Journal of Solid Mechanics Vol. 2, No. 1 (2010) pp. 63-78

A Semi-Analytical Solution for Free Vibration and Modal Stress Analyses of Circular Plates Resting on Two-Parameter Elastic Foundations

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Received 2 March 2010; accepted 1 May 2010

ABSTRACT

In the present research, free vibration and modal stress analyses of thin circular plates with arbitrary edge conditions, resting on two-parameter elastic foundations are investigated. Both Pasternak and Winkler parameters are adopted to model the elastic foundation. The differential transform method (DTM) is used to solve the eigenvalue equation yielding the natural frequencies and mode shapes of the circular plates. Accuracy of obtained results is evaluated by comparing the results with those available in the well-known references. Furthermore, effects of the foundation stiffness parameters and the edge conditions on the natural frequencies, mode shapes, and distribution of the maximum in-plane modal stresses are investigated.

Keywords: Free vibration; Thin circular plates; DTM; Two-parameter elastic foundation; Modal stress

1 INTRODUCTION

MANY practical examples of plate-type engineering components with specified edge conditions resting on elastic foundations may be found in the aerospace, marine, civil, automotive, electronic, and nuclear industries. In some circumstances, the plate may be modeled as a plate resting on elastic foundations. Face sheets of sandwich structures with honey comb or very soft cores are some examples. Numerous free vibration studies of the circular plates may be found in literatures. An extensive survey of the early investigations on the free vibration of the circular plates was given by Leissa [1]. Some researchers have developed analytical solutions for axisymmetric vibration of circular plates with edge supports and expressed the natural frequencies in terms of Bessel's functions [2-5]. In this regard, natural frequencies of circular plates with different combinations of free, simply-supported, and clamped boundary conditions have been reported by Irie et al. [6]. It is worth to mention that other methods, such as the Homotopy [7-9] and the differential quadrature methods [10, 11], have been used by other researchers to solve free vibration and nonlinear dynamic problems.

In recent years, Rokni et al. [12] developed an axisymmetric vibration analysis for variable thickness circular plates with various combinations of the edge boundary conditions. Liew and Yang [13] used a three dimensional elasticity approach to analyze axisymmetric vibration of a circular plate with uniform thickness using Ritz solution procedure. Zhou et al. [14] used the three dimensional elasticity approach for free axisymmetric vibration analysis of circular plates with simple edge conditions, employing Chebyshev-Ritz method to solve the resulted governing equations. Recently, the subject of plates resting on elastic foundations has been adopted by many researchers to model the interaction between a metal plate and its elastic substrate, especially when the substrate exhibits both shear and transverse flexibilities. It is known that Winkler model of an elastic foundation is the most preliminary one. In this model, the vertical displacement is assumed to be proportional to the contact pressure at each arbitrary

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point. Investigations on free vibration, buckling and bending behaviors of plates with Winkler-type foundations have been performed by many researchers. For instance, Chen et al. [15] derived solutions of various orders for axisymmetric vibrations of the thin plates, Berger plate, and Winkler plate. Gupta et al. [16] investigated axisymmetric buckling and vibration of polar orthotropic circular plates resting on Winkler foundations. Gupta et al. [17] analyzed free axisymmetric vibrations of non-homogeneous isotropic circular plates with nonlinear thickness variations on the basis of the classical plate theory employing the differential quadrature method (DQM). Although Winkler model is simple and widely used, it is not accurate enough [18]. To overcome this problem, some researchers have proposed various two-parameter foundation models, which may capture the real behavior more precisely, such as Vlasov foundation [19], the generalized foundation [20], and Pasternak foundation [21]. Many other researchers have used the mentioned models. For instance, Zhou et al. [22] presented an excellent investigation on the 3-D free vibration of thick circular plates resting on a Pasternak-type foundation by using the Chebyshev–Ritz method.

In the present study, another powerful method called the differential transform method (DTM) is used to analyze free vibration of FG circular plates resting on two-parameter elastic foundations, analytically. In contrast to many available researches in the field, it is focused on the non-axisymmetric vibration and the modal stress analysis. DTM is a semi-analytical technique that formulates the Taylor series in a totally different manner. With this technique, the given differential equation and the relevant initial conditions are transformed into recurrence equations that finally lead to a system of algebraic equations which give the coefficients of the power series solution. This method is useful for obtaining exact and approximate solutions for linear and nonlinear differential equations. There is no need to use linearization or perturbation techniques, and significant computational times and round-off errors are avoided. This method is well addressed in [23-25]. In the present study, the convergence and accuracy of the applied method are evaluated and compared with Refs. [1, 8]. Then effects of the elastic foundation on the natural frequencies, mode shapes, and modal stresses are evaluated for different boundary conditions.

