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Surface Effects on Free Vibration Analysis of Nanobeams Using Nonlocal Elasticity: A Comparison Between Euler-Bernoulli and Timoshenko

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ABSTRACT
In this paper, surface effects including surface elasticity, surface stress and surface density, on the free vibration analysis of Euler-Bernoulli and Timoshenko nanobeams are considered using nonlocal elasticity theory. To this end, the balance conditions between nanobeam bulk and its surfaces are considered to be satisfied assuming a linear variation for the component of the normal stress through the nanobeam thickness. The governing equations are obtained and solved for Silicon and Aluminum nanobeams with three different boundary conditions, i.e. Simply-Simply, Clamped-Simply and Clamped-Clamped. The results show that the influence of the surface effects on the natural frequencies of the Aluminum nanobeams follows the order CC<CS<SS while this is not the case for Silicon nanobeams. On the other hand, the influence of the nonlocal parameter is opposite and follows the order SS<CS<CC. In addition, it is seen that considering rotary inertia and shear deformation has more effect on the surface effects than the nonlocal parameter.

Keywords: Surface effects; Nonlocal elasticity; Free vibration; Nanobeam; Euler-Bernoulli theory; Timoshenko theory

1 INTRODUCTION

IN order to study the mechanical behaviors of nanostructures, the surface effects and nonlocal elasticity are two important fields which are investigated by researchers separately, or simultaneously.

The surface of a solid is a region with small thickness which has different properties from the bulk. If the surface energy-to-bulk energy ratio is large, for example in the case of nanostructures, the surface effects cannot be ignored [1]. On the other hand, the nonlocal elasticity theory which is initiated in the papers of Eringen and Eringen and Edelen [2-5] expresses that the stress at a point is a function of strains at all points in the continuum.

To account for the effect of surfaces/interfaces on mechanical deformation, the surface elasticity theory is presented by modeling the surface as a two dimensional membrane adhering to the underlying bulk material without slipping [6, 7]. There are many studies related to the wave propagation, static, buckling and free linear and nonlinear vibration analysis of nanobeams and carbon nanotubes (CNTs) based on different beam theories [8-20]. For example, Wang et al. [9] studied the surface buckling of a bending beam using the surface elasticity theory. The corresponding buckling wave number was analytically obtained in their work. They also reported that surfaces with

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positive surface elastic modulus may buckle under compression, while surfaces with negative surface elastic modulus are possible to wrinkle irrespective of the sign of surface strain.

The nonlocal continuum mechanics is suitable for modeling submicro- or nano-sized structures because it avoids enormous computational efforts when compared with discrete atomistic or molecular dynamics simulations. Many researchers have applied the nonlocal elasticity concept for the wave propagation [21,22], bending, buckling, and vibration [23-30] analyses of beam-like elements in micro- or nano electromechanical systems. For example, Reddy [27] applied the Eringen nonlocal elastic constitutive relations to derive the equation of motion of various kinds of beam theories (i.e., Euler–Bernoulli, Timoshenko, Reddy and Levinson) and proposed analytical and numerical solutions on static deflections, buckling loads, and natural frequencies of nano-beams. It can also be mentioned to the Refs. [31-34] which they used the nonlocal elasticity theory for analysis of the plate-like structures.

From literature, it can be found that most of the works considered the influences of the surface effects and the nonlocal effect separately. There are a few papers in which both surface and small scale effects on static and dynamic behaviors of nanostructures and CNTs are taken into account. Lee and Chang [35] studied the surface and small-scale effects on the free vibration analysis of a non-uniform nano-cantilever Euler-Bernoulli beam. Wang and Wang [36] considered the influences of the surface effects on the free vibration behaviors of simply supported Kirchhoff and Mindlin nanoscale plates using the nonlocal elasticity theory. Lei et al. [37] investigated the surface effects on the vibrational frequency of double-walled carbon nanotubes using the nonlocal Timoshenko beam model. The influence of the surface and small-scale effects on electromechanical coupling behavior of piezoelectric nanowires is also studied in Refs. [38, 39].

It is reported in literature that the surface effects include the surface elasticity, the surface stress and the surface density. In addition, some literatures introduce relations for satisfying the balance condition between the nano-structure bulk and its surfaces. Most of the studies examine only the surface elasticity and stress effects and there is no work focusing on the influences of the surface density and satisfying the balance condition as well as the surface elasticity and stress effects. For example, Gheshlaghi and Hasheminejad [12] considered the surface elasticity and stress on the nonlinear free vibration of simply supported Euler-Bernoulli nanobeams without satisfying the balance condition. The similar situation can be found in Ref.[40] that the nonlinear free vibration of non-uniform nanobeams in the presence of the nonlocal effect as well as the surface elasticity and stress effects is studied using differential quadrature method (DQM). The other similar work is the one done by Mahmood et al. [41]. In this work ,the static behavior of nonlocal Euler-Bernoulli nanobeams is considered using finite element approach.

In the present work, for considering the surface and small scale effects on the free vibration analysis of nanobeams with different end conditions in details, a comprehensive analytical model proposes to study all the surface effects, including the surface elasticity, the surface tension and the surface density, on the free vibration of nanoscale Euler-Bernoulli and Timoshenko beams, made from aluminum and silicon, using nonlocal elasticity. The surface density is introduced into the governing equations assuming a linear variation through the nanobeam thickness for the component of normal stress, σ_{zz}. This also satisfies the balance conditions between the nanobeam bulk and its surfaces. An exact solution is used to obtain the natural frequencies of nanobeams. Lastly, the surface and small scale effects on the natural frequencies of nanobeams are examined for different boundary conditions, nanobeam lengths and mode numbers. It is found out that considering the surface density and satisfying the balance conditions between the nanobeam bulk and its surfaces have considerable influences on the natural frequencies of nanobeams.

2 PROBLEM FORMULATIONS

To obtain the governing equations of nanobeams in the presence of the surface effects via nonlocal elasticity, a nanobeam with length L ($0 \le x \le L$), width b ($-0.5b \le y \le 0.5b$) and height H=2h ($-h \le z \le h$) is considered.

