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Torsional Stability of Cylindrical Shells with Functionally Graded Middle Layer on the Winkler Elastic Foundation

A.H. Sofiyev*, S. Adiguzel

Department of Civil Engineering of Suleyman Demirel University, 32260 Isparta, Turkey

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ABSTRACT

In this study, the torsional stability analysis is presented for thin cylindrical with the functionally graded (FG) middle layer resting on the Winker elastic foundation. The mechanical properties of functionally graded material (FGM) are assumed to be graded in the thickness direction according to a simple power law and exponential distributions in terms of volume fractions of the constituents. The fundamental relations and basic equations of three-layered cylindrical shells with a FG middle layer resting on the Winker elastic foundation under torsional load are derived. Governing equations are solved by using the Galerkin method. The numerical results reveal that variations of the shell thickness-to-FG layer thickness ratio, radius-to-shell thickness ratio, lengths-to-radius ratio, foundation stiffness and compositional profiles have significant effects on the critical torsional load of three-layered cylindrical shells with a FG middle layer. The results are verified by comparing the obtained values with those in the existing literature.

Keywords: FG layer; Torsional stability; Three-layered cylindrical shells; Critical torsional load; Elastic foundation

1 INTRODUCTION

LAYERED composite structures are commonly used in many kinds of engineering structures. In conventional laminated composite structures, homogeneous elastic laminas are bonded together to obtain enhanced mechanical properties. However, the abrupt change in material properties across the interface between different materials can result in large inter-laminar stresses leading to delamination. One way to overcome these adverse effects is to use FGMs in which material properties vary continuously. This may be achieved by gradually changing the volume fraction of the constituent materials, usually in the thickness direction only. This advantage eliminates interface problems of composite materials and thus the stress distribution becomes smooth. Used as interfacial zones, they can help to reduce mechanically and thermally induced stresses caused by the material property mismatch and to improve the bonding strength. The concept of FGMs was first introduced by a group of Japanese scientists in 1984 [1, 2]. Several notable studies have been performed to analyze the behavior of FG materials [3-5].

The solution of torsional stability problems of cylindrical shells have been the interest of many researchers since members have high strength and rigidity in torsion. Torsional analysis of functionally graded cylindrical shells may be one of the main concerns in the practical usage of FGMs in modern aerospace, marine, nuclear and defense industries, and pressure vessels, water ducts, pipelines, casing pipes and in other applications. As the cylindrical shell subjected to torsional loading, the solution of the stability problem become more cumbersome in comparison with the other types of loading cases (axial load, lateral or hydrostatic pressures). There are some important

* Corresponding author. Tel.: +90 246 2111195; Fax.: +90 246 2370859.
E-mail address: asofiyev@mmf.sdu.edu.tr (A.H. Sofiyev).

publications related to the torsional stability of FG cylindrical shells [6-13]. Comprehensive reviews of FG shells research can be found in the monograph by Shen [14].

The considerable advantages offered by FGMs over conventional materials and the need of overcoming the technical challenges involving high temperature environments have prompted an increased use of multilayer FG systems, in recent years. In the majority of practical applications involving functionally graded materials, they are used as inter-layer between two homogeneous materials [15]. Studies on the stability and vibration of three-layered shells with a FG middle layer are limited in the open literature. Li and Batra [16] studied the linear buckling of axially compressed thin cylindrical shells with a FG middle layer. Liew et al. [17] studied the non-linear vibration of a coating-FGM-substrate cylindrical panel subjected to a temperature gradient. Sofiyev [18] examined the linear vibration and stability of the three-layered cylindrical shells containing a FG layer subjected to various loads.

In recent years, FG cylindrical shells are widely used in modern engineering structures such as tunnels, storage tanks, pressure vessels, water ducts, pipelines, and casing pipes, process equipment and in other applications. Such FG cylindrical shells are usually laid on or placed in a soil medium as an elastic foundation, thus there is a great interest in the stability and vibration analysis of FG cylindrical shells on the elastic foundation. Currently, some investigations on the stability analysis of FG cylindrical shells with FG layer resting on elastic foundations have been published in the open literature [20-25].

To the authors' knowledge, the stability of the three-layered cylindrical shell with a FG middle layer resting on the Winkler elastic foundation and subjected to torsional load has not been reported yet. In the current study, an attempt is made to address this problem. The basic equations of the three-layered cylindrical shell with a FG middle layer and resting on the Winkler elastic foundation are derived and closed form solution is obtained. The results reveal that variations of cylindrical shell characteristics parameters, compositional profiles of a FG layer and Winkler foundation stiffness have significant effects on the values of the critical torsional load.

