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An Exact Solution for Kelvin-Voigt Model Classic Coupled Thermo Viscoelasticity in Spherical Coordinates

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ABSTRACT

In this paper, the classic Kelvin-Voigt model coupled thermo-viscoelasticity model of hollow and solid spheres under radial symmetric loading condition is considered. A full analytical method is used and an exact unique solution of the classic coupled equations is presented. The thermal and mechanical boundary conditions, the body force, and the heat source are considered in the most general forms and where no limiting assumption is used. This generality allows simulate varieties of applicable problems. At the end, numerical results are presented and compared with classic theory of thermoelasticity.

Keywords : Coupled thermo viscoelasticity; Hollow sphere; Exact solution; Kelvin-Voigt method.

1 INTRODUCTION

AT the classical uncoupled theory of thermo-elasticity predicts, such as the heat equation is of a parabolic type, predicting infinite speeds of propagation for heat waves and the equation of heat conduction of this theory does not contain any elastic terms contrary to the fact that the elastic changes produce heat effects, etc. which that are not compatible with physical observations. Therefore the theory coupled thermo-elasticity has received much attention in the literature during the past several decades [1].

The numbers of papers that present the closed-form or analytical solution for the coupled thermoelasticity function. Hetnarski [2] found the solution of the thermo elasticity in the form of a series function. Hetnarski and Ignaczak presented a study of the one-dimensional thermo elastic waves produced by an instantaneous plane source of heat in homogeneous isotropic infinite and semi-infinite bodies of the Green-Lindsay type [3]. These authors also presented an analysis for laser-induced waves propagating in an absorbing thermo elastic semi-space of the Green-Lindsay type [4]. Georgiadis and Lykotrafitis obtained a three-dimensional transient thermo elastic solution for Rayleigh-type disturbances propagating on the surface of the half-space [5]. Wagner [6] presented the fundamental matrix of a system of partial differential operators that governs the diffusion of heat and the strains in elastic media. This method can be used to predict the temperature distribution and the strains by an instantaneous point heat, point source of heat, or by a suddenly applied dilate force. Bahtui and Eslami [7] studied the coupled thermoelastic response of a functionally graded circular cylindrical shell, and used a Galerkin finite element formulation in the space domain and the Laplace transform in the time domain. Bagri and Eslami [8] presented a solution for one-dimensional generalized thermoelasticity of a disk. They employed the Laplace transform and Galerkin finite element method to solve the governing equations.

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The Kelvin-Voigt model at problem of magneto-thermo-viscoelasticity has extensive uses. This theory uses in divers fields, such as geophysics for understanding the effect of the Earth`s magnetic field on seismic wave, damping of acoustic waves in a magnetic field, development of a highly sensitive superconducting magnetometer, electrical power engineering, optics, supersonic airplanes, etc. [9]. Knopoff [10], Chadwick [11] and Nowacki [12] studied these types of problems at the beginning. Misra et al. [13, 14] Abd-Alla et al. [15] and Kaliski [16] studied these types of problems considering viscoelastic solid of Kelvin-Voigt type. Abd-Alla and Mahmoud [17] presented an analytical solution for magneto-thermo-viscoelastic non-homogeneous medium with a spherical cavity subjected to periodic loading. Song et al. [18] studied the problems of a plane harmonic wave at the interface between to viscoelastic media under generalized thermo-viscoelastic theory when the media permeate a uniform magnetic field. S.M. Abo-Dahab [19] studied the effects of the thermally induced vibration, magnetic field and viscoelasticity in an isotropic homogeneous unbounded body with a spherical cavity.

Sharma et al. [20, 21] employed kelvin-Voigt model of viscoelasticity to study Rayleigh-Lamb waves in thermo-elastic plates in the context of generalized (GL and LS) and coupled theories of thermoelasticity. Roy-Chudhuri and Mukhopdhyay [22] studied the effect of rotation and relaxation time on plane waves in an infinite generalized thermoviscoelastic solid of Kelvin-Voigt type with the entire medium rotating with a uniform angular velocity. M. I. A. Othman and I. A. Abbas [23] presented an investigation of the temperature, displacement, and stress in a viscoelastic half space of Kelvin-Voigt type which the no dimensional governing equations are solved by the finite element method. Avijt Kar and M. Kanoria [24] presented an interaction due to step input of temperature on the stress free boundaries of a homogeneous visco-elastic isotropic spherical shell in the context of generalized theories of thermo-elasticity. Ezzat et al. [25, 26] applied the state space approach to one-dimensional problems of generalized thermo-visco- elasticity.

In the present work a full analytical method is used to obtain the response of the governing equations, therefore an exact solution is presented. The method of solution is based on the Fourier's expansion and Eigen- function methods, which are traditional and routine methods in solving the partial differential equations. Since the coefficients of equations are not functions of the time variable (t), an exponential form is considered for the general solution matched with the physical wave properties of thermal and mechanical waves. For the particular solution, that is the response to mechanical and thermal shocks, the Eigen-function method and Laplace transformation is used. This work is the extension of the previous paper that presented an exact solution in the spherical coordinates [27].

2 GOVERNING EQUATIONS

A hollow sphere with inner and outer radius r_i and r_o, respectively, made of isotropic material subjected to radial-symmetric mechanical and thermal shock loads, is considered. If u is the displacement component in the radial direction, the strain-displacement relations in spherical coordinates are as follow:

$$\begin{aligned}
&\varepsilon_{rr} = u_{,r} \\
&\varepsilon_{\theta\theta} = r^{-1}\left(u + v_{,\theta}\right)u \\
&\varepsilon_{\varphi\varphi} = r^{-1}\left(\frac{1}{\sin\theta}\omega_{,\varphi} + u + v\cot\theta\right) \\
&\varepsilon_{r\theta} = 1/2\left(r^{-1}u_{,\theta} + v_{,r} - r^{-1}v\right) \\
&\varepsilon_{\theta\varphi} = 1/2\, r^{-1}\left(\frac{1}{\sin\theta}v_{,\varphi} + \omega_{,\theta} - v\cot\theta\right) \\
&\varepsilon_{r\varphi} = 1/2\left(\frac{1}{r\sin\theta}u_{,\varphi} - \frac{\omega}{r} + \omega_{,r}\right) \\
&\varepsilon_{\theta\varphi} = 1/2\, r^{-1}\left(\frac{1}{\sin\theta}v_{,\varphi} + \omega_{,\theta} - v\cot\theta\right)
\end{aligned} \tag{1}$$

where $(,)$ denotes partial derivative. The non-vanishing displacement component is $u_r = u\left(r, \mathrm{t}\right)$, so that,

$$\varepsilon_{\theta\theta} = \varepsilon_{\varphi\varphi} = \frac{u}{r} \qquad \varepsilon_{rr} = u_{,r} \tag{2}$$

The stress-strain-temperature relation for generalized thermo-viscoelastic Kelvin-voigt material type is

$$\sigma_{ij} = \left(\lambda\left(1+\tau_0\frac{\partial}{\partial t}\right)\varepsilon_{kk} - \gamma\left(T+\tau_2\dot{T}\right)\right)\delta_{ij} + 2\mu\left(1+\tau_0\frac{\partial}{\partial t}\right)\varepsilon_{ij} \tag{3}$$

where $i, j = r,\theta,\varphi,\lambda$ and μ are Lame's constants, $\gamma = \alpha_t\left(3\lambda+2\mu\right), \alpha_t$ is the thermal expansion coefficient, T is the absolute temperature, τ_2 is the thermal relaxation, τ_0 is the mechanical relaxation time (sensitive part of the term of the viscosity).