2.1 Nonlocal elasticity theory and surface effects
2.1.1 Nonlocal elasticity theory

Nonlocal elasticity is first considered by Eringen [2-5] assumes that the stress field at a point **x** in an elastic continuum not only depends on the strain field at the point but also on strains at all other points of the body. For homogeneous and isotropic elastic solids, the constitutive relations can be expressed in a differential form as:

$$(1-\tau^2 L^2 \nabla^2)\sigma_{ij} = C_{ijmn}\varepsilon_{mn} \tag{1}$$

where $\tau = e_0 a / L$ is a material constant that depends on internal (*a*) and external (*L*) characteristic length, e_0 is a material constant, ∇^2 is the Laplacian operator and *C* is the fourth-order elasticity tensor. The nonlocal constitutive relations in Eq.(1), for the macroscopic stress take the following special relations for nanobeams [27]

$$\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E\varepsilon_{xx} \quad , \quad \sigma_{xz} - \mu \frac{\partial^2 \sigma_{xz}}{\partial x^2} = 2G\varepsilon_{xz} \, (\mu = e_0^2 a^2) \tag{2}$$

2.1.2 Surface effects

At the micro/nanoscale, the fraction of energy stored in the surfaces becomes comparable with that in the bulk, because of the relatively high ratio of surface area to volume of nanoscale structures; therefore the surface and the induced surface forces cannot be ignored. The constitutive relations of the surface layers S^+ and S^- given by Gurtin and Murdoch [6, 42] can be expressed as:

$$\tau_{\alpha\beta}^{\pm} = \tau_0^{\pm}\delta_{\alpha\beta} + \left(\mu_0^{\pm} - \tau_0^{\pm}\right)\left(u_{\alpha,\beta}^{\pm} + u_{\beta,\alpha}^{\pm}\right) + \left(\lambda_0^{\pm} + \tau_0^{\pm}\right)u_{\gamma,\gamma}^{\pm}\delta_{\alpha\beta} + \tau_0^{\pm}u_{\alpha,\beta}^{\pm} \quad , \quad \tau_{\alpha 3}^{\pm} = \tau_0^{\pm}u_{3,\alpha}^{\pm} \tag{3}$$

where $\alpha, \beta, \gamma = 1, 2$, $\tau_0^{\pm}$ are residual surface tensions under unconstrained conditions, $\lambda_0^{\pm}$ and $\mu_0^{\pm}$ are the surface Lame constants on the surfaces S^+ and S^- which can be determined from atomistic calculations [43], $\delta_{\alpha\beta}$ the Kronecker delta and $u_\alpha^{\pm}$ are the displacement components of the surfaces S^+ and S^-. If the top and bottom layers have the same material properties, the stress–strain relations of the surface layers, i.e. Eq.(3), can be reduced to the following relation for nanobeams

$$\tau_{xx} = \tau_0 + E^S u_{x,x} \quad , \quad E^S = 2\mu_0 + \lambda_0, \; \tau_{nx} = \tau_0 u_{n,x} \tag{4}$$

where n denotes the outward unit normal. The equilibrium relations for the surface layer can be expressed in terms of the surface and bulk stress components as [44]

$$\tau_{i\alpha,\alpha} - T_i = \rho_0 \ddot{u}_i^s \tag{5}$$

where $i = x, n, m; \alpha = x, m$; ρ_0 denotes the density of surface layer; T is the contact tractions on the contact surface between the bulk material and the surface layer; *m* is the tangent unit vector; and $\ddot{u}_i^s$ denotes the acceleration of the surface layer in the *i*- direction.

In classical beam theory, the stress component σ_{zz} is neglected. However, σ_{zz} must be considered to satisfy the surface equilibrium equations of the Gurtin–Murdoch model. Following Lu et al. [45], it is assumed that the bulk stress σ_{zz} varies linearly through the nanobeam thickness. Therefore,

$$\sigma_{zz} = \frac{1}{2}\left(\sigma_{zz}^{+} + \sigma_{zz}^{-}\right) + \frac{z}{H}\left(\sigma_{zz}^{+} - \sigma_{zz}^{-}\right) \tag{6}$$

Substituting of Eq. (4) and Eq. (5) into Eq. (6) yields

$$\begin{aligned} \sigma_{zz} &= \frac{1}{2}\left(\tau_{nx,x}^{+} + \tau_{nx,x}^{-} - \rho_0\ddot{u}_n^{+} - \rho_0\ddot{u}_n^{-}\right) + \frac{z}{H}\left(\tau_{nx,x}^{+} - \tau_{nx,x}^{-} - \rho_0\ddot{u}_n^{+} + \rho_0\ddot{u}_n^{-}\right) \\ &= \frac{1}{2}\left(\tau_0 u_{z,xx}^{+} - \tau_0 u_{z,xx}^{-} - \rho_0\ddot{u}_z^{+} + \rho_0\ddot{u}_z^{-}\right) + \frac{z}{H}\left(\tau_0 u_{z,xx}^{+} + \tau_0 u_{z,xx}^{-} - \rho_0\ddot{u}_z^{+} - \rho_0\ddot{u}_z^{-}\right) \end{aligned} \tag{7}$$

2.2 Governing equations for nanobeams
2.2.1 Timoshenko beam theory

The bending moment and vertical force equilibrium equations including rotary inertia, shear deformation and the surface effects can be expressed as follow [15]

$$\frac{dM}{dx}+\int_{s}\tau_{xx,x}zds-Q=\int_{A}\rho\ddot{u}_{x}zdA+\int_{s}\rho_{0}\ddot{u}_{x}^{s}zds \tag{8}$$

$$\frac{dQ}{dx}+\int_{s}\tau_{nx,x}n_{z}ds-q(x)=\int_{A}\rho\ddot{u}_{z}dA+\int_{s}\rho_{0}\ddot{u}_{n}^{s}n_{z}ds \tag{9}$$

where τ_{xx} and τ_{nx} are nonzero membrane stresses due to surface energy; q(x) is the transverse distributed force; *Q* and *M* are the stress resultants defined as follow:

$$Q=\int_{A}\sigma_{xz}dA\ M=\int_{A}\sigma_{xx}zdA \tag{10}$$

Bulk stress–strain relations of the nanobeam can be expressed as [15]

$$\sigma_{xx}=E\varepsilon_{xx}+v\sigma_{zz}\ ;\ \ \sigma_{xz}=2G\varepsilon_{xz} \tag{11}$$

where *E* is the elastic modulus, *v* is Poisson's ratio and *G* is the shear modulus. Defining the displacement fields as Timoshenko beam theory [15]

$$u_{x}=z\phi(x,t)\ ;\ \ u_{z}=w(x,t) \tag{12}$$

where $\phi(\mathrm{x},\mathrm{t})$ and w(x, t) denote the rotation of cross section and vertical displacement of mid-plane at time *t*, respectively. So, the nonzero strains are given by

$$\varepsilon_{xx}=\frac{\partial u_{x}}{\partial x}=z\frac{\partial\phi(x,t)}{\partial x}\quad ;\quad \varepsilon_{xz}=\frac{1}{2}(\frac{\partial u_{x}}{\partial z}+\frac{\partial u_{z}}{\partial x})=\frac{1}{2}(\phi(x,t)+\frac{\partial w(x,t)}{\partial x}) \tag{13}$$

