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Prediction of Crack Initiation Direction for Inclined Crack Under Biaxial Loading by Finite Element Method

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ABSTRACT

This paper presents a simple method based on strain energy density criterion to study the crack initiation angle by finite element method under biaxial loading condition. The crack surface relative displacement method is used to eliminate the calculation of the stress intensity factors which are normally required. The analysis is performed using higher order four node quadrilateral element. The results by finite element method are compared with DET (determinant of stress tensor criterion) and strain energy density criteria. Finite element results are in well agreement with the experimental and analytical results.

Keywords: Biaxial loading; mixed mode; Crack initiation; Finite element method; Crack tip displacement.

1 INTRODUCTION

In the past few years, several research papers have been published on the finite element methods dealing with self similar crack growth under uniaxial loading condition. The problem of non-self similar crack growth was first discussed in details by Erdogan and Sih [1]. Since then, several theories have been developed to study the fracture behaviour under mixed mode loading conditions. These theories include, the maximum tangential stress (MTS) criterion [1], minimum strain energy (SED) theory [2], T-criterion [3], maximum energy release rate theory [4], determinant of stress tensor criterion (DET criterion) [5], etc. The use of finite element method (FEM) for the solution of elastic crack problem has been the interest of many researchers in the recent years. The use of FEM for simulating crack propagation requires an analysis in two steps. The first step is the determination of state of stress & deformations in a loaded structure with an inclined crack. The second is to determine the trajectory of the crack extension under combined loading. Under mixed mode (mode I and mode II) conditions, Shih [6] has presented a full field finite element computation method for crack tip stress and strain fields for power-law hardening materials and perfectly plastic materials. The results of Shih's full field finite element computations indicate the direction of initial crack extension and critical load in terms of elastic stress intensity factors for mixed mode problems under very small scale yielding and plane strain condition. Obta [7], Saka et al. [8] and Dong and Pan [9, 10] have also studied the mixed mode crack tip stress fields for perfectly plastic materials.

Many investigators [11, 12] have demonstrated regarding the accuracy and sensitivity of solution to mesh configuration and type of element used in the analysis. It is demonstrated that minimum element size, node spacing in the near tip region and slenderness ratio of individual element are some of the important parameters, which affects the accuracy of the result. The result of Guydish and Fleming [12] shows that an optimum size of about 0.005 times the crack length (a) near the crack tips predicts the results within an error of 1% with analytical value. The direction of initial crack extension in compact tension specimen made of D16AT Aluminium alloy has been determined by

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Maiti and Mahanty [13, 14] by using four node quadrilateral element having element size 2.8% of the initial crack length. Ju [15] has performed the finite element analysis coupled with least-squares method to find the stress intensity factors of a notch formed from several elastic anisotropic materials. Réthoré et al. [16] have proposed a hybrid analytical and extended finite element method to study the mixed-mode crack propagation in a double edge notched concrete sample. Benrahou et al [17] have used finite element method to estimate the plastic zone under mixed mode (I and II) loading. The important aspect of mixed mode crack problem is the prediction of the angle of initial crack extension. Ghorbanpoor and Zhang [18] have studied the mixed mode crack growth using the boundary element method. They have predicted the crack growth direction by the maximum principal stress theory. However, crack surface relative displacement (CSRD) method has been used to eliminate the calculation of the stress intensity factors. The investigation of Wang and Chow [19] shows the effect of proportional & non-proportional loading on the mixed mode fracture behavior. Sun and Xu [20] have reported some numerical and experimental results on mixed mode crack extension for angled crack problem. They have concluded that crack mouth opening angle (CMOA) remains constant throughout the range of crack extension, but not the crack tip opening angle (CTOA). The direction of initial crack extensions predicted by Maiti and Mahanty [14], based on an elastic finite element analysis and maximum tangential principal stress criterion shows quite close results to the experimental. Lee et al [21] have determined the angle of crack growth by the crack tip force criterion. They have shown that crack growth is always perpendicular to the resultant crack tip force. They have developed an elastic-finite element program to determine the angle iteratively. Most of these studies on the prediction of initial crack extension angles by FEM are for uni-axial loading conditions. Most of the publications, experimental or theoretical investigations dealing with the mixed mode problems are restricted to uniaxial loading conditions only but in practical situation most of the structure or components are subjected to biaxial loading also. A very few experimental investigations for biaxial loading are available in literature (Seibi and Zamrik [22], Ling and Woo [23], Shlyannikov et al. [24]).