2 FORMULATION OF THE PROBLEM

As shown in Fig. 1a, three-layered cylindrical shell with a FG middle layer surrounded by an elastic medium and subjected to the torsional load, S, is considered. Three-layered cylindrical shell is of length L and radius R. The origin of the coordinate system is taken as the midpoint of length of the reference surface of the cylindrical shell. The x axis is taken along a generator, y axis is taken tangential directions and z axis normal to them. The cross-section of the three-layered cylindrical shell is shown in Fig. 1b. The cylindrical shell is composed of three elastic layers, namely, layer (1), layer (2), and layer (3) from bottom to top of the cylindrical shell. The FG layer extends from $z=-a$ to $z=+a$ and, for continuous property assumptions to be valid; the thickness of this layer must be significantly larger than its dominant micro-structural length scale (e.g. grain size). The interfaces between different layers are assumed to be perfectly bonded at all times and the multilayer system behavior to be linear elastic. h is the total thickness of the three-layered cylindrical shell and $h_{Fg}=2a$ is the thickness of a FG layer. The vertical ordinates of the top, the two interfaces, and the bottom are denoted by $h_1=-h/2,\ h_2=-a,\ h_3=a,\ h_4=h/2$, respectively. For the brevity, the ratio of the thickness of each layer from bottom to top is denoted by three numbers, i.e. (1)-(2)-(3) Fig 1b. The Winkler model is used to describe the reaction of the elastic medium on the cylindrical shell. If the effects of damping and inertia force in the foundation are neglected, the foundation interface pressure $p(x,y)$ may be expressed as

$$p(x,y)=K_w w \tag{1}$$

where w is the displacement of the reference surface in the normal direction and positive towards the axis of the cylinder and assumed to be much smaller than the thickness and K_w (in $\mathrm{N/m^3}$) is the elastic foundation stiffness [26]. The volume fraction of three-layer system with a FG middle layer is assumed to obey

$$\begin{aligned} &V^{(1)}=0 && \text{at} && z\in[h_1,h_2] \\ &V^{(2)}(\overline{z}) && \text{at} && z\in[h_2,h_3] \\ &V^{(3)}=1 && \text{at} && z\in[h_3,h_4] \end{aligned} \tag{2}$$

Let the volume fraction of the layer 2 material within the FGM vary as a function of the coordinate, $\overline{z} = z/2a$, and be arbitrarily defined by a generic function, $V^{(2)}(\overline{z})$, which satisfies the following conditions at the homogeneous layers' interfaces,

$$V^{(2)}(\overline{z}) = \begin{cases} 0 & at \quad \overline{z} = -0.5 \\ \\ 1 & at \quad \overline{z} = 0.5 \end{cases} \tag{3}$$

The compositional gradation of a FG layer is defined by the volume fraction of the ceramic phase, $V^{(2)}(\overline{z}) = V_c(\overline{z})$. Here, the following functions of $V^{(2)}(\overline{z})$ will be considered [15]:

Linear: $$V^{(2)}(\overline{z}) = \overline{z} + 0.5 \tag{4a}$$
Quadratic: $$V^{(2)}(\overline{z}) = (\overline{z} + 0.5)^2 \tag{4b}$$
Inverse Quadratic: $$V^{(2)}(\overline{z}) = 1 - (0.5 - \overline{z})^2 \tag{4c}$$

The effective material properties, like Young's modulus $E^{(k)}$ and Poisson's ratio $\nu^{(k)}$, then can be expressed by the rule of mixture as [15]:

$$P^{(k)}(\overline{\zeta}) = P_{cm}V^{(k)} + P_m, \tag{5}$$

where $P^{(k)}$ is the effective material property of FGM of layer *k*, $P_{cm} = P_c - P_m$, P_c and P_m are properties of the ceramic and metal surfaces of a FG layer.

From Eqs. (2) and (5), the effective Young's modulus and Poisson's ratio of a FG layer can be written as

$$E^{(2)}(\overline{z}) = E_{cm}V^{(2)}(\overline{z}) + E_m, \qquad \nu^{(2)}(\overline{z}) = \nu_{cm}V^{(2)}(\overline{z}) + \nu_m \tag{6}$$

where the following definitions apply:

$$E_{cm} = E_c - E_m, \qquad \nu_{cm} = \nu_c - \nu_m \tag{7}$$

There is also another FG layer whose material properties obey the exponential distribution which can be expressed as [3]:

$$E^{(2)}(\overline{z}) = E_m e^{(\overline{z}+0.5)\ln(E_c/E_m)}, \qquad \nu^{(2)}(\overline{z}) = \nu_m e^{(\overline{z}+0.5)\ln(\nu_c/\nu_m)} \tag{8}$$