For a spherical radial-symmetric system the non-vanishing stresses components may written as:

$$\begin{aligned} \sigma_{rr} &= (\lambda+2\mu)\left(1+\tau_0\frac{\partial}{\partial t}\right)u_{,r} + \lambda\left(1+\tau_0\frac{\partial}{\partial t}\right)\frac{2u}{r} - \gamma\left(T+\tau_2\dot{T}\right) \\ \sigma_{\theta\theta} &= (\lambda+\mu)\left(1+\tau_0\frac{\partial}{\partial t}\right)\frac{2u}{r} + \lambda\left(1+\tau_0\frac{\partial}{\partial t}\right)u_{,r} - \gamma\left(T+\tau_2\dot{T}\right) \end{aligned} \tag{4}$$

The equation of motion in the radial direction is

$$\sigma_{rr,r} + \frac{2}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) + F(r,t) = \rho\ddot{u} \tag{5}$$

where $F\left(r,t\right)$ is the body force in the radial direction. Substituting Eq. (4) into Eq. (5), the Navier equation in terms of the displacement components is obtained as:

$$\left(1+\tau_0\frac{\partial}{\partial t}\right)u_{,rr} + \left(1+\tau_0\frac{\partial}{\partial t}\right)\frac{2u_{,r}}{r} - \left(1+\tau_0\frac{\partial}{\partial t}\right)\frac{2u}{r^2} - \frac{\gamma}{(\lambda+2\mu)}\left(T_{,r}+\tau_2\dot{T}_{,r}\right) - \frac{\rho}{(\lambda+2\mu)}\ddot{u} = -\frac{1}{(\lambda+2\mu)}F(r\,t) \tag{6}$$

Heat conduction equation in radial-symmetric direction with the mechanical coupling term is

$$T_{,rr} + \frac{2}{r}T_{,r} - \frac{\rho C_v}{k}(\dot{T}+\tau_1\ddot{T}) - \frac{\gamma T_0}{k}(\dot{u}_{,r} + \frac{2}{r}\dot{u}) = -\frac{1}{k}Q(r,t) \tag{7}$$

where ρ is density of the material, k is thermal conductivity, C_v is specific heat of the material per unit mass, τ_1 is thermal relaxation parameter, T_0 is reference temperature solid, $Q\left(r,t\right)$ is heat generation source. Mechanical and thermal boundary conditions are

$$\begin{aligned} &C_{11}u\left(r_i,t\right) + C_{12}u_{,r}\left(r_i,t\right) + C_{13}T\left(r_i,t\right) = f_1(t) \\ &C_{21}u\left(r_0,t\right) + C_{22}u_{,r}\left(r_0,t\right) + C_{23}T\left(r_0,t\right) = f_2(t) \\ &C_{31}T\left(r_i,t\right) + C_{32}T_{,r}\left(r_i,t\right) = f_3(t) \\ &C_{41}T\left(r_0,t\right) + C_{42}T_{,r}\left(r_0,t\right) = f_4(t) \end{aligned} \tag{8}$$

where C_{ij} are mechanical and thermal coefficients and by assigning different values for them, different type of mechanical and thermal boundary condition may be obtained. These boundary conditions include the displacement, strain, stress (for the first and second boundary conditions), specified temperature, and convection, heat flux condition (for the third and fourth boundary conditions). The f_1 to f_4 are arbitrary functions which show mechanical and thermal shocks, respectively. The initial boundary conditions are assumed in the following general form

$$u(r,0)=f_5(r) \qquad u_{,t}(r,0)=f_6(r) \qquad T(r,0)=f_7(r) \tag{9}$$

where f_5 to f_7 are arbitrary functions which show initial distributions of displacement and temperature, respectively.

3 SOLUTION

The Eq. (1) and Eq. (3) constitute a system of nonhomogeneous partial differential equations with non-constant coefficients (functions of the radius only) has general and particular solution.

3.1 General solution with homogeneous boundary conditions

A form of solution can be suitable for Eq. (1) and Eq. (3) may be assumed for the general solution as:

$$u(r,t)=\left[u^*(r)\right]e^{\alpha t} \qquad T(r,t)=\left[\theta^*(r)\right]e^{\alpha t} \tag{10}$$

By substituting Eq. (10) into the homogeneous parts of Eqs. (6) and (7) yields,

$$\begin{gathered}(1+\alpha\tau_0)u^{*''}+(1+\alpha\tau_0)\frac{2}{r}u^{*'}-(1+\alpha\tau_0)\frac{2}{r^2}u^*-\left[\frac{\gamma(1+\alpha\tau_2)}{(\lambda+2\mu)}\right]\theta^{*'}-\left[\frac{\rho}{(\lambda+2\mu)}\right]\alpha^2u^*=0\\ \theta^{*''}+\frac{2}{r}\theta^{*'}-\left[\frac{\rho C_v(1+\alpha\tau_1)}{k}\right]\alpha\theta^*-\left(\frac{\gamma T_0}{k}\right)\alpha\left(u^{*'}+\frac{2}{r}u^*\right)=0\end{gathered} \tag{11}$$

Eq. (11) is a system of ordinary differential equations, where the prime symbol (') shows differentiation with respect to radial variable r and if suppose:

$$d_1=\frac{-\gamma(1+\alpha\tau_2)}{(1+\alpha\tau_0)(\lambda+2\mu)} \quad , \quad d_2=\frac{-\rho}{(1+\alpha\tau_0)(\lambda+2\mu)} \quad , \quad d_3=\frac{-\rho C_v(1+\alpha\tau_1)}{k} \quad , \quad d_3=-\frac{\gamma T_0}{k} \tag{12}$$

3.2 Changes in dependent variables

To obtain a solution for Eq. (11), the dependent variables are changed as:

$$u^*(r)=r^{-1/2}u(r) \qquad \theta^*(r)=r^{-1/2}\theta(r) \tag{13}$$

Substituting Eq. (13) into Eq. (11) gives

$$\begin{gathered}u''+\frac{1}{r}u'-\frac{9}{4}\frac{1}{r^2}u-d_1\frac{1}{2}\frac{1}{r}\theta+d_1\theta'+d_2\alpha^2u=0\\ \theta''+\frac{1}{r}\theta'-\frac{1}{4}\frac{1}{r^2}\theta+d_3\alpha\theta+d_4\alpha u'+d_4\alpha\frac{3}{2}\frac{1}{r}u=0\end{gathered} \tag{14}$$

3.3 Solution justification

The first solution u_1 and θ_1 are considered for the solid sphere as:

$$u_1(r) = A_1 J_{3/2}(\beta r) \qquad \theta_1(r) = B_1 J_{1/2}(\beta r) \tag{15}$$

Substituting Eq. (15) into Eq. (14) and using the formulas for derivatives of the Bessel function, such as: $J_n'(\beta r) = -\beta J_{n+1}'(\beta r) + (n/r) J_n(\beta r)$ and $J_n'(\beta r) = \beta J_{n-1}'(\beta r) - (n/r) J_n(\beta r)$, yield

$$\begin{aligned} &\left\{\left(-\beta^2 + d_2\alpha^2\right)A_1 - d_1\beta B_1\right\} J_{3/2}(\beta r) = 0 \\ &\left\{\alpha d_4 \beta A_1 + \left(-\beta^2 + d_3\alpha\right)B_1\right\} J_{1/2}(\beta r) = 0 \end{aligned} \tag{16}$$

Eq. (16) shows that u_1 and θ_1 can be the solution of Eq. (14) if and only if

$$\begin{bmatrix} \left(-\beta^2 + d_2\alpha^2\right) & -d_1\beta \\ \alpha d_4\beta & \left(-\beta^2 + d_3\alpha\right) \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{17}$$