The study under biaxial loading condition is limited whereas it is experimentally shown by Kibler and Roberts [25] that initial crack extension angle depends on this biaxial factor. Hilton [26] reported the biaxial effect on strain intensity factor for mode I case. Lee and Liebowitz [27] performed a nonlinear finite element analysis on finite center-cracked specimen subjected to biaxial mode I loading and found that there is a significant biaxial effect on the energy rate (global), J integral, stress and strain intensity factor. The importance of retaining the second, and constant stress term of the series expression for local stress is demonstrated both experimentally and numerically by many investigators [28]. It is shown that the standard singular expressions for the stress and displacement field in the vicinity of the crack tip needs to be corrected. It is well demonstrated that crack growth depends upon the initial extension of crack. The crack initiation angle depends upon several factors, such as loading configuration, material properties, etc. Keeping view of the importance of the crack initiation angle under mixed mode loading, the problem has been studied using finite element method.

In this paper, a thin plate with central inclined crack was analyzed by finite element method and crack initiation angles under different biaxial loading conditions were determined. The FEM results are compared with the theoretically obtained results. Available DET and SED criteria are modified taking higher order stress solution terms. The single, two and three terms of the series stress solution are included in these criteria and comparisons have been made.

2 CRITERION FOR PREDICTION OF CRACK INITIATION ANGLE

There are several criteria available in the literature to predict the crack initiation angle under mixed mode loading. The maximum tangential stress (MTS) criterion and Strain Energy Density (SED) Criterion are mostly used by many researchers [29-32]. MTS criterion states that the direction of crack initiation coincides with the direction of maximum tangential stress along a constant radius around the crack tip. This criterion is the simplest amongst the others and is based on the assumption that material behaves ideally brittle. The SED criterion proposed by Sih states that the direction of the crack initiation coincides with the direction of minimum strain energy density along a constant radius around the crack tip. The application of the *S*-factor to fracture prediction is based on two hypotheses.

i. Crack initiation occurs when the strain energy density factor reaches a critical value.
ii. The initial crack growth takes place in the direction along which the strain energy density factor possesses a stationary value.

SED criterion is the only criterion which shows the dependence of crack initiation angle on material properties. The stress intensity factor K, is equivalent to the strain energy density factor (functions) S, when the crack problem is linear/elastic. In LEFM (or under the SSY assumption), it is obvious that the SED criterion would predict

precisely the same critical stress as what is obtained using either K or G. So, SED criterion begins to differ when one goes outside the province of the simplest LEFM case or incase of larger plastic zone around the crack tip. For mixed mode (mode I and II) loading, the SED criterion can be expressed mathematically as

$$\frac{\partial S}{\partial \theta}=0, \qquad \frac{\partial^2 S}{\partial \theta^2}<0 \tag{1}$$

where

$$S=a_{11}K_I^2+a_{12}K_IK_{II}+a_{22}K_{II}^2 \tag{2}$$

The factor S depends on θ through coefficient a_{ij} and therefore gives the description of the local energy density on any radial plane intersecting the crack tip. Expressions for a_{ij} are available in literature [2]. Differentiating Eq. (2) with respect to θ and setting the result to zero the crack initiation angle θ_0, for given loading and boundary condition can be obtained from

$$K_I^2 a_{11}^{'}+K_{II}^2 a_{22}^{'}+K_IK_{II}a_{12}^{'}=0 \tag{3}$$

where $(..)^{'}$ denotes the differentiation with respect to θ. After differentiating the coefficients and substituting in Eq. (3), it becomes,