Substituting Eq. (12) and Eq. (13) into Eq. (4) and Eq. (7), then substituting the results into Eq. (11), the constitutive relations are obtained as follow

$$\begin{aligned}&\sigma_{xx}=E(z\frac{\partial\phi}{\partial x})+\frac{2\upsilon z}{H}(\tau_{0}\frac{\partial^{2}w}{\partial x^{2}}-\rho_{0}\ddot{w})\\&\sigma_{zz}=\frac{2z}{H}(\tau_{0}\frac{\partial^{2}w}{\partial x^{2}}-\rho_{0}\ddot{w})\quad ;\quad \sigma_{xz}=Gk(\frac{\partial w}{\partial x}+\phi)\\&\tau_{xx}=\tau_{0}+z(2\mu_{0}+\lambda_{0})\frac{\partial\phi}{\partial x}\quad ;\quad \tau_{nx}=\tau_{0}\frac{\partial w}{\partial x}n_{z}\quad ;\quad \tau_{nx}^{\pm}=\pm\tau_{0}\frac{\partial w}{\partial x}\end{aligned} \tag{14}$$

where *k* denotes shear correction coefficient. Substituting Eq. (14) into Eq. (2) and considering Eq.(10), the nonlocal constitutive relations can be written as:

$$\left(1-\mu\frac{\partial^{2}}{\partial x^{2}}\right)\left[M+\int_{s}\tau_{xx}zds\right]=(EI+E^{s}I^{*})\frac{\partial\phi}{\partial x}+\frac{2\upsilon\tau_{0}I}{H}\frac{\partial^{2}w}{\partial x^{2}}-\frac{2\upsilon\rho_{0}I}{H}\frac{\partial^{2}w}{\partial t^{2}} \tag{15}$$

$$\left(1-\mu\frac{\partial^{2}}{\partial x^{2}}\right)\left[Q+\int_{s}\tau_{nx}n_{z}ds\right]=kGA(\phi+\frac{\partial w}{\partial x})+2b\tau_{0}\frac{\partial w}{\partial x} \tag{16}$$

where, we have $I^*=2bh^2+4h^3/3$ for rectangular cross section. Substituting Eq. (15) and Eq. (16) in Eq. (8) and Eq. (9) yields the nonlocal governing equations for Timoshenko nanobeam in presence of the surface effects as follow

$$\left(1-\mu\frac{\partial^2}{\partial x^2}\right)\left[(\rho A+2b\rho_0)\frac{\partial^2 w}{\partial t^2}\right]=kGA(\frac{\partial\phi}{\partial x}+\frac{\partial^2 w}{\partial x^2})+2b\tau_0\frac{\partial^2 w}{\partial x^2} \tag{17}$$

$$\left(1-\mu\frac{\partial^2}{\partial x^2}\right)\left[\left(\rho I+\rho_0 I^*\right)\frac{\partial^2\phi}{\partial t^2}\right]+kGA(\phi+\frac{\partial w}{\partial x})=(EI+E^s I^*)\frac{\partial^2\phi}{\partial x^2}+\frac{2\nu\tau_0 I}{H}\frac{\partial^3 w}{\partial x^3}-\frac{2\nu\rho_0 I}{H}\frac{\partial^3 w}{\partial x\partial t^2} \tag{18}$$

It can be seen that Eqs. (17) and (18) can be reduced to the governing equation of Timoshenko nanobeam with only the nonlocal effect by setting the surface effects equal zero (i.e. $E^s=\tau_0=\rho_0=0$); and in the case of $\mu=0$, Eqs. (17) and (18) will be reduced to the governing equations of Timoshenko nanobeam with only the surface effects. Finally, the resultant moment and shear force of a nanobeam cross section, including the surface layer contributions and the nonlocal parameter, are given by

$$M^T=\left(EI+E^s I^*\right)\frac{\partial\phi}{\partial x}+\frac{2\upsilon\tau_0 I}{H}\frac{\partial^2 w}{\partial x^2}-\frac{2\upsilon\rho_0 I}{H}\frac{\partial^2 w}{\partial t^2}+\mu\left[\left(\rho I+\rho_0 I^*\right)\frac{\partial^3\phi}{\partial x\partial t^2}+\left(\rho A+2b\rho_0\right)\frac{\partial^2 w}{\partial t^2}-2b\tau_0\frac{\partial^2 w}{\partial x^2}\right] \tag{19}$$

$$Q^T=kGA(\phi+\frac{\partial w}{\partial x})+2b\tau_0\frac{\partial w}{\partial x}+\mu(\rho A+2b\rho_0)\frac{\partial^3 w}{\partial x\partial t^2} \tag{20}$$

2.2.2 Euler-Bernoulli beam theory

In the case of Euler–Bernoulli beam theory, the rotational displacement of the cross section is related to the slope of the vertical deflection and the shear deformation effect is ignored, i.e. $\phi=-\partial w/\partial x$ [46, 47]. In addition, the rotational inertia effects are neglected. Thus, the governing equation of nonlocal Euler–Bernoulli nanobeam in the presence of the surface effects can be obtained from Eq. (17) and Eq. (18) in terms of the transverse deflection as:

$$\left(EI+E^s I^*-\frac{2\upsilon\tau_0 I}{H}\right)\frac{\partial^4 w}{\partial x^4}-2b\tau_0\frac{\partial^2 w}{\partial x^2}+\frac{2\upsilon\rho_0 I}{H}\frac{\partial^4 w}{\partial x^2\partial t^2}=\left(1-\mu\frac{\partial^2}{\partial x^2}\right)\left[-(\rho A+2b\rho_0)\frac{\partial^2 w}{\partial t^2}\right] \tag{21}$$

where the M^E and Q^E are given by

$$M^E=-\left(EI+E^s I^*-\frac{2\nu\tau_0 I}{H}\right)\frac{\partial^2 w}{\partial x^2}-\frac{2\nu\rho_0 I}{H}\frac{\partial^2 w}{\partial t^2}+\mu\left[(\rho A+2b\rho_0)\frac{\partial^2 w}{\partial t^2}-2b\tau_0\frac{\partial^2 w}{\partial x^2}\right] \tag{22}$$

$$Q^E=-\left(EI+E^s I^*-\frac{2\nu\tau_0 I}{H}\right)\frac{\partial^3 w}{\partial x^3}-\frac{2\nu\rho_0 I}{H}\frac{\partial^3 w}{\partial x\partial t^2}+2b\tau_0\frac{\partial w}{\partial x}+\mu(\rho A+2b\rho_0)\frac{\partial^3 w}{\partial x\partial t^2} \tag{23}$$

According to Eqs. (19), (20), (22) and (23), it can be expected that the nonlocal characteristic equations of nanobeams in presence of the surface effects will be different from the classic ones which are stated in Ref. [46].