The non-trivial solution of Eq. (17) is obtained by equating the determinant of this equation to zero and brings the first characteristic equation. The second solutions of u_1 and θ_1 are considered as:

$$\begin{aligned} &u_2(r) = A_2 J_{3/2}(\beta r) + A_3 r J_{5/2}(\beta r) \\ &\theta_2(r) = B_2 J_{1/2}(\beta r) + B_3 r J_{3/2}(\beta r) \end{aligned} \tag{18}$$

Substituting Eq. (18) into Eq. (14) yields

$$\begin{aligned} &\left\{(-\beta^2 + d_2\alpha^2)A_3 - d_1\beta B_3\right\} r J_{1/2}(\beta r) + \left\{(-\beta^2 + d_2\alpha^2)A_2 + (d_2\alpha^2 \frac{3}{\beta} - \beta)A_3 - d_1\beta B_2 - d_1 B_3\right\} J_{3/2}(\beta r) = 0 \\ &\left\{\alpha d_4\beta A_2 + \left(-\beta^2 + d_3\alpha\right)B_2 + 2\beta B_3 + 2\beta B_3\right\} J_{1/2}(\beta r) + \left\{\alpha d_4\beta A_3 + \left(-\beta^2 + d_3\alpha\right)B_3\right\} r J_{3/2}(\beta r) = 0 \end{aligned} \tag{19}$$

The expressions for u_2 and θ_2 can be the solution of Eq. (14) if and only if

$$\begin{bmatrix} \left(-\beta^2 + d_2\alpha^2\right) & -d_1\beta \\ \alpha d_4\beta & \left(-\beta^2 + d_3\alpha\right) \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{20}$$

$$\left(-\beta^2 + d_2\alpha^2\right)A_2 + \left(d_2\alpha^2 \frac{3}{\beta} - \beta\right)A_3 - d_1\beta B_2 - d_1 B_3 = 0 \tag{21}$$

$$\alpha d_4\beta A_2 + \left(-\beta^2 + d_3\alpha\right)B_2 + 2\beta B_3 = 0 \tag{22}$$

The non-trivial solution of Eq. (20) is obtained by equating the determinant to zero as:

$$\left(-\beta^2 + d_2\alpha^2\right)\left(-\beta^2 + d_3\alpha\right) + d_1 d_4 \alpha\beta^2 = 0 \tag{23}$$

The equality of Eq. (17) with Eq. (20) is interesting as it prevents mathematical dilemma and complexity, and a single value for the eigenvalue β simultaneously satisfies both characteristic equations yielded by Eq. (17) and Eq. (20). Eqs. (21) and (22) give the relation between A_2, A_3, B_2 and B_3, and they play as the balancing ratios that help Eq. (18) to be the second solution of the system of Eqs. (14). The complete unique general solutions for the solid sphere are

$$\begin{aligned} u^g(r) &= A_1 J_{3/2}(\beta r) + A_2\left[J_{3/2}(\beta r) + \xi_1 r J_{5/2}(\beta r)\right] \\ \theta^g(r) &= A_1 \xi_2 J_{1/2}(\beta r) + A_2\left[\xi_3 J_{1/2}(\beta r) + \xi_4 r J_{3/2}(\beta r)\right] \end{aligned} \tag{24}$$

Those for the hollow sphere are

$$\begin{aligned} u^g(r) &= A_1 J_{3/2}(\beta r) + A_2\left[J_{3/2}(\beta r) + \xi_1 r J_{5/2}(\beta r)\right] + A_3 Y_{3/2}(\beta r) + A_4\left[Y_{3/2}(\beta r) + \xi_1 r Y_{5/2}(\beta r)\right] \\ \theta^g(r) &= A_1 \xi_2 J_{1/2}(\beta r) + A_2\left[\xi_3 J_{1/2}(\beta r) + \xi_4 r J_{3/2}(\beta r)\right] + A_3 \xi_2 Y_{1/2}(\beta r) + A_2\left[\xi_3 Y_{1/2}(\beta r) + \xi_4 r Y_{3/2}(\beta r)\right] \end{aligned} \tag{25}$$

where $\xi_1 - \xi_4$ are ratios obtained from Eqs.(17), (20)-(22) and are given in Appendix A. Substituting u^g and θ^g in the homogeneous form of the boundary conditions (Eq.(8)), four linear algebraic equations are obtained as:

$$\begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} \\ \mu_{21} & \mu_{22} & \mu_{23} & \mu_{24} \\ \mu_{31} & \mu_{32} & \mu_{33} & \mu_{34} \\ \mu_{41} & \mu_{42} & \mu_{43} & \mu_{44} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{26}$$

where μ_{ij} are given in the Appendix. Setting the determinant of the coefficients of Eq. (26) equal to zero, the second characteristic equation is obtained. A simultaneous solution of this equation and non-trivial solution of Eq. (17) results in an infinite number of two eigenvalues, β_n and λ_n. Therefore, u^g and θ^g are rewritten as:

$$\begin{aligned} u^g(r) &= A_1\left[J_{3/2}(\beta r) + [\xi_5 J_{3/2}(\beta r) + \xi_6 r J_{5/2}(\beta r)] + \xi_7 Y_{3/2}(\beta r) + [\xi_8 Y_{3/2}(\beta r) + \xi_9 r Y_{5/2}(\beta r)]\right] \\ \theta^g(r) &= A_1\left[\xi_{10} J_{1/2}(\beta r) + [\xi_{11} J_{1/2}(\beta r) + \xi_{12} r J_{3/2}(\beta r)] + \xi_{13} Y_{1/2}(\beta r) + [\xi_{14} Y_{1/2}(\beta r) + \xi_{15} r Y_{3/2}(\beta r)]\right] \end{aligned} \tag{27}$$

where ξ_5 to ξ_{15} are ratios presented in Appendix A and are obtained from Eq.(26). Let us show the functions in the brackets of Eq. (27) by functions H_1 and H_0 as:

$$\begin{aligned} H_1(\beta_n r) &= J_{3/2}(\beta r) + [\xi_5 J_{3/2}(\beta r) + \xi_6 r J_{5/2}(\beta r)] + \xi_7 Y_{3/2}(\beta r) + [\xi_8 Y_{3/2}(\beta r) + \xi_9 r Y_{5/2}(\beta r)] \\ H_0(\beta_n r) &= \xi_{10} J_{1/2}(\beta r) + [\xi_{11} J_{1/2}(\beta r) + \xi_{12} r J_{3/2}(\beta r)] + \xi_{13} Y_{1/2}(\beta r) + [\xi_{14} Y_{1/2}(\beta r) + \xi_{15} r Y_{3/2}(\beta r)] \end{aligned} \tag{28}$$

According to the Sturm–Liouville theories, these functions are orthogonal with respect to the weight function r as:

$$\int_{r_1}^{r_0} H(\beta_n r) H(\beta_m r) r dr = \begin{Bmatrix} 0 & n \neq m \\ \|H(\beta_n r)\|^2 & n = m \end{Bmatrix} \tag{29}$$

where $\|H(\beta_n r)\|$ is the norm of the H function and equals

$$\left\|H(\beta_n r)\right\| = \left[\int_{r_1}^{r_0} rH^2(\beta_n r)dr\right]^{1/2} \tag{30}$$

Due to the orthogonality of function *H*, every piecewise continuous function, such as $f\left(r\right)$, can be expanded in terms of the function *H* (either for H_0 or H_1) and is called the *H*-Fourier series as:

$$f(r) = \sum_{n=1}^{\infty} e_n H(\beta_n r) \tag{31}$$

where e_n equals

$$e_n = \frac{1}{\left\|H_1(\beta_n r)\right\|^2}\int_{r_i}^{r_0} f(r)H(r)rdr = \tag{32}$$