$$K_I^2(-4\kappa_{1,2}\sin\theta+\sin 2\theta)+K_{II}^2(4\kappa_{1,2}\sin\theta-3\sin 2\theta)+K_IK_{II}(-4\kappa_{1,2}\cos\theta+4\cos 2\theta)=0 \tag{4}$$

where $\kappa_1=3-4\nu$ for plane strain case and $\kappa_2=(3-\nu)/(1+\nu)$ for plane stress case. Dividing Eq. (4) by $K_I K_{II}$, we get

$$\frac{K_I}{K_{II}}(-4\kappa_{1,2}\sin\theta+\sin 2\theta)+\frac{K_{II}}{K_I}(4\kappa_{1,2}\sin\theta-3\sin 2\theta)+(-4\kappa_{1,2}\cos\theta+4\cos 2\theta)=0 \tag{5}$$

Because of the stress singularity at the crack tip, obtaining an accurate stress intensity factor from the stress field involves very tedious and time consuming computation. Advances in numerical techniques, such as finite element method, allows relatively rapid determination of the displacement field in the region close to the crack tip with good accuracy. Several investigators have made efforts to derive displacement correlation techniques using FEM results for the crack problems [33, 34]. Displacement analysis such as crack surface relative displacement (CSRD) method, has proved reliable and easy to use. Consider two points, A and B, on the opposite crack surfaces which are at the same distance from the crack tip, Fig 1. The resulting CSRD for mixed mode I and II problems can be obtained by simple superposition of the results from the two separate single modes, as shown in Fig. 1.

$$\mathrm{CSRD}^2=\mathrm{COD}^2+\mathrm{CSD}^2 \tag{6}$$

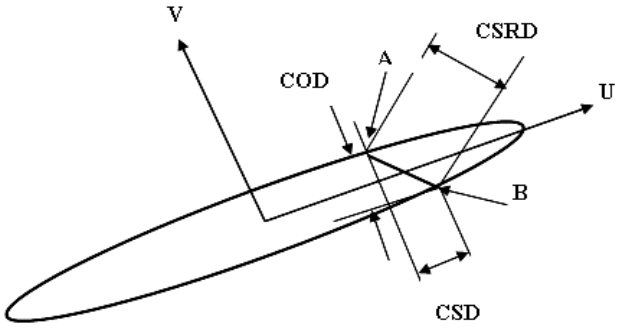

Fig. 1
The crack surface relative displacement (CSRD) for mixed mode I and II fracture problem (COD- Crack Opening Distance, CSD- Crack Sliding Distance).

where COD is the relative crack opening distance and CSD is the relative crack sliding distance. The crack surface relative displacement may be related to the stress intensity factors by two approaches, the conic section simulation and displacement method. The conic section method assumes that a crack surface relative displacement can be described by an elliptic function. Using the expression given by Sneddon [35], COD, CSD and CSRD may be related to the stress intensity factors K_I and K_{II}, for mode I and mode II, as follows:

$$\mathrm{COD}\left(\frac{x}{a}\right)=\frac{4\sqrt{a}}{E\sqrt{\pi}}\sqrt{1-\left(\frac{x}{a}\right)^2}\,K_I \tag{7}$$

$$\mathrm{CSD}\left(\frac{x}{a}\right)=\frac{4\sqrt{a}}{E\sqrt{\pi}}\sqrt{1-\left(\frac{x}{a}\right)^2}\,K_{II} \tag{8}$$

$$\mathrm{CSRD}(r)=\frac{1}{E}\sqrt{\frac{2r}{\pi}}\sqrt{K_I^2+K_{II}^2}, \qquad \mathrm{CSRD}\left(\frac{x}{a}\right)=\frac{4\sqrt{a}}{E\sqrt{\pi}}\sqrt{1-\left(\frac{x}{a}\right)^2}\sqrt{K_I^2+K_{II}^2} \tag{9}$$

where a is the crack length, E is the material constant (modulus of elasticity) and x represents the position at which the crack surface relative displacement is computed. The displacement method directly uses the displacement field at the front of the crack tip, as expressed by the Westergaard equation. The expression for CSRD is similar to that obtained from the conic section method, or

