0.0005	8.7523	13.7206	11.4879	11.4925

Table 8
Convergence and comparison of natural frequencies of square plates with free corners, when adjacent δ-points are introduced on grid points.

Plate	Method	$n \times m$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
CSFF	Present	15 × 13	5.4017	19.1158	24.5975	43.0245	52.7025
		17 × 15	5.3921	19.1119	24.6081	43.0248	52.7076
		19 × 17	5.3835	19.1073	24.6192	43.0308	52.7096
	FE-Ritz [40]		5.3511	19.0752	24.6705	43.0876	52.7075
CCFF	Present	15 × 13	6.9945	23.8898	26.5280	47.5824	62.7195
		17 × 15	6.9864	23.8897	26.5381	47.5761	62.7146
		19 × 17	6.9780	23.8917	26.5466	47.5789	62.7117
	FE-Ritz [40]		6.9195	23.9040	26.5851	47.6519	62.7063
CFFF	Present	15 × 13	3.5073	8.5945	21.2331	27.2385	30.8367
		17 × 15	3.4952	8.5859	21.2412	27.2397	30.8427
		19 × 17	3.4864	8.5751	21.2494	27.2375	30.8566
	FE-Ritz [40]		3.4711	8.5067	21.2850	27.1989	30.9563
SSFF	Present	15 × 17	3.3675	17.3163	19.2946	38.2134	51.0352
		17 × 19	3.3671	17.3163	19.2934	38.2119	51.0352
		19 × 21	3.3669	17.3163	19.2928	38.2112	51.0353
	FE-Ritz [40]		3.3670	17.3164	19.2929	38.2112	51.0354
SFFF	Present	15 × 17	6.6444	14.9017	25.3783	26.0024	48.4492
		17 × 19	6.6437	14.9016	25.3765	26.0010	48.4491
		19 × 21	6.6434	14.9016	25.3755	26.0003	48.4491
	FE-Ritz [40]		6.6437	14.9015	25.3757	26.0005	48.4495
FFFF	Present	15 × 17	13.4706	19.5959	24.2726	34.8069	34.8071
		17 × 19	13.4683	19.5960	24.2711	34.8024	34.8034
		19 × 21	13.4672	19.5961	24.2704	34.8003	34.8014
	FE-Ritz [40]		13.4682	19.5961	24.2702	34.8009	34.8009

6 FORMULATION FOR FORCED VIBRATION ANALYSIS OF RECTANGULAR PLATES

6.1 Governing equation

Consider the forced vibration problem of an isotropic thin rectangular plate with length a, width b, mass per unit area ρh , and flexural rigidity D subjected to a dynamic transverse load f (x, y, t). The governing differential equation of motion of the plate is given by

$$\rho h \frac{\partial^2 w}{\partial t^2} + D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = f(x, y, t) \tag{95}$$

where $w(x, y, t)$ is the lateral deflection of the plate, and t is the time.

6.2 Discretization of plate governing equation using the DQM

Consider n grid points with coordinates $x_1, x_2, \ldots, x_n$ in the x-direction, and m grid points with coordinates $y_1, y_2, \ldots, y_m$ in the y-direction. Satisfying Eq. (95) at any grid point $x = x_i$ and substituting the quadrature rule, given in Eq. (1), into results gives

$$\rho h[I]\frac{\partial^2}{\partial t^2}\{w(y,t)\} + D\left([A]^{(4)}\{w(y,t)\} + 2[A]^{(2)}\frac{\partial^2}{\partial y^2}\{w(y,t)\} + [I]\frac{\partial^4}{\partial y^4}\{w(y,t)\}\right) = \{f(y,t)\} \tag{96}$$

where the matrices $[I]$, $[A]^{(2)}$ and $[A]^{(4)}$ are defined in Section 5.2. Furthermore,

$$\{w(y,t)\} = [w(x_1,y,t) \quad w(x_2,y,t) \quad . \quad . \quad . \quad w(x_n,y,t)]^T \tag{97}$$

$$\{f(y,t)\} = [f(x_1,y,t) \quad f(x_2,y,t) \quad . \quad . \quad . \quad f(x_n,y,t)]^T \tag{98}$$