3 SOLUTIONS FOR FREE VIBRATION OF NANOBEAMS

In this section, the approach of solving governing equations of nanobeams is presented for three different boundary conditions, simply-simply (SS), clamped-simply (CS) and clamped-clamped (CC).

3.1 Free vibration of timoshenko nanobeams

In order to solve the nonlocal governing equations of Timoshenko nanobeam in the presence of the surface effects, we consider $\phi(\mathrm{x},\mathrm{t})$ and $w(\mathrm{x}, \mathrm{t})$ as follow:

$$\begin{cases} w(x,t) = W(x)e^{i\lambda t} \\ \phi(x,t) = \Phi(x)e^{i\lambda t} \end{cases} \tag{24}$$

where, λ denotes the vibration frequency. Considering Eq.(24), the solution of Eq. (17) and Eq. (18) can be expressed as:

$$W(x) = c_1 \sin(\xi_1 x) + c_2 \cos(\xi_1 x) + c_3 \sinh(\xi_2 x) + c_4 \cosh(\xi_2 x) \tag{25}$$

$$\Phi(x) = c_5 \sin(\xi_1 x) + c_6 \cos(\xi_1 x) + c_7 \sinh(\xi_2 x) + c_8 \cosh(\xi_2 x) \tag{26}$$

where

$$\eta_1 = R/Q,\ \eta_2 = k_1 k_6/Q\ ,\ \ \xi_1 = \left(\frac{\eta_1 + \sqrt{\eta_1^2 - 4\eta_2}}{2}\right)^{\frac{1}{2}}\ ,\ \ \xi_2 = \left(\frac{-\eta_1 + \sqrt{\eta_1^2 - 4\eta_2}}{2}\right)^{\frac{1}{2}} \tag{27}$$

and

$$\begin{gathered} \mathrm{Q} = k_3 k_5 - k_2 k_7\ ,\ R = k_4 k_5 + k_2 k_6 - k_1 k_7 - k_5^2\ ,\ k_1 = -(\rho I + \rho_0 I^*)\lambda^2 + kGA \\ k_2 = \mu\lambda^2(\rho I + \rho_0 I^*) - (EI + E^s I^*)\ ,\ k_3 = 2\nu\tau_0 I/H\ ,\ k_4 = 2\upsilon\rho_0 I/H\,\lambda^2, k_5 = kGA \\ k_6 = (\rho A + 2b\rho_0)\lambda^2\ ,\ k_7 = \mu(\rho A + 2b\rho_0)\lambda^2 - kGA - 2b\tau_0 \end{gathered} \tag{28}$$

Substituting Eq. (25) and Eq. (26) into Eq. (18) gives

$$c_5 = k_a c_2 \quad , \quad c_6 = -k_a c_1 \quad , \quad c_7 = k_b c_4 \quad , \quad c_8 = k_b c_3 \tag{29}$$

where

$$k_a = \frac{(k_5 - k_4)\xi_1 + k_3 \xi_1^3}{k_1 - k_2 \xi_1^2},\ k_b = \frac{k_3 \xi_2^3 - (k_5 - k_4)\xi_2}{k_1 + k_2 \xi_2^2} \tag{30}$$

The boundary conditions of SS, CS and CC nanobeams are given, respectively, by

$$\begin{gathered} W(0) = \mathrm{M}(0) = 0;\ W(L) = \mathrm{M}(L) = 0 \\ W(0) = \Phi(0) = 0;\ W(L) = \mathrm{M}(L) = 0 \\ W(0) = \Phi(0) = 0;\ W(L) = \Phi(L) = 0 \end{gathered} \tag{31}$$

Considering mentioned boundary conditions, the characteristic equations of SS, CS and CC nanobeams are obtained, respectively, as:

$$\begin{gathered} \sin(\xi_1 L)\sinh(\xi_2 L) = 0 \\ \sin(\xi_1 L)\cosh(\xi_2 L) + \frac{k_a}{k_b}\cos(\xi_1 L)\sinh(\xi_2 L) = 0 \\ (k_a^2 - k_b^2)\sin(\xi_1 L)\sinh(\xi_2 L) + 2k_a k_b\left(1 - \cos(\xi_1 L)\cosh(\xi_2 L)\right) = 0 \end{gathered} \tag{32}$$

3.2 Free vibration of euler-bernoulli nanobeams

In order to solve the nonlocal governing equation of Euler-Bernoulli nanobeam in the presence of the surface effects, i.e. Eq.(21), it is assumed that

$$\mathrm{w}(\mathrm{x},\mathrm{t}) = W(x)e^{i\lambda t} \tag{33}$$

Considering Eq.(33), the solution of Eq. (21) can be expressed as:

$$W(x) = c_1 \sin(\xi_1 x) + c_2 \cos(\xi_1 x) + c_3 \sinh(\xi_2 x) + c_4 \cosh(\xi_2 x) \tag{34}$$

where

$$\eta_1 = R/Q,\ \eta_2 = \frac{\lambda^2(\rho A + 2b\rho_0)}{Q},\ \xi_1 = \left(\frac{-\eta_1 + \sqrt{\eta_1^{\,2} + 4\eta_2}}{2}\right)^{\frac{1}{2}},\ \xi_2 = \left(\frac{\eta_1 + \sqrt{\eta_1^{\,2} + 4\eta_2}}{2}\right)^{\frac{1}{2}} \tag{35}$$

and

$$\begin{aligned} Q &= (EI + E^s I^*) - \frac{2\nu\tau_0 I}{H} \\ R &= \frac{2\upsilon\rho_0 I}{H}\lambda^2 + 2b\tau_0 - \mu\lambda^2(\rho A + 2b\rho_0) \end{aligned} \tag{36}$$

According to the mentioned boundary conditions, Eq. (31), the characteristic equations of Euler–Bernoulli nanobeams with SS, CS and CC end conditions are obtained, respectively, as:

$$\begin{aligned} &\mathrm{Sin}(\xi_1 L) = 0 \\ &-(\xi_1^{\,2} + \xi_2^{\,2})\mathrm{Sin}(\xi_1 L)\mathrm{Cosh}(\xi_2 L) + \left(\frac{\xi_1^{\,3}}{\xi_2} + \xi_1\xi_2\right)\mathrm{Sinh}(\xi_2 L)\mathrm{Cos}(\xi_1 L) = 0 \\ &2\xi_1 + \left(\xi_2 - \frac{\xi_1^{\,2}}{\xi_2}\right)\mathrm{Sin}(\xi_1 L)\mathrm{Sinh}(\xi_2 L) - 2\xi_1 \mathrm{Cos}(\xi_1 L)\mathrm{Cosh}(\xi_2 L) = 0 \end{aligned} \tag{37}$$

As previously mentioned, it is seen that the nonlocal characteristic equations of nanobeams in the presence of the surface effects are different from the classic ones, except for SS boundary condition.