$$\mathrm{COD}(r)=\frac{1}{E}\sqrt{\frac{2r}{\pi}}K_I \tag{10}$$

$$\mathrm{CSD}(r)=\frac{1}{E}\sqrt{\frac{2r}{\pi}}K_{II} \tag{11}$$

$$\mathrm{CSRD}(r)=\frac{1}{E}\sqrt{\frac{2r}{\pi}}\sqrt{K_I^2+K_{II}^2} \tag{12}$$

where r is the distance from the crack tip. In the numerical analysis, the CSRD values can be obtained from the computed displacement results at the nodal points on the crack surfaces. From the Eqs. (7) to (12), we get,

$$\alpha=\frac{K_I}{K_{II}}=\frac{\mathrm{COD}}{\mathrm{CSD}} \tag{13}$$

Now substituting Eq. (13) in Eq. (5), we get

$$\frac{\mathrm{COD}}{\mathrm{CSD}}(-4\kappa_{1,2}\sin\theta+\sin 2\theta)+\frac{\mathrm{CSD}}{\mathrm{COD}}(4\kappa_{1,2}\sin\theta-3\sin 2\theta)+(-4\kappa_{1,2}\cos\theta+4\cos 2\theta)=0 \tag{14}$$

Solving the Eq. (14) we can find out the direction of the initial crack extension angle θ_0, thus we may avoid the difficulties in determining the stress intensity factors by utilizing the surface relative displacement values, which can be easily obtained from FEM analysis to evaluate the crack growth direction.

3 RESULTS AND DISCUSSION

3.1 Experimental

Tests were conducted on 100 kN servo hydraulic universal testing machine (ADMET, USA Make) in static load on aluminum and mild steel sheet. The mechanical slits were cut at the centre of the rectangular plate of thickness 0.9 mm at the required angle of inclination to the loading axis by drilling a hole of diameter 1.5 mm and then cutting the crack by a band saw. The cracks at the ends of the slits were created by cutting the ends of the slits further with razor

blades by about 0.5 mm each size to make the total crack length of 20 mm. These portions of the crack can be considered as natural crack. The detailed of the specimen geometry is shown in Fig. 2. The specimen was fixed in the 100 kN servo hydraulic universal testing machine (USA made) in static mode and load was applied monotonically at 0.01 mm/sec crosshead speed. The crack initiation and stable crack growth during the static loading were monitored by traveling microscope (model RTM-500) and CCD camera (Model Sony Cyber Shot 8 M.P. DH-7) attached to another computer. The loading was continued till the specimen fractured. The crack initiation angle was then measured from the fractured specimen and the crack growth direction θ_0 is presented.

3.2 Material and specimen geometry for finite element analysis

A rectangular plate of 200 mm × 150 mm × 4 mm with an inclined crack is considered for 2-Dimensional finite element analysis Fig. 3. The material properties of C15 steel used in this study were taken as E=200 GPa and ν =0.3, where E and ν represents the modulus of elasticity and Poisson's ratio, respectively.

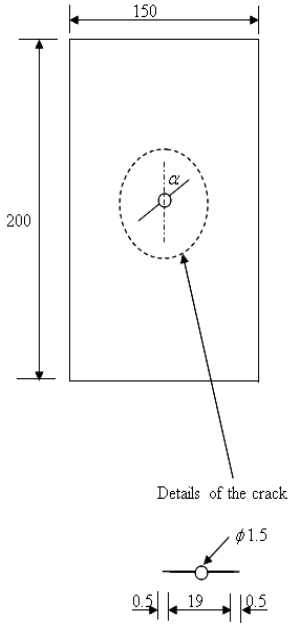

Fig. 2
Specimen geometry for determination of the crack initiation angle and details of the crack geometry. All dimensions are in mm and not to the scale.