According to the numerical results, there are three groups for eigenvalues λ_n, where the first λ_{n1} is real and negative, and the second and third ones, λ_{n2} and λ_{n3}, are conjugate complex with a negative real part, $-\zeta_n\omega_n$, and an imaginary part, $\pm\omega_{dn}$. Terms ω_{dn} and ω_n are the damped and nondamped thermal-mechanical natural frequencies, and ζ_n is the damping ratio for the *n*-th natural mode. The non-trivial solution of Eq. (15) is an algebraic equation in polynomial form, and the determinant of Eq. (24) is an algebraic equation in the Bessel function form. The exact analytical solution for this system of nonlinear algebraic equations is complicated, and the numerical method of solution is employed in this paper. Since the Bessel functions are periodic, the system has an infinite number of roots. The numerical results of β_n and λ_n for 50 roots are presented in Sec. 4. Using Eqs. (10), (13), (27) and (28), the displacement and temperature distributions due to the general solution become

$$\begin{aligned} u^p\left(r,t\right) &= r^{-1/2}\sum_{n=1}^{\infty}\left\{a_n e^{\lambda n_1 t} + e^{-\varsigma\omega_n t}\left[b_n \cos\omega_{dn}t + c_n \sin\omega_{dn}t\right]\right\}H_1(\beta_n r) \\ T^p\left(r,t\right) &= r^{-1/2}\sum_{n=1}^{\infty}\left\{a_n e^{\lambda n_1 t} + e^{-\varsigma\omega_n t}\left[b_n \cos\omega_{dn}t + c_n \sin\omega_{dn}t\right]\right\}H_0(\beta_n r) \end{aligned} \tag{33}$$

Using the initial conditions (Eq. (9)) and with the help of Eqs. (31)-(33), four unknown constants, a_n, b_n, c_n and d_n are obtained.

3.4 Particular solution with non-homogeneous boundary conditions

The general solutions may be used as proper functions for guessing the particular solution suitable to the non-homogeneous parts of the Eqs. (6) to (7) and the non-homogeneous boundary conditions (8) as:

$$\begin{aligned} u^p\left(r,t\right) &= r^{-1/2}\sum_{n=1}^{\infty}\left\{G_{1n}(t)J_{3/2}(\beta_n r) + G_{2n}(t)rJ_{5/2}(\beta_n r)] + r^2 G_{5n}(t)\right\} \\ T^p\left(r,t\right) &= r^{-1/2}\sum_{n=1}^{\infty}\left\{G_{3n}(t)J_{1/2}(\beta_n r) + G_{4n}(t)rJ_{3/2}(\beta_n r)] + r^2 G_{6n}(t)\right\} \end{aligned} \tag{34}$$

For solid sphere, the second type of Bessel function Y is excluded. It is necessary and suitable to expand the body force $F\left(r,t\right)$ and heat source $Q\left(r,t\right)$ in the H-Fourier expansion form as:

$$r^{-1/2}F(r,t)=\sum_{n=1}^{\infty}F_n(t)H_1(\beta_n r)$$
$$r^{-1/2}Q(r,t)=\sum_{n=1}^{\infty}Q_n(t)H_0(\beta_n r) \tag{35}$$

where $F_n\left(t\right)$ and $Q_n\left(t\right)$ are

$$F_n(t)=\frac{1}{\left\|H_1(\beta_n r)\right\|^2}\int_{r_i}^{r_0}F(r,t)H_1(\beta_n r)r^{3/2}dr$$
$$Q_n(t)=\frac{1}{\left\|H_0(\beta_n r)\right\|^2}\int_{r_i}^{r_0}Q(r,t)H_0(\beta_n r)r^{3/2}dr \tag{36}$$

Substituting Eqs. (34) and (35) into the nonhomogeneous form of Eqs.(6) and (7) yields

$$\begin{aligned}
&[-\beta_n{}^2G_{1n}(t)-\tau_0\beta_n{}^2\dot{G}_{1n}(t)-\beta_nG_{2n}(t)-\tau_0\beta_n\dot{G}_{2n}(t)-b_1\beta_n\left(G_{3n}(t)+\tau_2\dot{G}_{3n}(t)\right)\\
&-b_1\left(G_{4n}(t)+\tau_2\dot{G}_{4n}(t)\right)+b_2\ddot{G}_{1n}(t)+b_2\ddot{G}_{2n}(t)\frac{3}{\beta_n}+\left(1+\xi_5\right)b_7\left(G_{5n}(t)+\tau_0\dot{G}_{5n}(t)\right)\\
&+b_7\xi_6\frac{3}{\beta_n}\left(G_{5n}(t)+\tau_0\dot{G}_{5n}(t)\right)+b_8\left(1+\xi_5\right)\left(G_{6n}(t)+\tau_2\dot{G}_{6n}(t)\right)+b_8\xi_6\frac{3}{\beta_n}\left(G_{6n}(t)+\tau_2\dot{G}_{6n}(t)\right)\\
&+b_9\left(1+\xi_5\right)\ddot{G}_{5n}(t)+b_9\xi_6\frac{3}{\beta_n}\ddot{G}_{5n}(t)-b_3\left(1+\xi_5\right)F_n(t)-b_3\xi_6\frac{3}{\beta_n}F_n(t)]J_{3/2}(\beta r)\\
&+[\beta_n{}^2G_{2n}(t)+\tau_0\beta_n{}^2\dot{G}_{2n}(t)+b_1\beta_n\left(G_{4n}(t)+\tau_2\dot{G}_{4n}(t)\right)-b_2\ddot{G}_{2n}(t)\\
&-b_7\xi_6\left(G_{5n}(t)+\tau_0\dot{G}_{5n}(t)\right)-b_8\xi_6\left(G_{6n}(t)+\tau_2\dot{G}_{6n}(t)\right)-b_9\xi_6\ddot{G}_{5n}(t)+b_3\xi_6F_n(t)]rJ_{1/2}(\beta r)=0\\
&[b_5\beta_n\dot{G}_{1n}(t)-\beta_n{}^2G_{3n}(t)+b_4\left(\dot{G}_{3n}(t)+\tau_1\ddot{G}_{3n}(t)\right)+2\beta_nG_{4n}(t)+b_{11}(\xi_{10}+\xi_{11})\dot{G}_{5n}(t)\\
&+b_{10}(\xi_{10}+\xi_{11})G_{6n}(t)+b_{12}(\xi_{10}+\xi_{11})\left(\dot{G}_{6n}(t)+\tau_1\ddot{G}_{6n}(t)\right)-b_6(\xi_{10}+\xi_{11})Q_n(t)]J_{1/2}(\beta r)\\
&+[b_5\beta_n\dot{G}_{2n}(t)-\beta_n{}^2G_{4n}(t)+b_4\left(\dot{G}_{4n}(t)+\tau_1\ddot{G}_{4n}(t)\right)+b_{11}\xi_{12}\dot{G}_{5n}(t)\\
&+b_{10}\xi_{12}G_{6n}(t)+b_{12}\xi_{12}\left(\dot{G}_{6n}(t)+\tau_1\ddot{G}_{6n}(t)\right)-b_6\xi_{12}Q_n(t)]rJ_{3/2}(\beta r)=0
\end{aligned} \tag{37}$$

where b_6-b_{12} are the coefficients of H-expansion and b_1-b_5 are given in Appendix A. The initial boundary conditions for the particular solutions are assumed in the following general form

$$u\left(r,0\right)=0 \qquad u_{,t}\left(r,0\right)=0 \qquad T\left(r,0\right)=0 \tag{38}$$