Now, the only unknown value in the characteristic equations is the natural frequency (i.e., λ).Using a computer code written in mathematica software, the characteristic equations are solved and the natural frequencies are obtained.

4 NUMERICAL RESULTS AND DISSCUSSION

In order to investigate the influences of the surface effects and the nonlocal parameter on the free vibration of Euler-Bernoulli and Timoshenko nanobeams, in this section the numerical results of nanobeams made of Aluminum (Al) and Silicon (Si) are presented. The relevant bulk material properties are, E = 70GPa, ν = 0.3 and ρ =2700Kg/m^3 for Al [48] and E = 210GPa, ν =0.24 and ρ =2370 Kg/m^3 for Si [49]. Furthermore, the surface material properties are E^s=5.1882N/m , τ_0 = 0.9108 N/m and ρ_0 =5.46×10^{-7}kg/m^2 for Al [50]; and E^s=-10.6543N/m , τ_0= 0.6048 N/m, and ρ_0 =3.17×10^{-7}kg/m^2 for Si [50]. It should be noted that the value of the nonlocal parameter is set to be 4 nm^2 for the cases considering the nonlocal elasticity.

In the following discussion, NSF, SF, NF and CF stand for the natural frequency of the Euler-Bernoulli nanobeam with considering both of the surface effects and the nonlocal parameter, the natural frequency with only the surface effects, the natural frequency with only the nonlocal parameter and the classical natural frequency of the Euler-Bernoulli nanobeam, respectively. Also, T indicates the relevant results for Timoshenko nanobeams.

4.1 Comparison study

To confirm the reliability of the present formulation and results, comparison studies are conducted for the natural frequencies of the Euler-Bernoulli (EBT) and Timoshenko (TBT) nanobeams. Firstly, the accuracy of the nonlocal natural frequencies is investigated [27,40], then the validity of the natural frequencies of nanobeam with considering the surface effects is studied [15]. In Tables 1 and 2 , fundamental non-dimensional natural frequencies, $\lambda L^2\sqrt{\rho A/EI}$ and $(\lambda L^2\sqrt{\rho A/EI})^{1/2}$, of nonlocal nanobeams (without considering the surface effects) are listed. In addition, the results given by Reddy et al. [27] (Table 1.) and Malekzadeh and shojaee [40] (Table 2.) are provided for direct comparison. It is observed that the present results agree very well with those given by Refs. [27, 40].

Table 1

Comparison of non-dimensional fundamental natural frequencies ($\lambda L^2\sqrt{\rho A/EI}$) of Euler-Bernoulli (EBT) and Timoshenko (TBT) nanobeams (L=10, E = 30×10^6, v = 0.3, ρ =1, E^s= τ_0= ρ_0 =0)

Beam type	Thickness ratio 2h/L	Nonlocal parameter μ	Simply-Simply Present	Simply-Simply Ref.[27]	Clamped-Clamped Present	Clamped-Clamped Ref.[27]
EBT	0.01	0	9.86960	9.8696	22.37329	-
		2	9.01948	9.0195	20.03280	-
		4	8.35692	8.3569	18.28942	-
	0.05	0	9.86960	9.8696	22.37328	-
		2	9.01948	9.0195	20.03280	-
		4	8.35692	8.3569	18.28942	-
TBT	0.01	0	9.86793	9.8683	22.35776	-
		2	9.01795	9.0183	20.01929	-
		4	8.35550	8.3558	18.27725	-
	0.05	0	9.82812	9.8381	21.99526	-
		2	8.98158	8.9907	19.70368	-
		4	8.32180	8.3302	17.99314	-

Table 2

Comparison of non-dimensional fundamental natural frequencies $(\lambda L^2\sqrt{\rho A/EI})^{1/2}$ of Euler-Bernoulli (EBT) and Timoshenko (TBT) nanobeams (L=10, 2h=0.01L, E = 30×10^6, v = 0.3, ρ =1, μ=1, E^s= τ_0= ρ_0 =0)

Beam type	Simply-Simply EBT	Simply-Simply TBT	Clamped-Simply EBT	Clamped-Simply TBT	Clamped-Clamped EBT	Clamped-Clamped TBT
Present	3.06853	3.06207	3.82088	3.82014	4.59446	4.59289
Numerical[40]	3.0685	3.0683	3.8209	3.8201	4.5945	4.5929

Table 3
Comparison of the first four natural frequencies of the SS and CC Euler-Bernoulli (Thin nanobeam with L=120 nm) and Timoshenko (Thick nanobeam with L=50 nm) nanobeams incorporating the surface effects when, μ=0 and b=h=3 nm.

Beam type		1st(GHz) Present	1st(GHz) Ref.[15]	2nd(GHz) Present	2nd(GHz) Ref. [15]	3rd(GHz) Present	3rd(GHz) Ref. [15]	4th(GHz) Present	4th(GHz) Ref. [15]
SS	Thin	1.6616	1.66	5.1872	5.19	10.9565	10.96	19.0148	19.02
	Thick	7.0809	7.08	25.0636	25.07	51.3283	51.33	82.9234	82.92
CC	Thin	2.9283	2.94	7.6366	7.64	14.5885	14.59	23.8182	23.83
	Thick	14.1948	14.20	35.3994	35.41	62.7245	62.74	93.8411	93.86

Also, a comparative study for evaluation of the first four natural frequencies between the present solution, without considering the nonlocal parameter (i.e., μ=0), and the results given by Liu and Rajapakse [15] is carried out in Table 3 for Euler-Bernoulli (thin) and Timoshenko (thick) nanobeams made of silicon with SS and CC boundary conditions. The results given in Table 3 are obtained in the presence of the surface effects. Table 3 confirms the reliability of the present formulation and results.