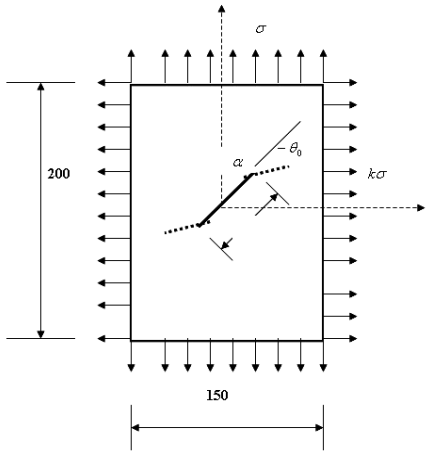

Fig. 3
Specimen geometry for finite element analysis. Dimensions are in mm and not to the scale.

3.3 Modeling of the crack region

The most important region in a fracture problem is the region around a crack tip and edge of the crack. In this investigation, the edge as well as the crack tip is modeled in a 2-D model. In linear elastic problem it has been shown that the displacement near the crack tip or crack front vary as $1/\sqrt{r}$, where r is the distance from the crack tip. The stresses and strains are singular at the crack tip, vary as $1/\sqrt{r}$. Hence, element around the crack tip and faces of the crack should be quadratic, with mid side nodes placed at the quarter points. In the present analysis PLANE82, four node quadrilateral elements have been used in Fig. 4. The singularity effects near the crack tip are solved by introducing 4 node Iso-parametric quadratic structural higher order element at the crack tip region and collapsing them into triangular elements at the crack tip. The discretized specimen is shown in Fig. 5. The FEM model contained very fine mesh in the vicinity of the crack. Table 1 shows the details of the element, degree of freedom etc., for the determination of crack ignition angles for various initial crack angles.

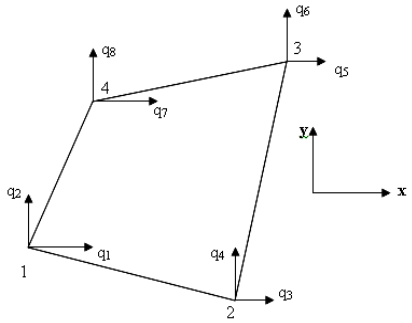

Fig. 4
Four node quadrilateral element, *q*: element displacements.

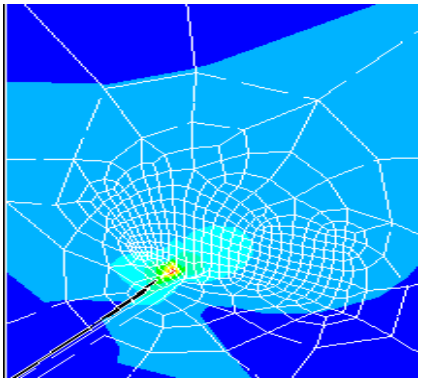

Fig. 5
Discretization and distribution of stresses for $\alpha = 45^0$.

Table 1
Details of discretization of the specimen

Specimen configuration	Degree of freedom (D.O.F)/node	Mesh generation	Purpose
$\alpha = 60^0$, *r*/*a*=.001, different biaxial factors	2	560 node, 372 element, and 1149 D.O.F	Crack initiation angle and stress analysis
$\alpha = 45^0$, *r*/*a*=.001, different biaxial factors	2	544 node, 524 element and 711 D.O.F	Crack initiation angle and stress analysis
$\alpha = 30^0$, *r*/*a*=.001, different biaxial factors	2	562 node, 467 element and 1046 D.O.F	Crack initiation angle and stress analysis
$\alpha = 15^0$, *r*/*a*=.001, different biaxial factors	2	546 node, 360 element and 1142 D.O.F	Crack initiation angle and stress analysis

The crack length, $2a$=20 mm is taken in the present investigation. The commercial package ANSYS has been used for analysis of the problem. The finite element solutions are compared with theoretical and experimental results. To compare the FEM results with theoretical results obtained from the strain energy density criterion (SED Criterion) and criterion based. On determinant of stress tensor (DET Criterion) have been used. These equations are modified by taking two and three terms of the stress solution. The derivations of these criteria are taken from the reference [36]. The plots of initial crack initiation angles θ_0, average from both crack ends and for various crack inclination angle α , are shown in Figs. 6-10. Fig. 6 compares the FEM results with experimental results for uniaxial loading condition. Experimental results of aluminum and mild steel are taken from reference [36-40]. The numerical results from FEM show good agreement with the experimental results. Fig. 7 shows the effect of biaxial factor on the initial crack extension angle for initial crack angle 60^0.