For such cases, the variation of the Young's modulus and Poisson's ratio in the three-layer system are given as [17]:

$$\left[E(\overline{z}),\nu(\overline{z})\right] = \begin{cases} E_{0m},\, \nu_{0m} & -h_1/(2a) \le \overline{z} \le -h_2/(2a) \\ \\ E^{(2)}(\overline{z}),\ \nu^{(2)}(\overline{z}) & -h_2/(2a) \le \overline{z} \le h_3/(2a) \\ \\ E_{0c},\ \nu_{0c} & h_3/(2a) \le \overline{z} \le h_4/(2a) \end{cases} \tag{9}$$

where E_{0m}, ν_{0m} and E_{0c}, ν_{0c} are the Young's modulus and Poisson's ratio of the pure metal and ceramic materials, respectively.

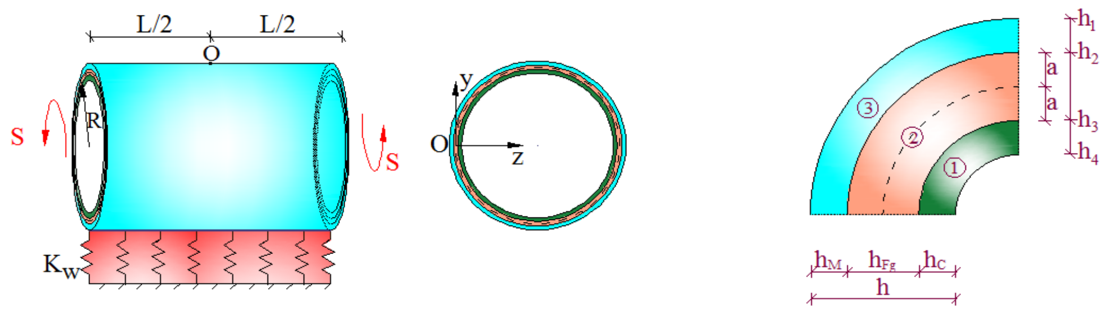

Fig. 1
The cross-section of the three-layered cylindrical shell with FG middle layer (1) ceramic, (2) FGM, (3) metal.

3 BASIC RELATIONS AND EQUATIONS

The stress-strain relations of the three-layered cylindrical shell with a FG middle layer are given as follows [18]:

$$\begin{bmatrix} \sigma_x^{(k)} \\ \sigma_y^{(k)} \\ \sigma_{xy}^{(k)} \end{bmatrix} = \begin{bmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} & 0 \\ Q_{12}^{(k)} & Q_{11}^{(k)} & 0 \\ 0 & 0 & Q_{66}^{(k)} \end{bmatrix} \begin{bmatrix} e_x - z\dfrac{\partial^2 w}{\partial x^2} \\ e_y - z\dfrac{\partial^2 w}{\partial y^2} \\ e_{xy} - z\dfrac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \tag{10}$$

where $\sigma_x^{(k)}, \sigma_y^{(k)}, \sigma_{xy}^{(k)}$ (k=1,2,3) are the stress components in the layers, k is the number of the layers, e_x, e_y are strains in x and y direction on the reference surface, respectively, e_{xy} is the shear strain on the reference surface and the quantities $Q_{ij}^{(k)}$ (i,j = 1,2,6) are

$$\begin{gathered} Q_{11}^{(1)} = Q_{22}^{(1)} = \frac{E_{0m}}{1-\nu_{0m}^2}, \quad Q_{12}^{(1)} = \frac{\nu_{0m} E_{0m}}{1-\nu_{0m}^2}, \quad Q_{66}^{(1)} = \frac{E_{0m}}{1+\nu_{0m}}, \quad Q_{11}^{(2)} = Q_{22}^{(2)} = \frac{E_{fg}}{1-\nu_{fg}^2}, \\ Q_{12}^{(2)} = \frac{\nu_{fg} E_{fg}}{1-\nu_{fg}^2}, \quad Q_{66}^{(2)} = \frac{E_{fg}}{1+\nu_{fg}}, \quad Q_{11}^{(3)} = Q_{22}^{(3)} = \frac{E_{0c}}{1-\nu_{0c}^2}, \quad Q_{12}^{(3)} = \frac{\nu_{0c} E_{0c}}{1-\nu_{0c}^2}, \quad Q_{66}^{(3)} = \frac{E_{0c}}{1+\nu_{0c}} \end{gathered} \tag{11}$$