4.2 Benchmark results

Firstly, in order to investigate the influences of considering the surface density and satisfying the balance condition between the nanobeam bulk and its surfaces, the fundamental natural frequencies of Al nanobeam, normalized with respect to the fundamental natural frequency without the surface and small scale effects, i.e. CF and TCF, versus the nanobeam length are presented in Figs. 2 (a) and (b) for two different cases. In Case 1, the results are obtained without considering the surface density and satisfying balance condition while in Case 2, the surface density is considered and the balance condition is satisfied. In Fig. 2 (a), only the surface effects are considered whereas in Fig. 2(b) the nonlocal parameter as well as the surface effects is considered. Also, $b=h=0.1L$. Comparison between the Case 1 curves with those of Case 2 indicates that there are significant differences between the results of two cases, especially in low lengths. It can be seen that all of the Case 2 curves are located bellow the Case 1 curves, implying that considering the surface density and satisfying the balance condition lead to decreasing the natural frequency of nanobeam. In other words, considering the mentioned parameters makes the nanobeam more flexible. The behavior is observed in both of Figs. 2(a) and (b).

Next, the surface and small scale effects on the natural frequencies are examined in Figs. 3 and 4 for various values of the nanobeam length. Figs.3 (a-d) show variations of frequency ratio of Al nanobeam versus the nanobeam length. In Figs. 3(a) and (b), the influences of the surface effects and the nonlocal parameter on the natural frequencies are separately studied while in Figs. 3(c) and (d), the influences of the surface effects and the nonlocal parameter are simultaneously investigated. It should be noted here that in Figs. 3(a-c), the nanobeam cross section is considered to be dependent on the nanobeam length, $b=h=0.1L$, whereas in Fig. 3(d), the nanobeam cross section is considered to be constant, $b=h=2.5$ nm.

It is seen from Fig. 3(a) that considering the surface effects increases the natural frequencies of Euler-Bernoulli and Timoshenko nanobeams made of Al while Fig. 3(b) illustrates that the nonlocal parameter shows a decreasing effect on the natural frequencies. In addition, Fig. 3(a) shows that, for both nanobeam types, the influences of the surface effects on the natural frequency of Al nanobeam follows the order CC<CS<SS, implying that using the softer boundary conditions causes an increase in the influence of the surface effects. However, the situation is reverse when only the nonlocal effect is considered. In this case, Fig. 3(b)displays that the influences of the nonlocal parameter for nanobeams with stiffer boundary conditions are more pronounced and follow the order SS<CS<CC.

Comparison between Euler-Bernoulli and Timoshenko frequency ratio curves in Figs. 3(a) and (b) shows that the increasing influence of the surface effect and the decreasing influence of the nonlocal parameter are more pronounced for Euler-Bernoulli nanobeams than Timoshenko ones, indicating that regarding shear deformation and rotary inertia cause a reduction in the influences of the surface effects and the nonlocal parameter. Moreover, these differences are more considerable for nanobeams with stiffer boundary conditions.

Fig. 3(c) illustrates that the effects of the nonlocal parameter become dominant for shorter nanobeams with stiffer end condition while it is not the case for the influences of the surface effects. In other words, the influences of the surface effects become dominant for nanobeams with softer boundary conditions and for all values of the nanobeam length.

As seen from Figs. 3(a-c), by increasing the nanobeam length all frequency ratio curves gradually approach the classic solution line which entails an expected decrease in the influences of the surface and small scale effects. However, in Fig. 3(d), increasing the nanobeam length increases the surface and small scale effects. This is due to this fact that since in Figs. 3(a-c) the cross section sizes depend on the length of the nanobeam, i.e. $b=h=0.1L$, the surface energy-to-bulk energy ratio decreases by increasing the length of nanobeam and the internal length scale becomes much smaller than the nanobeam sizes while this is the other way round for Fig. 3(d). Here, it is worth to note that the general conclusion of some literatures [8, 12, 40] is that "increasing the nanobeam length results in decreasing the influence of the surface effects" while this conclusion is not generally true because it depends on the relation between the nanobeam cross section and length as is shown in Figs. 3(a-d).

Similar findings as those found in Figs. 3(a-d) for nanobeams made of Al can be observed in Figs. 4(a-d) for nanobeams made of Si. It is worth noting here that unlike Figs. 3(a-d) where the surface effects and the nonlocal parameter have opposite influences on the fundamental natural frequencies of Al nanobeams, it can be observed from Figs. 4(a-d) that the surface effects and the nonlocal parameter exhibit similar influences on the fundamental natural frequencies of Si nanobeams. This effect can be attributed to the negative value of the Si surface elasticity.

Lastly,variations of the frequency ratios versus mode number for Al and Si simply supported nanobeams are presented in Figs. 5(a) and 5(b), respectively, when L=50 nm and b=h=2.5 nm. From Figs. 5(a) and 5(b), it is observed that at low mode numbers the NSF/CF curve is close to the SF/CF curve and away from the NF/CF curve. This implies that the influence of the surface effects on the natural frequencies is dominant at low mode numbers. As the mode number increases, the difference between the NSF/CF and SF/CF curves becomes more and the NSF/CF curve approaches the NF/CF one. This indicates that by increasing the mode number the surface effects decrease and the nonlocal parameter effect becomes dominant. It is worthwhile to point out that the aluminum SF/CF curve does not reach to the classic solution curve as the mode number increases. This behavior is also observed from Fig. 1(b) in Ref. [12] for the nonlinear free vibration of Euler-Bernoulli nanobeams. Ref. [12] concludes that this behavior may be due to the nonlinear effects. But the similar behavior is also seen from Fig. 5(a) for the linear free vibration of Al nanobeams with the surface effects. Therefore, the behavior is not due to the nonlinear effects.

A final point to note is that the frequency ratio curves of both Euler-Bernoulli and Timoshenko nanobeams made of Al and Si with only the nonlocal parameter effect (i.e., NF/CF and TNF/TCF) have the same trend for all mode numbers whereas this is not the case for the frequency ratio curves with only the surface effects. At lower mode numbers, the frequency ratio curves of Euler-Bernoulli and Timoshenko nanobeams with only the surface effects have the same trend, indicating that considering rotary inertia and shear deformation does not change the influence of the surface effects on the natural frequency. While by increasing the mode, number the difference between the frequency ratio curves of Timoshenko and Euler-Bernoulli nanobeams becomes more. It can be deduced from this difference that considering rotary inertia and shear deformation decreases the influence of the surface effects. This reduction is such a way that at higher mode numbers the surface effects decrease the natural frequencies of the Al Timoshenko nanobeams.