2 THE GOVERNING EQUATIONS

Consider a thin circular plate resting on a two-parameter elastic foundation, as shown in Fig. 1. Based on the classical bending theory of the plates [26], the displacement field is described as follows:

$$\begin{aligned} u_z &= u_z(r,\theta,t) = w(r,\theta,t) \\ u_r &= u_r(r,\theta,t) = -zw_{,r} \\ u_\theta &= u_\theta(r,\theta,t) = -zw_{,\theta} \end{aligned} \tag{1}$$

The strain-displacement relations of the plate in a non-axisymmetric deformation may be written as

$$\begin{aligned} \varepsilon_{rr} &= -zw_{,rr} \\ \varepsilon_{\theta\theta} &= -z\left(\frac{1}{r^2}w_{,\theta\theta} + \frac{1}{r}w_{,r}\right) \\ \varepsilon_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}u_{r,\theta} + u_{\theta,r} - \frac{u_\theta}{r}\right) \end{aligned} \tag{2}$$

Based on the generalized Hooke's law, the stress-strain and stress-displacement relations may be expressed as

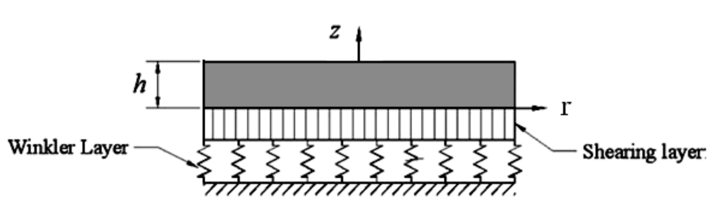


Fig. 1
A circular plate resting on a two-parameter elastic foundation.

$$\sigma_{rr}=\frac{E}{1-\nu^2}(\varepsilon_{rr}+\nu\varepsilon_{\theta\theta})=-\frac{Ez}{1-\nu^2}\left(w_{,rr}+\frac{\nu}{r}w_{,r}+\frac{\nu}{r^2}w_{,\theta\theta}\right)$$
$$\sigma_{\theta\theta}=\frac{E}{1-\nu^2}(\varepsilon_{\theta\theta}+\nu\varepsilon_{rr})=-\frac{Ez}{1-\nu^2}\left(\nu\, w_{,rr}+\frac{1}{r}w_{,r}+\frac{1}{r^2}w_{,\theta\theta}\right) \tag{3}$$
$$\tau_{r\theta}=-\frac{Ez}{1+\nu}\left(\frac{1}{r}w_{,r\theta}-\frac{1}{r^2}w_{,\theta}\right)$$

The two-parameter elastic foundation model $p(x,y)=k_w w-k_s\nabla^2 w$, introduces a more realistic model for the foundations by taking into account the curvature and the displacement of the foundation layer. k_s is usually referred to as the shear modulus of the foundation. Therefore, based on Eq. (1), the resulted equation of motion of the plate is given as [26]

$$D\nabla^4 w-k_s\nabla^2 w+k_w w+\rho\frac{\partial^2 w}{\partial t^2}=0 \tag{4}$$

where w is the transverse deflection, ∇^4 is the biharmonic operator, k_w is the Winkler elastic coefficient of the foundation. ρ and D are the mass density per unit area and the flexural rigidity of the plate respectively. When free vibrations are considered, deflection of a circular plate in the polar coordinates may be expressed as follows:

$$w=f(r)\mathrm{e}^{\mathrm{i}(m\theta+\omega t)} \tag{5}$$

where m is an integer number of the circumferential half waves, ω is the natural frequency, and $i=\sqrt{-1}$ is the imaginary number. Substituting Eq. (5) into Eq. (4) yields

$$\frac{d^4 f}{dr^4}+\frac{2}{r}\frac{d^3 f}{dr^3}-\frac{B}{r^2}\frac{d^2 f}{dr^2}+\frac{B}{r^3}\frac{df}{dr}+\frac{A}{r^4}f=\Omega^2 f-K_w f+K_s\nabla^2 f \tag{6}$$

where

$$A=m^4-4m^2 \quad \text{and} \quad B=2m^2+1 \tag{6a}$$

and the other dimensionless parameters are defined as follow:

Dimensionless frequency: $\Omega=a^2\omega\sqrt{\rho/D}$ (6b)

Dimensionless Winkler normal stiffness of the foundation: $K_w=(a^4/D)k_w$ (6c)

Dimensionless stiffness of the shearing layer of the foundation: $K_s=(a^2/D)k_s$ (6d)

In Eq. (6), it is assumed that f is the dimensionless deflection, a is the plate radius, r is the dimensionless radial coordinate, and θ is the dimensionless tangential coordinate. Three types of the edge conditions are defined at the outer edge of the circular plate (r=1). These conditions may be written in terms of the dimensionless deflection function $f(r)$ as follows:

Free edge:

$$M_r\big|_{r=1}=-D\left[\frac{d^2 f}{dr^2}+\nu\left(\frac{1}{r}\frac{df}{dr}+\frac{m^2}{r^2}f\right)\right]=0 \tag{7}$$
$$V_r\big|_{r=1}=\left[\frac{d^3 f}{dr^3}+\frac{1}{r}\frac{d^2 f}{dr^2}+\left(\frac{m^2\nu-2m^2-1}{r^2}\right)\frac{df}{dr}+\left(\frac{3m^2-m^2\nu}{r^3}\right)f\right]=0$$