The force and moment resultants are defined in the following manner [26]:

$$\left[(N_x, N_y, N_{xy}), (M_x, M_y, M_x)\right] = \sum_{k=1}^{3} \int_{h_k}^{h_{k+1}} (1, \mathrm{z}) \left[\sigma_x^{(k)}, \sigma_y^{(k)}, \sigma_{xy}^{(k)}\right] \mathrm{dz} \tag{12}$$

The relations between the forces and the Airy stress function, $\Psi = \Psi_1 / h,$ are given by

$$(N_x, N_y, N_{xy}) = \left(\frac{\partial^2 \Psi_1}{\partial y^2}, \frac{\partial^2 \Psi_1}{\partial x^2}, -\frac{\partial^2 \Psi_1}{\partial x \partial y} \right) \tag{13}$$

Substituting relations (10) in (12) after some rearrangements, the relations found for moments and strains, being substituted in the basic equations of cylindrical shells [26] together with relations (1) and (13), the system of differential equations for w and Ψ_1 can be obtained as

$$\begin{aligned} & c_2 \frac{\partial^4 \Psi_1}{\partial x^4} + 2(c_1 - c_5)\frac{\partial^4 \Psi_1}{\partial x^2 \partial y^2} + c_2 \frac{\partial^4 \Psi_1}{\partial y^4} + \frac{1}{R}\frac{\partial^2 \Psi_1}{\partial x^2} - c_3 \frac{\partial^4 w}{\partial x^4} - 2(c_4 + c_6)\frac{\partial^4 w}{\partial x^2 \partial y^2} \\ & \quad - c_3 \frac{\partial^4 w}{\partial y^4} - 2Sh\frac{\partial^2 w}{\partial x \partial y} - K_w w = 0 \end{aligned} \tag{14}$$

$$\mathrm{b}_1 \frac{\partial^4 \Psi_1}{\partial x^4} + 2(b_2 + b_5)\frac{\partial^4 \Psi_1}{\partial x^2 \partial y^2} + b_1 \frac{\partial^4 \Psi_1}{\partial y^4} - b_4 \frac{\partial^4 w}{\partial x^4} - 2(b_3 - b_6)\frac{\partial^4 w}{\partial x^2 \partial y^2} - \mathrm{b}_4 \frac{\partial^4 w}{\partial y^4} + \frac{1}{\mathrm{R}}\frac{\partial^2 w}{\partial x^2} = 0 \tag{15}$$

where the following definitions apply:

$$\begin{aligned} & c_1 = A_{11}b_1 + A_{21}b_2, \quad c_2 = A_{11}b_2 + A_{21}b_1, \quad c_3 = A_{11}b_3 + A_{21}b_4 + A_{12}, \quad c_4 = A_{11}b_4 + A_{21}b_3 + A_{22}, \\ & c_5 = A_{61}b_5, \quad c_6 = A_{61}b_6 + A_{62}, \quad b_1 = A_{10}L_0^{-1}, \quad b_2 = A_{20}L_0^{-1}, \quad b_3 = (A_{20}A_{21} - A_{11}A_{10})L_0^{-1}, \\ & b_4 = (A_{20}A_{11} - A_{21}A_{10})L_0^{-1}, \quad b_5 = 1/A_{60}, \quad b_6 = -A_{61}/A_{60}, \quad L_0 = A_{10}^2 - A_{20}^2 \end{aligned} \tag{16}$$

in which

$$\begin{aligned} A_{1k_1} &= \frac{E_{0m}}{1-\nu_{0m}^2}\int_{-h_1}^{-h_2} z^{k_1}\mathrm{d}z + \int_{-h_2}^{h_3} z^{k_1}\frac{E_{fg}}{1-\nu_{fg}^2}\mathrm{d}z + \frac{E_{0c}}{1-\nu_{0c}^2}\int_{h_3}^{h_4} z^{k_1}\mathrm{d}z \\ A_{2k_1} &= \frac{\nu_{0m}E_{0m}}{1-\nu_{0m}^2}\int_{-h_1}^{-h_2} z^{k_1}\mathrm{d}\zeta + \int_{-h_2}^{h_3} z^{k_1}\frac{\nu_{fg}E_{fg}}{1-\nu_{fg}^2}\mathrm{d}z + h_4\int_{h_3}\frac{\nu_{0c}E_{0c}}{1-\nu_{0c}^2}z^{k_1}\mathrm{d}z \\ A_{6k_1} &= \frac{E_{0m}}{1+\nu_{0m}}\int_{-h_1}^{-h_2} z^{k_1}\mathrm{d}z + \int_{-h_2}^{h_3} z^{k_1}\frac{E_{fg}}{1+\nu_{fg}}\mathrm{d}z + \frac{E_{0c}}{1+\nu_{0c}}\int_{h_3}^{h_4} z^{k_1}\mathrm{d}z; \quad k_1 = 0,1,2 \end{aligned} \tag{17}$$

Eqs. (14) and (15) are torsional stability equations of the three-layered cylindrical shell with a FG middle layer resting on the Winkler elastic foundation.