Satisfying Eq. (96) at any grid point $y = y_i$ and substituting the quadrature rule, given in Eq. (1), into results gives

$$[\tilde{\boldsymbol{M}}]\{\ddot{\tilde{\boldsymbol{w}}}(t)\} + [\tilde{\boldsymbol{K}}]\{\tilde{\boldsymbol{w}}(t)\} = \{\tilde{\boldsymbol{f}}(t)\} \tag{99}$$

where the $n \times n$ sub-matrices $[\tilde{\boldsymbol{M}}_{ij}]$ and $[\tilde{\boldsymbol{K}}_{ij}]$ are given by

$$[\tilde{\boldsymbol{M}}_{ij}] = \rho h \bar{I}_{ij}[I], \qquad i, j = 1,2,...,m \tag{100}$$

$$[\tilde{\boldsymbol{K}}_{ij}] = D\left(\bar{I}_{ij}[A]^{(4)} + 2\bar{A}_{ij}^{(2)}[A]^{(2)} + \bar{A}_{ij}^{(4)}[I]\right) \tag{101}$$

where $\bar{I}_{ij}$, $\bar{A}_{ij}^{(2)}$, and $\bar{A}_{ij}^{(4)}$ are defined in Section 5.2. Furthermore,

$$\{\tilde{\boldsymbol{w}}(t)\} = [\{w(y_1,t)\}^T \quad \{w(y_2,t)\}^T \quad . \quad . \quad . \quad \{w(y_m,t)\}^T]^T, \qquad \{\ddot{\tilde{\boldsymbol{w}}}(t)\} = \frac{d^2}{dt^2}\{\tilde{\boldsymbol{w}}(t)\} \tag{102}$$

$$\{\tilde{\boldsymbol{f}}(t)\} = [\{f(y_1,t)\}^T \quad \{f(y_2,t)\}^T \quad . \quad . \quad . \quad \{f(y_m,t)\}^T]^T \tag{103}$$

Eq. (99) can be solved for unknowns if the boundary conditions of the plate are also applied. The procedure for applying the plate boundary conditions is similar to that described in Sections 4.2 and 4.3 for the beam problem. To save the space, this procedure is not detailed here.

6.3 Numerical results

To demonstrate the applicability of the proposed approach for the forced vibration analysis of rectangular plates, numerical experiments are carried out for the dynamic analysis of a simply supported plate subjected to a harmonic concentrated load acting at a point $(x, y) = (x_f, y_f)$. The time-dependent concentrated harmonic load is assumed to be in the form:

$$f(x,y,t) = f_0 \Delta(x - x_f)\Delta(y - y_f)\sin\omega_f t \tag{104}$$

where f_0 and ω_f are the amplitude and frequency of the applied load, respectively, and $\Delta(x)$ or $\Delta(y)$ is the Dirac-delta function. An exact analytic solution for the present test problem can be easily obtained using the modal technique [35, 36]. Therefore, the accuracy of the proposed method can be easily verified by comparing the calculated numerical results with those of analytical ones.

The point discretization methods like the DQM may encounter some difficulties in mathematical modeling and treatment of the singular Dirac-delta function. This is mainly caused by the particular properties of the Dirac-delta function which are in the form of integrals [14]. A way for overcoming such difficulty is to combine the DQM with integral quadrature method [14]. Alternatively, it is possible to regularize the Dirac-delta function with simple mathematical functions in order to achieve a smoother representation of the singular function. In this regard, various forms of the regularized Dirac-delta function have been proposed in the literature (for example, see Wei et al. [42]). Among them, the following form of the regularized Dirac-delta function is used here due to its excellent numerical properties such as simplicity, smoothness and regularity:

$$\Delta_\alpha(\eta) = \frac{1}{\pi}\frac{\alpha}{\eta^2 + \alpha^2}, \qquad \eta = x \qquad or \qquad \eta = y \tag{105}$$

where α is the regularization parameter that defines the relationship between the smoothness and the desired accuracy of the approximation. To achieve a good approximation of the Dirac-delta function using above function, the parameter α should be as small as possible.

Fig.1 shows the effects of parameter α on accuracy and smoothness of the regularized Dirac-delta function. It can be seen that the shape of approximate function approaches to that of the real Dirac-delta function by decreasing the value of the parameter α. In other words,

$$\Delta_\alpha(\eta) \to \Delta(\eta) \quad \text{when} \quad \alpha \to 0 \tag{106}$$

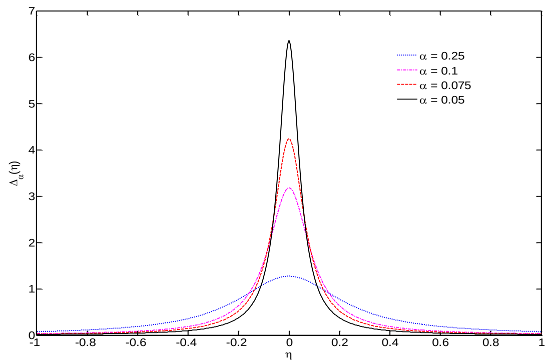

Fig.1
Variations of the regularized Dirac-delta function with η for different values of α.