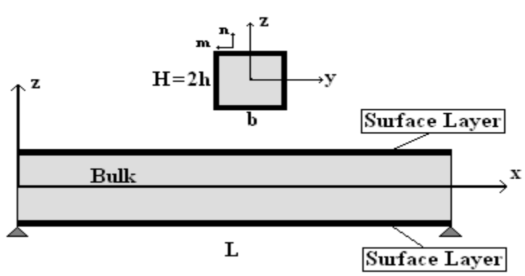
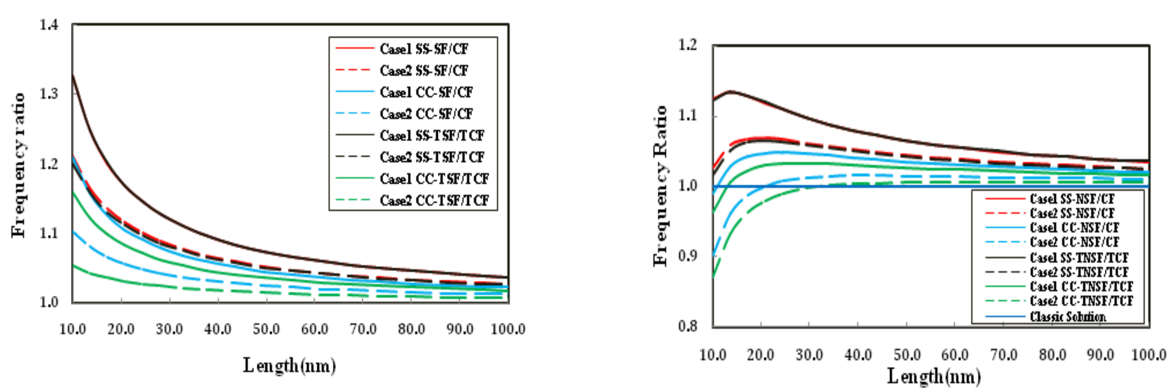


Fig. 1
Schematic of problem.



(a) (b)

Fig.2
Variations of the frequency ratios of Aluminum nanobeam versus the length with b=h=0.1L and considering, (a) only surface effects, (b) both of the surface effects and nonlocal parameter. Case1: without surface density and satisfying balance condition, Case2: with surface density and satisfying balance condition.

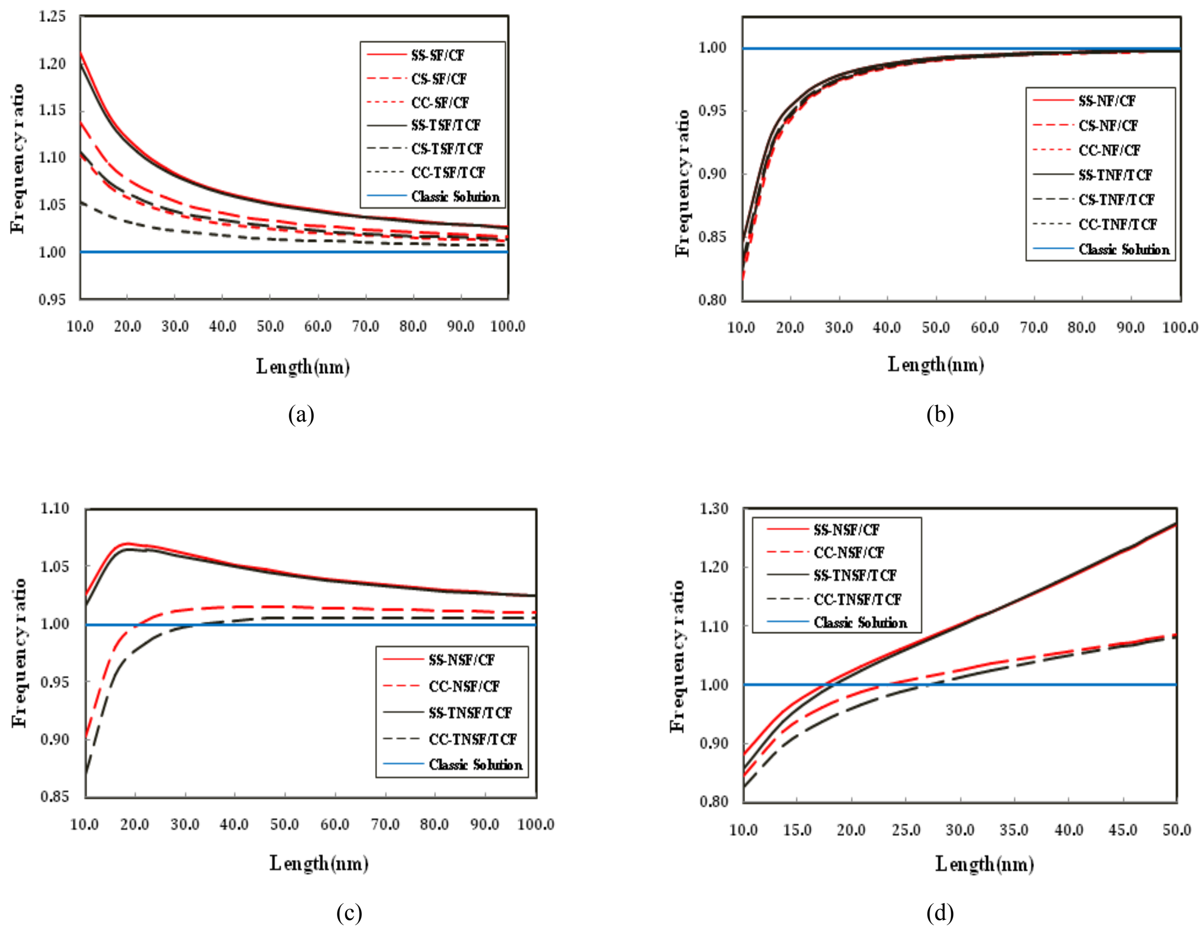


Fig.3
Variations of the frequency ratios of Aluminum nanobeam versus the length with considering, (a) surface effects, b=h=0.1L, (b) nonlocal parameter, b=h=0.1L, (c) both of the surface effects and nonlocal parameter b=h=0.1L, (d) both of the surface effects and nonlocal parameter, b=h=2.5 nm.