4 SOLUTION OF THE PROBLEM

The edge conditions of a cylindrical shell are mixed boundary conditions. The solution of Eqs. (14) and (15) is sought in the following form [26]:

$$w = \xi_1 \cos\frac{\pi x}{L}\cos\frac{n}{R}(y + \gamma x), \qquad \Psi_1 = \xi_2 \cos\frac{\pi x}{L}\cos\frac{n}{R}(y + \gamma x) \tag{18}$$

where n is the wave number in the direction of y axis, γ is tangent of the angle between the waves and x axis of the cylindrical shell, and ξ_1 and ξ_2 are amplitudes.

As $x = \pm L/2$, , in expressions (18), the condition w=0 is satisfied, whereas, $\partial w/\partial x$ and $\partial^2 w/\partial x^2$ boundary conditions are not satisfied.

$$\left.\frac{\partial w}{\partial x}\right|_{x=\pm\frac{L}{2}} = \pm\xi_1 \frac{\gamma n}{R}\cos\frac{n}{R}\left(y \pm \gamma\frac{L}{2}\right), \qquad \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=\pm\frac{L}{2}} = \mp\xi_1\left(\frac{\pi^2}{L^2} + \frac{\gamma^2 n^2}{R^2}\right)\sin\frac{n}{R}\left(y \pm \gamma\frac{L}{2}\right) \tag{19}$$

Namely, neither simple support nor clamped boundary conditions are satisfied, whereas, the following boundary conditions are satisfied:

$$\int_0^{2\pi R}\left(\frac{\partial w}{\partial x}\right)_{x=\pm L/2}\mathrm{d}y=\int_0^{2\pi R}\left(\frac{\partial^2 w}{\partial x^2}\right)_{x=\pm L/2}\mathrm{d}y=0 \tag{20}$$

So both boundary conditions are satisfied in the integral sense (see, Wolmir 1967). Substituting Eq. (18) into Eqs. (15) and (16), and then applying the Galerkin method, in the ranges $0\le y\le 2\pi R$ and $-L/2\le x\le L/2$, after integrating for the critical torsional load of layered cylindrical shell with a FG middle layer resting on the Winkler elastic foundation, the following equation is obtained:

$$\begin{aligned} S_{crw}=\frac{1}{2hn^2\gamma R^2}\Bigg\{&\left[R(\gamma^2n^2+m_1^2)-c_2(m_1^4+n^4+6m_1^2n^2\gamma^2+n^4\gamma^4)-2(c_1-c_5)(n^4\gamma^2+m_1^2n^2)\right]\\ &\times\frac{b_4(m_1^4+n^4+6m_1^2n^2\gamma^2+\gamma^4n^4)+2(b_3-b_6)(n^4\gamma^2+m_1^2n^2)+(\gamma^2n^2+m_1^2)R}{b_1n^4+b_1(m_1^4+6m_1^2n^2\gamma^2+\gamma^4n^4)+2(b_2+b_5)(n^4\gamma^2+m_1^2n^2)}\\ &+c_3(6m_1^2n^2\gamma^2+m_1^4+n^4+\gamma^4n^4)+2(c_4+c_6)(n^4\gamma^2+m_1^2n^2)+K_wR^4\Bigg\} \end{aligned} \tag{21}$$

where the following definition applies:

$$m_1=\pi R/L \tag{22}$$

As $K_w=0,$ from Eq. (21), the expression is obtained for the critical torsional load of the unconstrained layered cylindrical shell with a FG middle layer.

The minimum values of critical torsional loads of the thee-layered cylindrical shell with a FG middle layer with or without an elastic foundation are obtained by minimizing Eq. (21) with respect to (γ,n).