Therefore

$$G_{1n}(0)=G_{2n}(0)=G_{3n}(0)=G_{4n}(0)=G_{5n}(0)=G_{6n}(0)=0$$
$$\dot{G}_{1n}(0)=\dot{G}_{2n}(0)=\dot{G}_{5n}(0)=0 \tag{39}$$

The guessed functions (Eq. (34)) can satisfy the nonhomogeneous part of navier equation and heat equation Eq. (37) if and only if

$$
\begin{aligned}
&-\beta_n{}^2 G_{1n}(t)-\tau_0\beta_n{}^2\dot{G}_{1n}(t)-\beta_n G_{2n}(t)-\tau_0\beta_n\dot{G}_{2n}(t)-b_1\beta_n\left(G_{3n}(t)+\tau_2\dot{G}_{3n}(t)\right)-b_1\left(G_{4n}(t)+\tau_2\dot{G}_{4n}(t)\right)\\
&+b_2\ddot{G}_{1n}(t)+b_2\ddot{G}_{2n}(t)\frac{3}{\beta_n}+\left(1+\xi_5\right)b_7\left(G_{5n}(t)+\tau_0\dot{G}_{5n}(t)\right)+b_7\xi_6\frac{3}{\beta_n}\left(G_{5n}(t)+\tau_0\dot{G}_{5n}(t)\right)\\
&+b_8\left(1+\xi_5\right)\left(G_{6n}(t)+\tau_2\dot{G}_{6n}(t)\right)+b_8\xi_6\frac{3}{\beta_n}\left(G_{6n}(t)+\tau_2\dot{G}_{6n}(t)\right)+b_9\left(1+\xi_5\right)\ddot{G}_{5n}(t)+b_9\xi_6\frac{3}{\beta_n}\ddot{G}_{5n}(t)\\
&-b_3\left(1+\xi_5\right)F_n(t)-b_3\xi_6\frac{3}{\beta_n}F_n(t)=0\\
&\beta_n{}^2 G_{2n}(t)+\tau_0\beta_n{}^2\dot{G}_{2n}(t)+b_1\beta_n\left(G_{4n}(t)+\tau_2\dot{G}_{4n}(t)\right)-b_2\ddot{G}_{2n}(t)\\
&-b_7\xi_6\left(G_{5n}(t)+\tau_0\dot{G}_{5n}(t)\right)-b_8\xi_6\left(G_{6n}(t)+\tau_2\dot{G}_{6n}(t)\right)-b_9\xi_6\ddot{G}_{5n}(t)+b_3\xi_6 F_n(t)=0\\
&b_5\beta_n\dot{G}_{1n}(t)-\beta_n{}^2 G_{3n}(t)+b_4\left(\dot{G}_{3n}(t)+\tau_1\ddot{G}_{3n}(t)\right)+2\beta_n G_{4n}(t)+b_{11}(\xi_{10}+\xi_{11})\dot{G}_{5n}(t)\\
&+b_{10}(\xi_{10}+\xi_{11})G_{6n}(t)+b_{12}(\xi_{10}+\xi_{11})\left(\dot{G}_{6n}(t)+\tau_1\ddot{G}_{6n}(t)\right)-b_6(\xi_{10}+\xi_{11})Q_n(t)=0\\
&b_5\beta_n\dot{G}_{2n}(t)-\beta_n{}^2 G_{4n}(t)+b_4\left(\dot{G}_{4n}(t)+\tau_1\ddot{G}_{4n}(t)\right)+b_{11}\xi_{12}\dot{G}_{5n}(t)\\
&+b_{10}\xi_{12}G_{6n}(t)+b_{12}\xi_{12}\left(\dot{G}_{6n}(t)+\tau_1\ddot{G}_{6n}(t)\right)-b_6\xi_{12}Q_n(t)=0
\end{aligned}
\tag{40}
$$

Taking the Laplace transform of Eq. (38) and using two boundary conditions of Eq. (8) (for solid cylinders, only the second and fourth boundary conditions are applicable) give

$$
\begin{bmatrix}
-\beta_n^2-\tau_0\beta_n^2 s+b_2 s^2 & -\beta_n-\tau_0\beta_n s+b_2\frac{3}{\beta_n}s^2 & -b_1\beta_n(1+\tau_2 s) & -b_1(1+\tau_2 s) & \left((1+\xi_5)b_7+b_7\xi_6\frac{3}{\beta_n}\right)(1+\tau_0 s)+\left(b_9(1+\xi_5)+b_9\xi_6\frac{3}{\beta_n}\right)s^2 & \left(b_8(1+\xi_5)+b_8\xi_6\frac{3}{\beta_n}\right)(1+\tau_2 s)\\
0 & \beta_n^2+\tau_0\beta_n^2 s-b_2 s^2 & 0 & b_1\beta_n(1+\tau_2 s) & -\left[b_7\xi_6(1+\tau_0 s)+b_9\xi_6 s^2\right] & -b_8\xi_6(1+\tau_2 s)\\
b_5\beta_n s & 0 & -\beta_n^2+b_4(s+\tau_1 s^2) & 2\beta_n & b_{11}(\xi_{10}+\xi_{11})s & b_{10}(\xi_{10}+\xi_{11})+b_{12}(\xi_{10}+\xi_{11})(s+\tau_1 s^2)\\
0 & b_5\beta_n s & 0 & -\beta_n^2+b_4(s+\tau_1 s^2) & b_{11}\xi_{12}s & b_{10}\xi_{12}+b_{12}\xi_{12}(s+\tau_1 s^2)\\
b_{13} & b_{14} & b_{15} & b_{16} & b_{17} & b_{18}\\
0 & 0 & b_{19} & b_{20} & 0 & b_{21}
\end{bmatrix}
\begin{Bmatrix}
G_{1n}(s)\\ G_{2n}(s)\\ G_{3n}(s)\\ G_{4n}(s)\\ G_{5n}(s)\\ G_{6n}(s)
\end{Bmatrix}
=
\begin{Bmatrix}
-b_{23}F_n(s)\\ -b_3\xi_6 F_n(s)\\ b_6\left[(\xi_{10}+\xi_{11})\right]Q_n(s)\\ b_6\xi_{12}Q_n(st)\\ F_2(s)\\ F_4(s)
\end{Bmatrix}
\tag{41}
$$

where $b_{13} - b_{23}$ are given in Appendix A. Eq. (41) is solved for $G_{1n}(s) - G_{6n}(s)$ by the Cramer methods in the Laplace domain, where by the inverse Laplace transform the functions are transformed into the real time domain. In the process of solution, it is necessary to consider the following points:

1. Eq. (41) is in polynomial form function of the Laplace Parameter S (not the Bessel function form of S).Therefore, the exact inverse Laplace transform is possible and somehow simple.
2. For the hollow sphere, it is enough to include the second type of the Bessel function $Y(r)$ in the sequence of particular solution as:

$$\begin{aligned} &u^p(r,t) = \\ &r^{-1/2}\sum_{n=1}^{\infty}\{G_{1n}(t)J_{3/2}(\beta_n r) + G_{2n}(t)rJ_{5/2}(\beta_n r)] + [G_{3n}(t)Y_{3/2}(\beta_n r) + G_{4n}(t)rY_{5/2}(\beta_n r)] + r^2G_{5n}(t) + r^2G_{6n}(t)\} \\ &T^p(r,t) = \\ &r^{-1/2}\sum_{n=1}^{\infty}\left\{G_{7n}(t)J_{1/2}(\beta_n r) + G_{8n}(t)rJ_{3/2}(\beta_n r)] + [G_{9n}(t)Y_{1/2}(\beta_n r) + G_{10n}(t)rY_{3/2}(\beta_n r)] + r^2G_{11n}(t) + r^2G_{12n}(t)\right\} \end{aligned} \tag{42}$$

Substituting Eq. (42) in Eqs. (6) and (7), eight equations are obtained, where using the four boundary conditions (Eq. (8)), 12 functions are obtained for the hollow sphere.