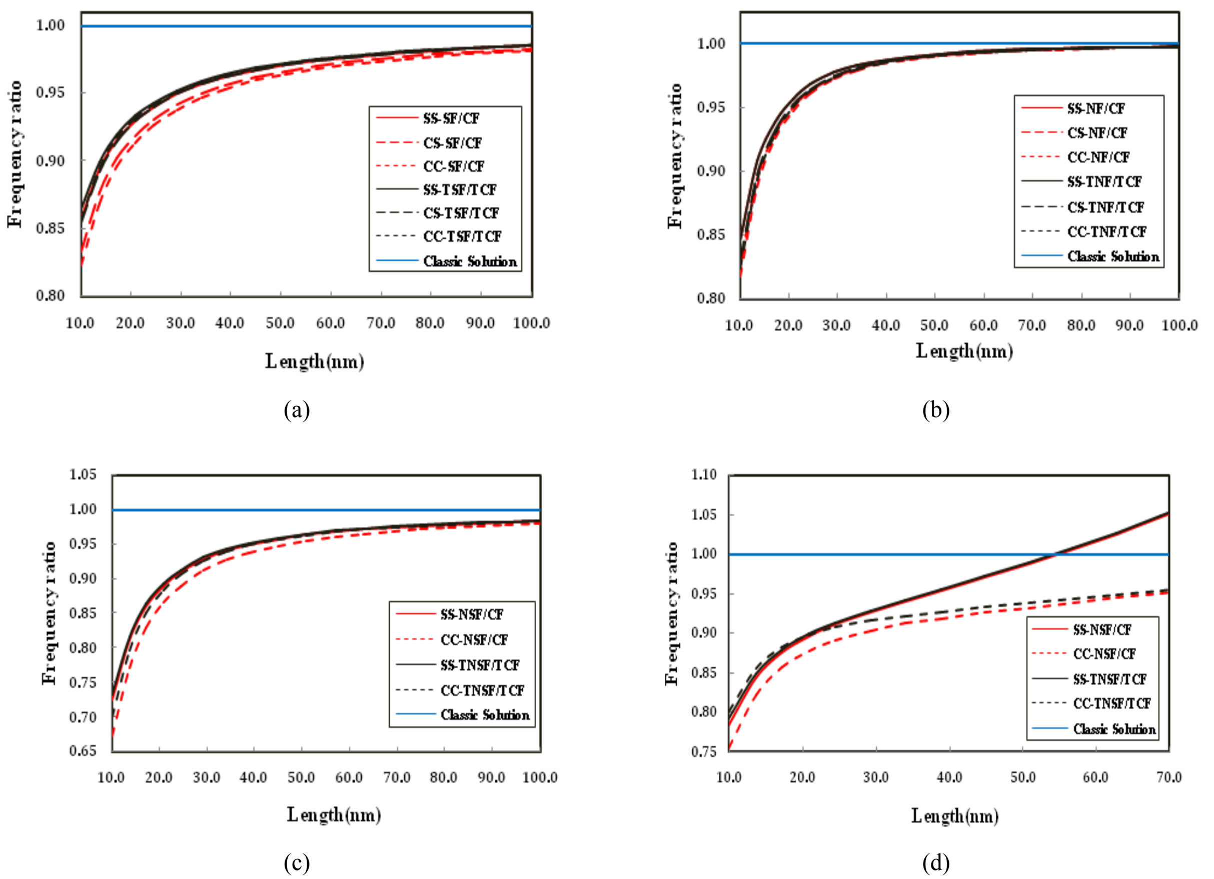


Fig.4
Variations of the frequency ratios of Silicon nanobeam versus the length with considering, (a) surface effects, b=h=0.1L, (b) nonlocal parameter, b=h=0.1L, (c) both of the surface effects and nonlocal parameter b=h=0.1L, (d) both of the surface effects and nonlocal parameter, b=h=2.5 nm.

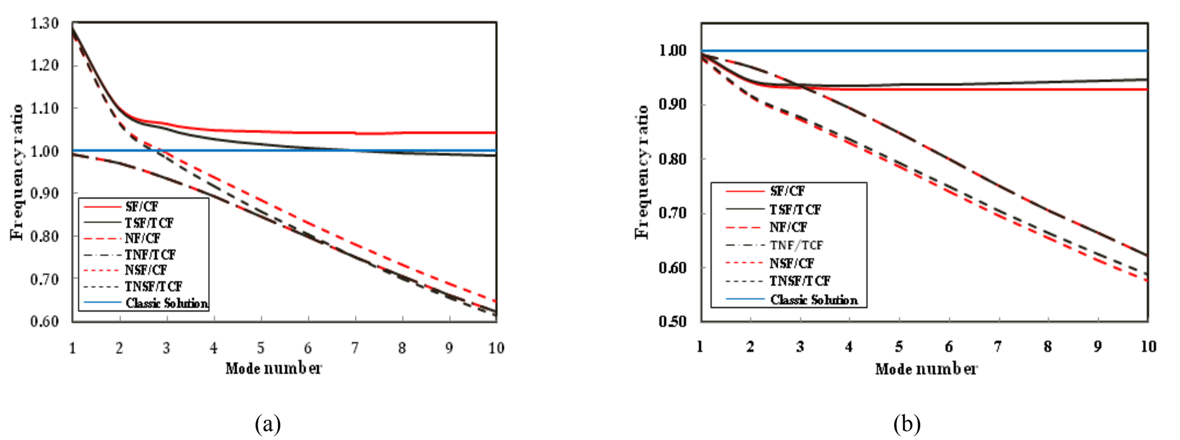


Fig.5
Variations of the frequency ratios of simply supported nanobeams versus mode number when, L=50 nm and b=h=2.5 nm, (a) Aluminum, (b) Silicon.

5 CONCLUSIONS

The influences of the surface and nonlocal effects on the vibrational behavior of Euler-Bernoulli and Timoshenko nanobeams are studied for three different boundary conditions, SS, CS and CC. The nanobeams are considered to be made of Al with positive surface elasticity and Si with negative surface elasticity. A linear variation for the normal stress along the nanobeam height is assumed to satisfy the balance condition between the nanobeam bulk and its surface layers. Furthermore, the surface density in addition to the surface elasticity and the surface tension is introduced into the governing equations where it was not considered in the previous works. Following conclusions are made from this study:

1. Considering the surface density and satisfying the balance condition between the nanobeam bulk and its surfaces cause significant differences in comparison with neglecting them.
2. The rotary inertia and shear deformation lead to a reduction in the influence of the surface effects on vibration frequencies of nanobeams. This is more significant in the case of stiffer boundary conditions than the softer ones.
3. The rotary inertia and shear deformation show notable effects on the influence of the surface effects on the natural frequencies at high mode numbers.
4. The rotary inertia and shear deformation do not show considerable effect on the influence of the nonlocal parameter on the natural frequencies of simply-simply nanobeams. While in the case of clamped-simply and clamped-clamped nanobeams, regarding shear deformation and rotary inertia cause a reduction in the decreasing effect of nonlocal parameter.
5. The surface-to-bulk ratio of nanobeams is an important parameter in determining the influence of the surface effects so that it can be said: if an increase in the nanobeam length leads to decreasing the surface-to-bulk ratio, the influence of the surface effects on frequency ratios will be diminished. Otherwise, the influence of the surface effects on frequency ratios will increase as the nanobeam length increases.

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