5 NUMERICAL ANALYSES

In order to validate the present study, the results for the critical torsional load of pure FG cylindrical shells are compared with results of Huang and Han [11] for different cylindrical shell characteristics and given in Table 1. As $h/h_{Fg}=1$, the three-layered cylindrical shell containing a FG layer transformed into the pure FG cylindrical shell. The ceramic-metal mixture is used as FGM that is the mixture of ZrO_2 and Ti-6Al-4V. Material properties of FG layer (Ti-6Al-4V/ZrO_2) as follow: $E_c=1.68063\times10^{11}$(Pa), $E_m=1.056982\times10^{11}$(Pa), $\nu_c=0.298$, ν_m=0.2981. The shell characteristics are taken from Huang and Han [11] and given in Table 1. It is noted that numbers in brackets indicate the (γ_{cr},n_{cr}) in the buckling case. It is seen that there is slight difference between present results and results of Huang and Han [11]. In our study, approximation functions for w and Ψ_1 are sought in the form of (18).

Table 1
Comparison of the present results with the results of Huang and Han [11]

	$S_{cr}^{FG}\times10^{-6}\,MPa\,(\gamma_{cr},n_{cr})$					
	Huang and Han [11]			Present study		
R/h	L/R=1	L/R=1.5	L/R =2	L/R=1	L/R=1.5	L/R=2
400	48.90	39.25	33.82	50.51	40.29	34.44
	(0.39, 15)	(0.33,13)	(0.31, 12)	(0.3,14)	(0.25,12)	(0.21,10)
500	36.78	29.61	25.58	37.92	30.33	25.93
	(0.36, 16)	(0.32, 14)	(0.30,13)	(0.29,15)	(0.22,12)	(0.2, 11)

Table 2
Variations of critical torsion loads of the pure metal, pure ceramic, pure FG andTi-6Al-4V/FG/ZrO_2 shells with and without an elastic foundation versus h/h_{Fg}

	$S_{cr}\times10^{-6}$ MPa (γ_{cr},n_{cr})					
h/h_{Fg}	ZrO_2	Ti-6Al-4V/FG/ZrO_2				Ti-6Al-4V
		Linear	Quad.	Inv. Quad.	Exp.	
1.0	363.18 (0.3,7)	202.04(0.3,7)	190.42(0.3,7)	213.62(0.3,7)	198.99(0.3,7)	182.24 (0.3,7)
1.5		234.92(0.31,7)	229.04(0.31,7)	240.75(0.3,7)	235.91(0.31,7)	
2.0		243.67(0.31,7)	239.77(0.31,7)	247.49(0.3,7)	244.46(0.31,7)	
5.0		252.97(0.3,7)	251.67(0.3,7)	254.25(0.3,7)	252.93(0.3,7)	
	$S_{crw}\times10^{-6}$ MPa (γ_{cr},n_{cr})					
1.0	485.88 (0.4,8)	305.60(0.45,9)	292.28(0.46,9)	318.75(0.45,9)	302.24(0.46,9)	282.37 (0.46,9)
1.5		345.43(0.45,9)	339.16(0.46,9)	351.63(0.45,9)	347.26(0.45,9)	
2.0		355.30(0.45,9)	351.28(0.45,9)	359.27(0.44,9)	356.72(0.45,9)	
5.0		364.68(0.44,9)	363.41(0.44,9)	365.94(0.44,9)	364.78(0.44,9)	

In study of Huang and Han [11], w is sought similarly and the expression for Ψ_1 is found by taking into account the expression of w in the compatibility equation. Therefore, Ψ_1 includes some additional terms in Ref. [11]. The difference 0.4%-3% is arisen from this detail. The influence of the FGM will be presented and discussed in detail, which will be useful for the design and manufacturing of layered shell structures containing a FG layer in the elastic medium. Numerical computations has been carried out for the values of critical torsional loads of the pure metal, pure ceramic and pure FG shells, and three-layered shell containing a FG layer with or without Winkler elastic foundation, which are explained below in details. The results are presented in Tables 2-3 and Figs. 2-3. The numbers in brackets (γ_{cr},n_{cr}) indicate tangent of the angle between the waves and x axis and the circumferential wave numbers corresponding to the critical torsional load. In the numerical analyses, layered cylindrical shells with the stacking sequences Ti-6Al-4V/FG/ZrO_2 are taken into account. In the middle layer, it is assumed that FGM is composed of ceramic-metal mixture, namely, Ti-6Al-4V/ZrO_2, whose material properties are given above. The material properties of the pure metal, namely, Titanium-alloy (Ti-6Al-4V) which is used in the outer layer as follows: $E_{0m}=1.2256\times10^{11}$(Pa), $\nu_{0m}=0.2884$; and material properties of the pure ceramic, namely, Zirconium Oxide (ZrO_2) which is used in the inner layer as follows: $E_{0c}=2.4427\times10^{11}$(Pa), $\nu_c=0.2882$.