4 RESULTS AND DISCUSSIONS

As an example, a solid sphere with radius one meter made of Aluminum is considered. The material properties are: $E = 70(GFa); v = 0.3; \alpha = 23\times10^{-6}(1/\mathrm{K}); \rho = 2707(kg/m^3); K = 204(W/mK); \tau_1 = \tau_2 = 0; \tau_0 = 10^{-12}; c = 903(J/kgK)$. The initial temperature T_0 is considered to be $293K$. Now, an instantaneous hot outside surface temperature $T(1,t) = 10^3\delta(t)$, where $\delta(t)$ is a unit Dirac function, is considered and the outside radius of the sphere is assumed to be fixed ($u(1,t) = 0$). Figs. 1-4 show the wave fronts for the displacement and temperature distributions along the radial direction, where the comparison is well justified between the elastic theory and viscoelastic theory. For the second example, a mechanical shock wave of the form $u(1,t) = 10^{12}\delta(t)$ is applied to the outside surface of the sphere, where surface is assumed to be at zero temperature ($T(1,t) = 0$). Figs. 5-8 show the wave fronts for the displacement and temperature, where the comparison is well justified between the elastic theory and viscoelastic theory. The convergence of the solutions for these examples is achieved by consideration of 2000 eigenvalues used for the *H*-Fourier expansion. More than these numbers of eigenvalues result in the increased round-off and truncation errors, which affect the quality of the graphs. The convergence of solution is faster for displacement in comparison with the temperature.

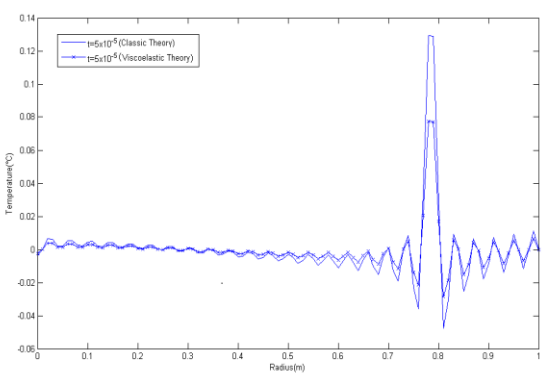

Fig.1
Temperature distribution due to input $T(1,t) = 10^3\delta(t)$ at 5×10^{-5} s.

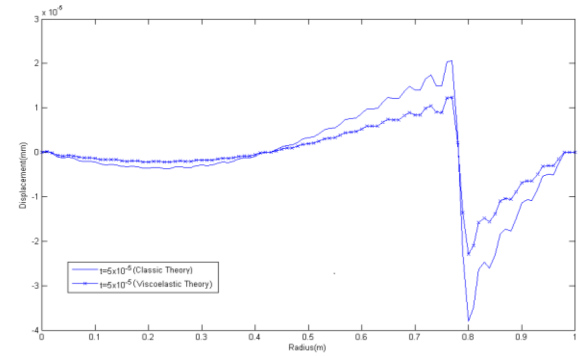

Fig.2
Displacement distribution due to input $T(1,t)=10^{3}\delta(t)$ at 5×10^{-5} s.

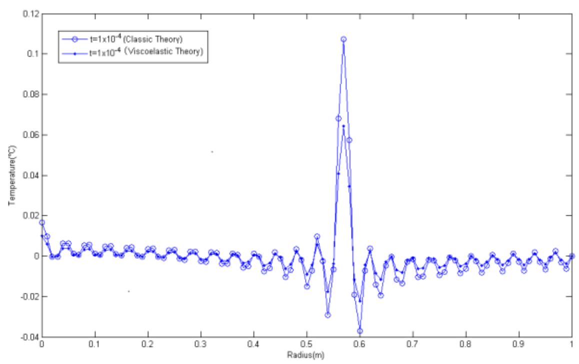

Fig.3
Temperature distribution due to input $T(1,t)=10^{3}\delta(t)$ at 1×10^{-4} s.

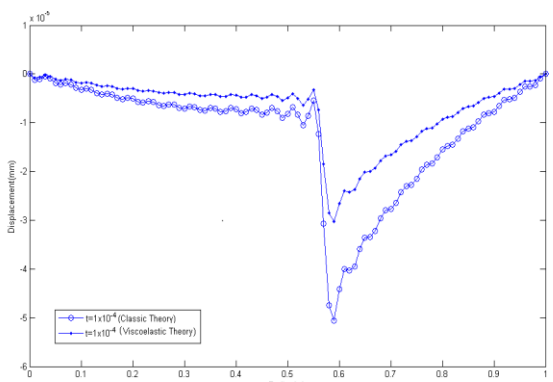

Fig.4
Displacement distribution due to input $T(1,t)=10^{3}\delta(t)$ at 1×10^{-4} s.

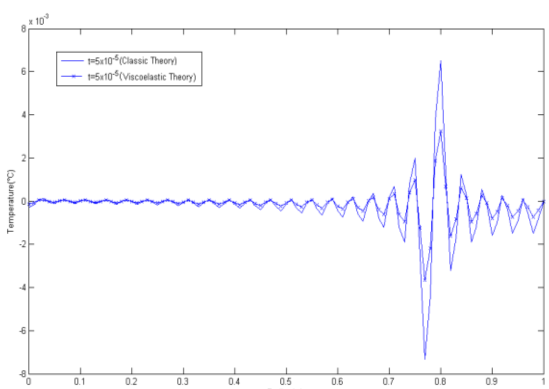

Fig.5
Temperature distribution due to input $u(1,t)=10^{12}\delta(t)$ at 5×10^{-5} s.

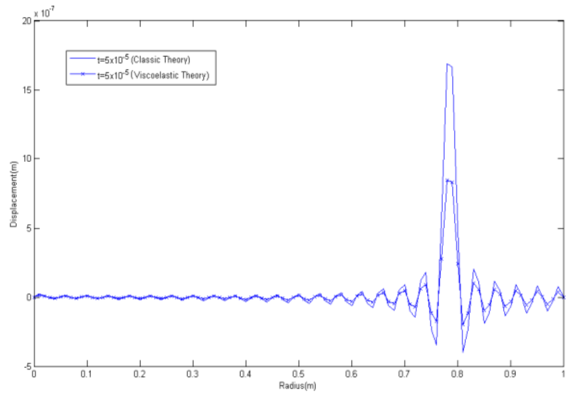

Fig.6
Displacement distribution due to input $u(1,t)=10^{12}\delta(t)$ at 5×10^{-5} s.

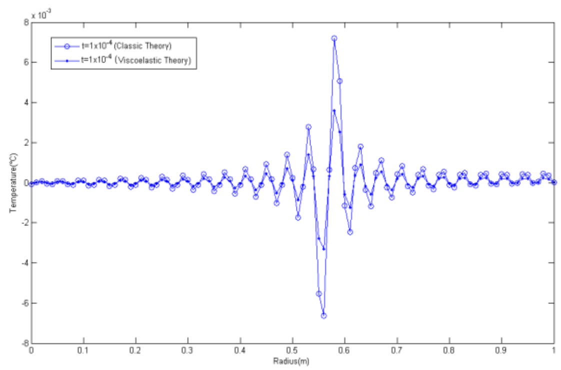

Fig.7
Temperature distribution due to input $u(1,t)=10^{12}\delta(t)$ at 1×10^{-4} s.