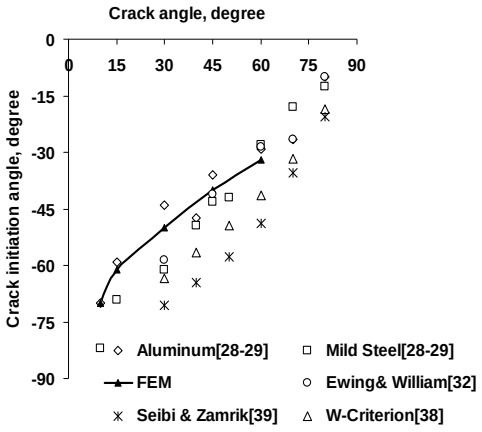

Fig. 6
Comparison of crack initiation angle predicted by Finite Element Method (FEM) with experimental results.

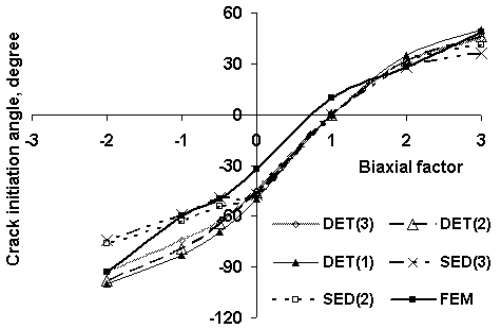

Fig. 7
Comparison of crack initiation angle predicted by Finite element method (FEM) with other theoretical predictions under different biaxial load for crack angle, $\alpha = 60^0$.

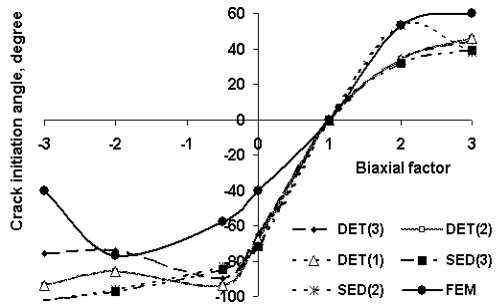

Fig. 8
Comparison of crack initiation angle predicted by finite element method (FEM) with other theoretical predictions under different biaxial load for crack angle, $\alpha = 45^0$.

3.4 Crack initiation angle

It is found that the variation between analytical (FEM) and theoretical results (obtained from two (DET (2) and SED (2) and three terms (DET (3) and SED (3) solution) are negligible for all biaxial factors considered in this investigation. Here, SED (2) indicates that the SED criterion is modified taking two terms of the series stress solution, whereas SED (3) is the strain energy density criterion modified taking three terms of the stress solution. Abbreviations for DET are also used in the similar manner. Fig. 8 compares the FEM results with theoretical results for initial crack position, α =45^0. It is seen that there is slight discrepancy between FEM and theoretical results for biaxial factor, k<0 and this discrepancy tends to narrow for k>0. It is also seen that all results are same under equal-equal biaxial loads The discrepancy under compressive loads may be due to non-inclusion of closing and frictional effects that exits due to the closing mechanism that takes place under compressive load. Under compressive load, the crack does not open up but it slides. Hence, this mechanism should be incorporated in the criteria for accurate prediction of crack initiation angle under such conditions. In the present investigation, SED and DET criteria are not modified to account such effect. It is also seen that the difference increases as the magnitude of compressive load increases.