In Table 2, variations of critical torsional loads of Ti-6Al-4V/FG/ ZrO_2 cylindrical shells containing a FG layer with different compositional profiles with and without Winkler elastic foundation versus the ratio, h/h_{Fg}, is presented. The cylindrical shell characteristics and Winkler foundation stiffness are taken to be $R/h=100,\ L/R=2;\ K_w=5\times10^7\ \mathrm{N/m^3}$. The compositional profiles of a FG layer are considered as linear, quadratic, inverse quadratic and exponential functions. As $h/h_{Fg}=1$, the three-layered cylindrical shell transformed into the pure FG cylindrical shell. As the ratio, h/h_{Fg}, increases, the critical torsional loads of Ti-6Al-4V/FG/ ZrO_2 cylindrical shell with and without Winkler elastic foundation increase, whereas, corresponding wave numbers γ_{cr} and n_{cr} remain constant. As h/h_{Fg} increases, the effect of compositional profiles of a FG layer on the critical torsional loads of the Ti-6Al-4V/FG/ ZrO_2 cylindrical shell with and without the Winkler elastic foundation decrease, but their percentage effects are different. Comparing the values of the critical torsional load of pure ceramic (ZrO_2) single-layer cylindrical shell with those of three-layered shells for all compositional profiles of a FG middle layer with and without an elastic foundation, respectively, the less effect is observed at inverse quadratic profile and the highest effect is observed at quadratic profile. For this case, when linear and exponential profiles are compared with each other, the highest effect on the dimensionless critical torsional load is observed in the exponential profile.

In Fig. 2, variations of critical torsional loads of the pure metal, pure ceramic and Ti-6Al-4V/FG/ZrO_2 cylindrical shells with and without the Winkler elastic foundation versus the ratio, *R/h*, is illustrated. The cylindrical shell characteristics and Winkler elastic foundation stiffness are taken to be $h/h_m = h/h_c = 4;\ L/R = 2;\ K_w = 1\times10^7\ \mathrm{N/m^3}$, and compositional profiles are quadratic and inverse quadratic. As the ratio, *R/h,* increases, the values of critical torsional loads of the pure metal, pure ceramic cylindrical shells and three-layered cylindrical shell containing a FG layer with different compositional profiles with and without an elastic foundation decrease. As the critical torsional loads of three-layered cylindrical shells containing a FG layer with the quadratic and inverse quadratic profiles with and without an elastic foundation are compared with appropriate values of cylindrical shell composed of the pure ceramic, effects on the critical torsional loads of cylindrical shells with and without Winkler elastic foundation are (33.44%), (31.51%) and (33.78%), (31.83%), respectively, for *R/h*=50, whereas, these effects on the critical torsional loads of shells with and without Winkler elastic foundation are (27.26%), (25.59%) and (34.19%), (31.89%), respectively, for *R/h*=200. Consequently, as *R/h* increases, the influence of the elastic foundation on the values of critical torsional load for Ti-6Al-4V/FG/ZrO_2 cylindrical shell increases. Furthermore, it is found that the effect of compositional profiles on the values of critical torsional loads of Ti-6Al-4V/FG/ZrO_2 cylindrical shells with different profiles of a FG middle layer is not depending on the variation of the ratio, *R/h*, for unconstrained shells, whereas, this effect increases for the cylindrical shells with an elastic foundation, as the ratio, *R/h*, increases.

The variation of critical torsional loads of three-layered Ti-6Al-4V/FG/ZrO_2 cylindrical shell with different profiles of a FG middle layer versus the Winkler elastic foundation stiffness, K_w, are presented in Table 3. The cylindrical shell characteristics are taken to be $h/h_m = h/h_c = 4;\ R/h = 100,\ L/R = 2$. A FG layer has three types of profiles, i.e. quadratic, inverse quadratic and exponential. As the foundation stiffness, K_w, increases, the values of critical torsional loads, n_{cr} and γ_{cr}, increase. The influences of compositional profiles on critical torsional loads of three-layered Ti-6Al-4V/FG/ZrO_2 cylindrical shell with different types of a FG layer decrease, as K_w increases.

Fig. 3 shows, variations of critical torsional loads of the pure metal, pure ceramic shells and Ti-6Al-4V/FG/ZrO_2 cylindrical shells with different profiles of a FG middle layer with and without an elastic foundation versus the ratio, *L/R*. The compositional profiles of a FG layer are taken into account as quadratic and inverse quadratic. The cylindrical shell characteristics and Winkler elastic foundation stiffness are taken to be $h/h_m = h/h_c = 4;\ R/h = 100;\ K_w = 1\times10^7\ \mathrm{N/m^3}$. It is seen that the values of critical torsional loads for cylindrical shells with and without an elastic foundation decrease, as the ratio, *L/R,* increase.