For the numerical experiment, the following data are used:

$$\frac{\rho h}{D}=1, \quad \frac{f_0}{D}=1, \quad \omega_f = 20, \quad a=b=1, \quad x_f = 1/2, \quad y_f = 1/2 \tag{107}$$

The dynamic response of the simply supported plate subjected to harmonic concentrated load (104) is computed using the proposed approach and using the CBCGE and MWCM approaches. The Newmark method with a time step of Δt = 0.002 s is used to solve the resulting dynamic equations and the regularization parameter α is taken as α = 0.05. The numerical results are obtained using two different grid numbers ($n = m$ =15 and 19), and the coordinates of the grid points are selected as those given in Eq. (42). Fig. 2 presents the results. It can be seen that the numerical results of different DQM methodologies are the same order of accuracy. This also confirms our previous numerical observations for the beam problem (see Table 4).

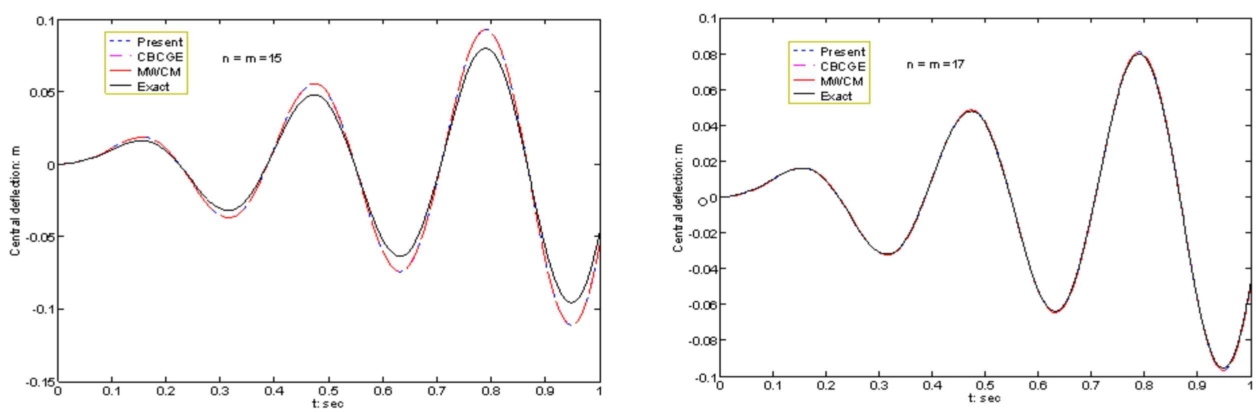

Fig.2
Comparison of central deflection of a simply supported square plate subjected to a concentrated harmonic load.

7 CONCLUSIONS

In this paper, a simple and systematic approach is proposed for imposing boundary conditions in the differential quadrature free and forced vibration analysis of beams and rectangular plates. The formulation of the proposed approach is similar to that of the CBCGE approach. But, unlike the CBCGE approach, the grid points near the boundaries are not treated as boundary points in the proposed approach. In other words, the degrees of freedom related to Dirichlet-type boundary conditions are only eliminated from the original discrete equations. This simplifies significantly the solution procedure and its programming. Numerical results reveal that the present approach is very efficient and reliable. This paper also suggests a simple scheme for handling problems involving the singular Dirac-delta function.

From the numerical results presented herein, the following general conclusions can be made:

1. In general, the MWCM approach can give better accuracy than the proposed approach and the CBCGE approach for beams and rectangular plates. However, it can produce spurious modes with zero frequency for beams with CC and FF boundary conditions, and spurious modes with non-zero frequency for rectangular plates involving clamped edges. This technique also cannot be applied to the plate problems with free edge and free corner boundary conditions.
2. In general, the CBCGE approach can produce satisfactory results for beams and rectangular plates with general boundary conditions. However, it can produce wrong and oscillatory results for rectangular plates involving free corners.
3. In general, the proposed technique is the same order of accuracy of the CBCGE approach. But it is much simpler than the CBCGE approach since the grid points near the boundaries are not treated as the boundary points. Besides, the proposed method can produce much better accuracy than the CBCGE approach for rectangular plates involving free corners.

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