Figs. 9 and 10 show the variation of crack initiation angle with different biaxial load factors for crack angles < 45^0. In these figures, it is seen that the differences between FEM and theoretical results under compressive biaxial load are significant. These results show that the stress component normal to the crack tip acting on an element plays important role in the crack opening and sliding mechanism. Hence, to correlate this phenomenon with the crack growth direction, the variation of stress normal to the crack tip σ_{yy} with θ, where θ is the polar coordinate, is shown in Fig. 11. Fig. 11 shows that for k=-2, σ_{yy} is negative for $-90^0 \le \theta \le 90^0$, whereas for k=-1, it is negative for $-60^0 < \theta < 0^0$, in which the crack initiation direction lies. Similar results are seen for other crack angles, $\alpha \le 45^0$. Hence, it can be concluded that the closing mechanism or crack sliding mechanism takes place for crack angle, $\alpha \le 45^0$ and k<0. Hence, the criteria for the prediction of crack initiation angle should be modified by incorporating suitable sliding and closing factor. This requires further investigation to find out these factors.

From these figures, it is also seen that self-similar crack growth occurs for a particular value of biaxiality factor. These critical values of biaxiality factors for initial crack angle, α =15^0, 30^0, 45^0, and 60^0 are 0.375, 0.625, 1.0 and 0.75, respectively.

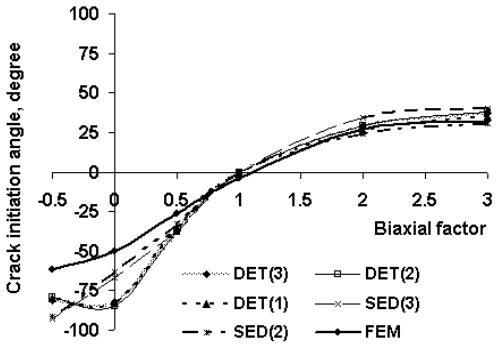

Fig. 9
Comparison of crack initiation angle predicted by finite element method (FEM) with other theoretical predictions under different biaxial load for crack angle, α = 30^0.

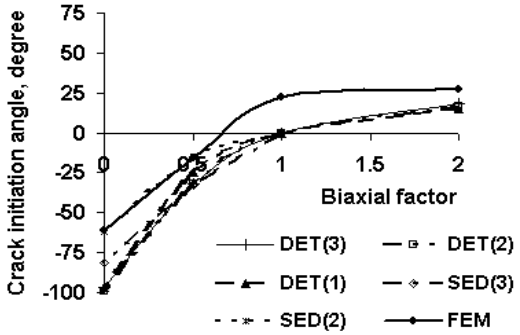

Fig. 10
Comparison of crack initiation angle predicted by Finite element method (FEM) with other theoretical predictions under different biaxial load for crack angle, α = 15^0.

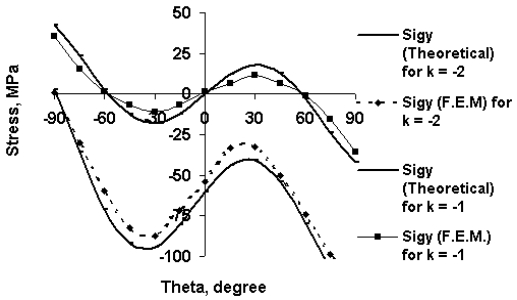

Fig. 11
Variation of σ_{yy} (r/a=0.1, $\alpha = 45^0$).

4 CONCLUSIONS

1. Finite element method supplemented with the crack closure relative displacement method used in SED criterion can be used effectively to determine the crack initiation angle under mixed mode loading condition. The accuracy of the FEM method is satisfactory when it is compared with other theoretical results from two terms or three terms DET and SED solutions. It is satisfactory with the experimental results under uni-axial mixed mode condition also.
2. Self-similar crack growth occurs for a particular value of biaxiality factor 0.375, 0.625, 1.0 and 0.75 for 15^0, 30^0, 45^0 and 60^0, respectively.
3. For tensile –compressive loading and crack angle $\alpha \leq 45^0$, crack closure or crack sliding mechanism occurs. Hence, criterion for the prediction of crack initiation angle for above loading conditions should be modified by incorporating suitable crack closing or friction factor.

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