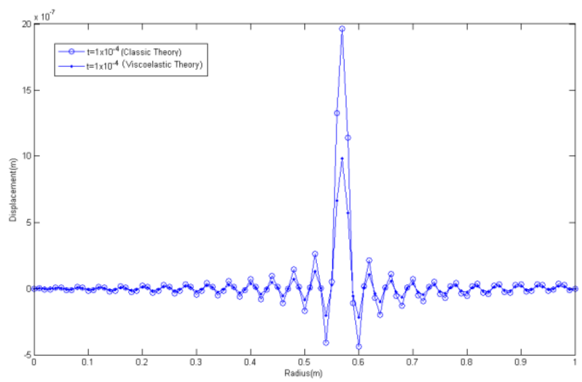

Fig.8
Displacement distribution due to input $u(1,t)=10^{12}\delta(t)$ at 1×10^{-4} s.

5 CONCLUSIONS

In the present paper, an analytical solution for the coupled thermoviscoelasticity of thick sphere under radial temperature is presented. Figs. 1 to 8 show relaxation time effect on variation of displacement and temperature. It is observed that the peak value of Classic coupled thermoelastic theory for displacement and temperature increases. The method is based on the eigenfunctions Fourier expansion, which is a classical and traditional method of solution of the typical initial and boundary value problems. The non-competetive strength of this method is its ability to reveal the fundamental mathematical and physical properties and interpretations of the problem under studying.

In the coupled thermoviscoelastic problem of radial-symmetric sphere, the governing equations constitute a system of partial differential equations with two independent variables, radius ® and time (t). The traditional procedure to solve this class of problems is to eliminate the time variable using the Laplace transform. The resulting system is a set of ordinary differential equations in terms of the radius variable, which solution falls in the Bessel functions family. This method of the analysis brings the Laplace parameter (s) in the argument of the Bessel functions, causing hardship or difficulties in carrying out the exact inverse of the Laplace transformation. As a result, the numerical inversion of the Laplace transformation is used in the papers dealing with this type of problems in literature. In the present paper, to prevent this problem, when the Laplace transform is applied to the particular solutions, it is postponed after eliminating the radius variable r by H-Fourier Expansion. Thus, the Laplace parameter (s) appears in polynomial function forms and hence the exact Laplace inversion transformation is possible.

APPENDIX A

$$b_1=\frac{-\gamma}{(\lambda+2\mu)}\quad,\quad b_2=\frac{-\rho}{(\lambda+2\mu)}\quad,\quad b_3=\frac{-1}{(\lambda+2\mu)}\quad,\quad b_4=\frac{-\rho C_v}{k}\quad,\quad b_5=-\frac{\gamma T_0}{k}\quad,\quad b_4=\frac{-1}{k}$$

$$b_7=\frac{1}{\|H_1(\beta_n r)\|^2}\int_0^1\frac{5}{2}H_1(\beta_n r)rdr\quad,\quad b_8=\frac{1}{\|H_1(\beta_n r)\|^2}\int_0^1\frac{3}{2}b_1H_1(\beta_n r)r^2dr$$

$$b_9=\frac{1}{\|H_1(\beta_n r)\|^2}\int_0^1 b_2r^3H_1(\beta_n r)r^3dr\quad,\quad b_{10}=\frac{1}{\|H_0(\beta_n r)\|^2}\int_0^1\frac{15}{4}H_0(\beta_n r)rdr$$

$$b_{11}=\frac{1}{\|H_0(\beta_n r)\|^2}\int_0^1\frac{7}{2}b_5H_0(\beta_n r)r^2dr\quad,\quad b_{12}=\frac{1}{\|H_0(\beta_n r)\|^2}\int_0^1 b_4H_0(\beta_n r)r^3dr$$

$$b_{13}=C_{21}J_{3/2}(\beta r_0)+C_{22}[\frac{-2}{r_0}J_{3/2}(\beta r_0)+\beta J_{1/2}(\beta r_0)]$$

$$b_{14}=C_{21}\left(\frac{3}{\beta}J_{3/2}(\beta r_0)-r_0J_{1/2}(\beta r_0)\right)+C_{22}\left(2J_{1/2}(\beta r_0)-\frac{6}{\beta}\frac{1}{r_0}J_{3/2}(\beta r_0)+\beta r_0J_{3/2}(\beta r_0)\right)$$

$$b_{15}=C_{23}J_{1/2}(\beta r_0)\quad,\quad b_{16}=C_{23}r_0J_{3/2}(\beta r_0)\quad,\quad b_{17}=C_{21}r_0^{\,2}+C_{22}\frac{3}{2}r_0\quad,\quad b_{18}=C_{23}r_0^{\,2}$$

$$b_{19}=C_{41}J_{1/2}(\beta r_0)-C_{42}\beta J_{3/2}(\beta r_0)\quad,\quad b_{20}=C_{42}\beta r_0J_{1/2}(\beta r_0)-C_{42}J_{3/2}(\beta r_0)+C_{41}r_0J_{3/2}(\beta r_0)$$

$$b_{21}=C_{41}r_0^{\,2}+C_{42}\frac{3}{2}r_0$$

$$\xi_1=\frac{\left(-\frac{\beta_n^{\,2}+\lambda_n^{\,2}d_2}{d_1}+\frac{-\lambda_nd_4}{-\beta_n^{\,2}+\lambda_nd_3}\right)}{\left(\frac{2\lambda_n^{\,2}d_2}{\beta_n^{\,2}d_1}-\frac{2\left(\beta_n^{\,2}-\lambda_n^{\,2}d_2\right)}{d_1\left(-\beta_n^{\,2}+\lambda_nd_3\right)}\right)}\quad,\quad \xi_2=\frac{-\beta_n^{\,2}+\lambda_n^{\,2}d_2}{\beta_n^{\,2}d_1}\quad,\quad \xi_3=\xi_2+\frac{2\lambda_n^{\,2}d_2}{\beta_nd_1}\xi_1$$

$$\xi_4=\frac{1}{2\beta_n}\left(\beta_n\lambda_nd_4+\xi_3\left(-\beta_n^{\,2}+\lambda_nd_3\right)\right)\quad,\quad \xi_5=\xi_{18}\quad,\quad \xi_6=\xi_1\xi_{18}\quad,\quad \xi_7=\xi_{17}\quad,\quad \xi_9=\xi_1\xi_{16}$$

$$\xi_{10}=\xi_2\quad,\quad \xi_{11}=\xi_3\xi_{18}\quad,\quad \xi_{12}=\xi_4\xi_{18}\quad,\quad \xi_{13}=\xi_2\xi_{17}\quad,\quad \xi_{14}=\xi_3\xi_{16}\quad,\quad \xi_{15}=\xi_4\xi_{18}$$

$$\xi_{16}=\frac{\left(-\frac{\mu_{31}}{\mu_{32}}+\frac{\mu_{41}}{\mu_{42}}\right)-\frac{\left(-\frac{\mu_{33}}{\mu_{32}}+\frac{\mu_{43}}{\mu_{42}}\right)\left(\mu_{21}+\frac{\mu_{22}\mu_{11}}{\mu_{12}}\right)}{\left(\mu_{23}+\frac{\mu_{22}\mu_{13}}{\mu_{12}}\right)}}{\left(-\frac{\mu_{34}}{\mu_{32}}+\frac{\mu_{44}}{\mu_{42}}\right)-\frac{\left(-\frac{\mu_{33}}{\mu_{32}}+\frac{\mu_{43}}{\mu_{42}}\right)\left(\mu_{24}-\frac{\mu_{22}\mu_{14}}{\mu_{12}}\right)}{\left(\mu_{23}+\frac{\mu_{22}\mu_{13}}{\mu_{12}}\right)}}$$