As Ti-6Al-4V/FG/ZrO_2 cylindrical shells with and without the Winkler elastic foundation are compared with those pure metal cylindrical shells with and without the Winkler elastic foundation, respectively, influences of quadratic and inverse quadratic compositional profiles of a FG middle layer on the values of critical torsional load remain constant for unconstrained cylindrical shells, whereas, these influences on the values of critical torsional loads for shells with the Winkler elastic foundation decrease, as *L/R* increases. In addition, as *L/R* increases, the influence of the Winkler elastic foundation on the values of critical torsional loads for pure metal, pure ceramic and Ti-6Al-4V/FG/ZrO_2 cylindrical shells increases.

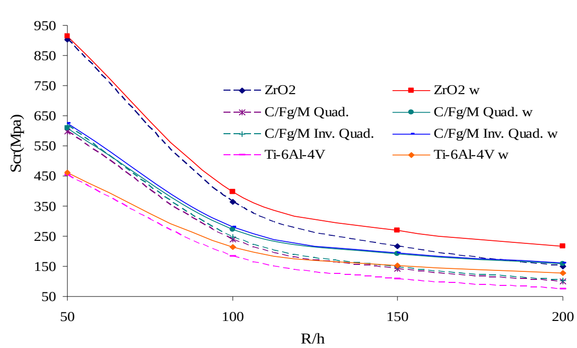

Fig. 2
Variations of critical torsional loads of cylindrical shells with and without an elastic foundation versus *R/h* for different compositional profiles of a FG layer.

Table 3
Variations of critical torsional loads of cylindrical shells versus the foundation stiffness, K_w, for different compositional profiles of a FG layer

K_w (N/m^3)	$S_{cr} \times 10^{-6}$ MPa (γ_{cr}, n_{cr})				
	ZrO_2	Ti-6Al-4V/FG/ZrO_2			Ti-6Al-4V
		Quad.	Inv. Quad	Exp	
0	363.18(0.3,7)	239.77(0.3,7)	247.49(0.3,7)	244.46(0.31,7)	182.24(0.3,7)
1×10^7	395.94(0.32,7)	271.65(0.33,7)	279.82(0.33,7)	276.39(0.33,7)	213.06(0.37,8)
5×10^7	485.88(0.4,8)	351.28(0.45,9)	359.27(0.44,9)	356.72(0.45,9)	282.37(0.46,9)
1×10^8	563.21(0.46,9)	415.97(0.5,9)	424.94(0.50,9)	421.66(0.50,9)	338.71(0.53,10)

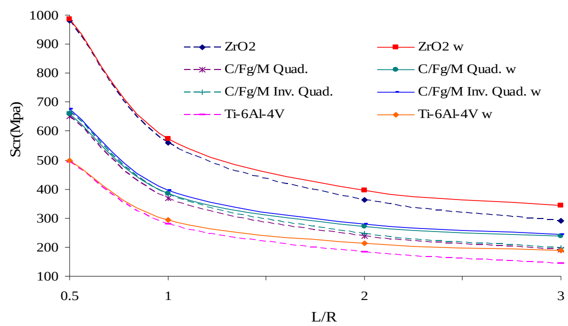

Fig. 3
Variations of critical torsional loads of cylindrical shells with and without an elastic foundation versus *L/R* for different compositional profiles of a FG layer
($h/h_m = h/h_c = 4$; $R/h = 100$, $K_w = 1\times10^7$ N/m^3).

6 CONCLUSIONS

In this study, the torsional stability analysis is presented for thin cylindrical with the functionally graded (FG) middle layer resting on the Winker elastic foundation. The mechanical properties of functionally graded material (FGM) are assumed to be graded in the thickness direction according to a simple power law and exponential distributions in terms of the volume fractions of the constituents. The fundamental relations, the basic equations of the three-layered cylindrical shell with a FG middle layer resting on the Winker elastic foundation under the torsional load are derived. The governing equations are solved by using the Galerkin method. The numerical results reveal that the variations of the shell thickness-to-FG layer thickness ratio, radius-to-shell thickness ratio, lengths-to-radius ratio, foundation stiffness and the compositional profiles have significant effects on the critical torsional load of three-layered cylindrical shells with a FG middle layer. The results are verified by comparing the obtained values with those in the existing literature.

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