$$\xi_{17}=\frac{\left(\mu_{21}-\frac{\mu_{22}\mu_{11}}{\mu_{12}}\right)}{\left(\mu_{23}+\frac{\mu_{22}\mu_{13}}{\mu_{12}}\right)}-\frac{\left(\mu_{24}+\frac{\mu_{22}\mu_{14}}{\mu_{12}}\right)}{\left(\mu_{21}-\frac{\mu_{22}\mu_{13}}{\mu_{12}}\right)}\xi_{16}\quad,\quad \xi_{18}=-\frac{\mu_{11}}{\mu_{12}}-\frac{\mu_{13}}{\mu_{12}}\xi_{17}-\frac{\mu_{14}}{\mu_{12}}\xi_{16}$$

$$\mu_{11}=C_{11}J_{3/2}(\beta_n r_i)+C_{12}\left(\beta_n J_{1/2}(\beta_n r_i)-\frac{1}{r_i}J_{3/2}(\beta_n r_i)\right)+C_{13}\xi_2 J_{3/2}(\beta_n r_i)$$

$$\mu_{12}=C_{11}\left(J_{3/2}(\beta_n r_i)+r_i\xi_1 J_{5/2}(\beta_n r_i)\right)$$

$$+C_{12}\left(\beta_n J_{1/2}(\beta_n r_i)-\frac{1}{r_i}J_{3/2}(\beta_n r_i)+\xi_1 J_{5/2}(\beta_n r_i)+r_i\xi_1 J_{3/2}(\beta_n r_i)-2\xi_1 J_{5/2}(\beta_n r_i)\right)$$

$$+C_{13}\left(\xi_3 J_{1/2}(\beta_n r_i)+r_i\xi_4 J_{3/2}(\beta_n r_i)\right)$$

$$\mu_{13}=C_{11}Y_{3/2}(\beta_n r_i)+C_{12}\left(\beta_n Y_{1/2}(\beta_n r_i)-\frac{1}{r_i}Y_{3/2}(\beta_n r_i)\right)+C_{13}\xi_2 Y_{3/2}(\beta_n r_i)$$

$$\mu_{14}=C_{11}\left(Y_{3/2}(\beta_n r_i)+r_i\xi_1 Y_{5/2}(\beta_n r_i)\right)$$

$$+C_{12}\left(\beta_n Y_{1/2}(\beta_n r_i)-\frac{1}{r_i}Y_{3/2}(\beta_n r_i)+\xi_1 Y_{5/2}(\beta_n r_i)+r_i\xi_1 Y_{3/2}(\beta_n r_i)-2\xi_1 Y_{5/2}(\beta_n r_i)\right)$$

$$+C_{13}\left(\xi_3 Y_{1/2}(\beta_n r_i)+r_i\xi_4 Y_{3/2}(\beta_n r_i)\right)$$

$$\mu_{21}=C_{21}J_{3/2}(\beta_n r_0)+C_{22}\left(\beta_n J_{1/2}(\beta_n r_0)-\frac{1}{r_0}J_{3/2}(\beta_n r_0)\right)+C_{23}\xi_2 J_{1/2}(\beta_n r_0)$$

$$\mu_{22}=C_{21}\left(J_{3/2}(\beta_n r_0)+r_0\xi_1 J_{5/2}(\beta_n r_0)\right)$$

$$+C_{22}\left(\beta_n J_{1/2}(\beta_n r_0)-\frac{1}{r_0}J_{3/2}(\beta_n r_0)+\xi_1 J_{5/2}(\beta_n r_0)+r_0\xi_1 J_{3/2}(\beta_n r_0)-2\xi_1 J_{5/2}(\beta_n r_0)\right)$$

$$+C_{13}\left(\xi_3 J_{1/2}(\beta_n r_0)+r_0\xi_4 J_{3/2}(\beta_n r_0)\right)$$

$$\mu_{23}=C_{21}Y_{3/2}(\beta_n r_0)+C_{22}\left(\beta_n Y_{1/2}(\beta_n r_0)-\frac{1}{r_0}Y_{3/2}(\beta_n r_0)\right)+C_{23}\xi_2 Y_{1/2}(\beta_n r_0)$$

$$\mu_{22}=C_{21}\left(Y_{3/2}(\beta_n r_0)+r_0\xi_1 Y_{5/2}(\beta_n r_0)\right)$$

$$+C_{22}\left(\beta_n Y_{1/2}(\beta_n r_0)-\frac{1}{r_0}Y_{3/2}(\beta_n r_0)+\xi_1 Y_{5/2}(\beta_n r_0)+r_0\xi_1 Y_{3/2}(\beta_n r_0)-2\xi_1 Y_{5/2}(\beta_n r_0)\right)$$

$$+C_{13}\left(\xi_3 Y_{1/2}(\beta_n r_0)+r_0\xi_4 Y_{3/2}(\beta_n r_0)\right)$$

$$\mu_{31}=C_{31}\xi_2 J_{1/2}(\beta_n r_i)+C_{32}\xi_2\beta_n J_{3/2}(\beta_n r_i)$$

$$\mu_{32}=C_{31}\left(\xi_3 J_{1/2}(\beta_n r_i)+r_i\xi_4 J_{3/2}(\beta_n r_i)\right)$$

$$+C_{32}\left(-\beta_n\xi_3 J_{3/2}(\beta_n r_i)+\xi_4 J_{1/2}(\beta_n r_i)+r_i\xi_4\beta_n J_{1/2}(\beta_n r_i)-\xi_4 J_{3/2}(\beta_n r_i)\right)$$

$$\mu_{33}=C_{31}\xi_2 Y_{1/2}(\beta_n r_i)-C_{32}\xi_2\beta_n Y_{3/2}(\beta_n r_i)$$

$$\mu_{32}=C_{31}\left(\xi_3 Y_{1/2}(\beta_n r_i)+r_i\xi_4 Y_{3/2}(\beta_n r_i)\right)$$

$$+C_{32}\left(-\beta_n\xi_3 Y_{3/2}(\beta_n r_i)+\xi_4 Y_{1/2}(\beta_n r_i)+r_i\xi_4\beta_n Y_{1/2}(\beta_n r_i)-\xi_4 Y_{3/2}(\beta_n r_i)\right)$$

$$\mu_{41} = C_{41}\xi_2 J_{1/2}(\beta_n r_0) - C_{42}\xi_2 \beta_n J_{3/2}(\beta_n r_0)$$
$$\mu_{42} = C_{41}\left(\xi_3 J_{1/2}(\beta_n r_0) + r_0 \xi_4 J_{3/2}(\beta_n r_0)\right)$$
$$+ C_{42}\left(-\beta_n \xi_3 J_{3/2}(\beta_n r_0) + \xi_4 J_{3/2}(\beta_n r_0) + r_0 \xi_4 \beta_n J_{1/2}(\beta_n r_0) - \xi_4 J_{3/2}(\beta_n r_0)\right)$$
$$\mu_{43} = C_{41}\xi_2 Y_{1/2}(\beta_n r_0) - C_{42}\xi_2 \beta_n Y_{3/2}(\beta_n r_0)$$
$$\mu_{44} = C_{41}\left(\xi_3 Y_{1/2}(\beta_n r_0) + r_0 \xi_4 Y_{3/2}(\beta_n r_0)\right)$$
$$+ C_{42}\left(-\beta_n \xi_3 Y_{3/2}(\beta_n r_0) + \xi_4 Y_{3/2}(\beta_n r_0) + r_0 \xi_4 \beta_n Y_{1/2}(\beta_n r_0) - \xi_4 Y_{3/2}(\beta_n r_0)\right)$$

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