Simply-supported edge:

$$f(r)\big|_{r=1}=0,\qquad M_r\big|_{r=1}=-D\left[\frac{d^2f}{dr^2}+\nu\left(\frac{1}{r}\frac{df}{dr}+\frac{m^2}{r^2}f\right)\right]=0 \tag{8}$$

Clamped edge:

$$f(r)\big|_{r=1}=0,\qquad \frac{df}{dr}\big|_{r=1}=0 \tag{9}$$

where M_r is the radial bending moment per unit length, V_r is the effective radial shear force per unit length, and ν is Poisson's ratio. It is obvious that Eq. (6) is a fourth-order differential equation and subsequently, four boundary conditions are required to determine the integration constants. One may obtain two of those from the boundary conditions of the outer edge of the circular plate. However, the remaining two conditions must be investigated within the regularity conditions at the center of the plate. Wu et al. stated two more boundary conditions at the center of the solid circular plate [11]. Denoting number of the circumferential half waves by m, circular plates with even m and odd m values have symmetric and antisymmetric modes, respectively. Therefore, one can obtain the regularity conditions at the center of the circular plate (r=0) in terms of the number of the nodal diameters (or number of the circumferential half waves) as follows:
Antisymmetric case:

$$f(r)\big|_{r=0}=0,\qquad M_r\big|_{r=0}=\frac{d^2f}{dr^2}\big|_{r=0}=0,\qquad \text{for}\ \ m=1,3,5,\ldots \tag{10}$$

Symmetric case:

$$\frac{df}{dr}\big|_{r=0}=0,\qquad V_r\big|_{r=0}=\frac{d^3f}{dr^3}\big|_{r=0}=0,\qquad \text{for}\ \ m=2,4,6,\ldots \tag{11}$$

3 THE TRANSFORMED FORM OF THE GOVERNING EQUATIONS

3.1 A brief description of the differential transform method (DTM)

The differential transformation technique, which is used in this paper was first proposed by Zhou in 1986 and is one of the analytical methods that has been proposed for solving ordinary and partial differential equations. This method has been developed based on the Taylor series expansion and is a useful tool to obtain analytical solutions for the differential equations. In this method, certain transformation rules are applied and the governing differential equations of the system and their relevant boundary conditions are transformed into a set of algebraic equations. The differential transforms of the original functions and the solution of these algebraic equations give the desired solution of the problem. The basic definitions and the application procedure of this method can be expressed as below. Consider a function $f(r)$ which is analytic in a domain R and let $r=r_0$ represent any point in R. The function $f(r)$ may be represented by a power series whose center is located at r_0. The differential transform of the kth derivative of the function $f(r)$ is given by

$$F_k=\frac{1}{k!}\left[\frac{\mathrm{d}^k f(r)}{\mathrm{d}r^k}\right]_{r=r_0} \tag{12}$$

where $f(r)$ is the original function and F_k is the transformed function. The inverse transformation of the function $f(r)$ is defined by

$$f(r)=\sum_{k=0}^{\infty}(r-r_0)^k F_k \tag{13}$$

Combining Eqs. (12) and (13), one may write:

$$f(r)=\sum_{k=0}^{\infty}\frac{(r-r_0)^k}{k!}\left[\frac{\mathrm{d}^k f(r)}{\mathrm{d}r^k}\right]_{r=r_0} \tag{14}$$

In Eq. (14), it is obvious that the concept of the differential transform is derived from Taylor's series expansion. In this study, the lower case letters represent the original functions and the upper case letters are used for the transformed functions. In practical applications, the function $f(r)$ may be expressed by a finite Taylor series as:

$$f(r)=\sum_{k=0}^{N}\frac{(r-r_0)^k}{k!}\left[\frac{\mathrm{d}^k f(r)}{\mathrm{d}r^k}\right]_{r=r_0} \tag{15}$$

which implies that $\sum_{k=N+1}^{\infty}(r-r_0)^k F_k$ is negligibly small. Here, choice the N value depends on the convergence criterion of the natural frequencies.

3.2 Basic mathematical operations of the differential transformation method

Some of the rules that are frequently used in the differential transformation method are introduced in Table 1.

3.3 Transformations of the free vibration governing equations

Employing the differential transformation rules defined in Table 1, transformation of Eq. (6) around r_0=0 leads to:

$$\begin{aligned}
&AF_k+B\sum_{l=0}^{k}\delta(l-1)(k-l+1)F_{k-l+1}-B\sum_{l=0}^{k}\delta(l-2)(k-l+1)(k-l+2)F_{k-l+2}\\
&\quad+2\sum_{l=0}^{k}\delta(l-3)(k-l+1)(k-l+2)(k-l+3)F_{k-l+3}\\
&\quad+\sum_{l=0}^{k}\delta(l-4)(k-l+1)(k-l+2)(k-l+3)(k-l+4)F_{k-l+4}\\
&\quad=(\Omega^2-K_w)\sum_{l=0}^{k}\delta(l-4)F_{k-l}\\
&\qquad+K_s\left[\sum_{l=0}^{k}\delta(l-4)(k-l+1)(k-l+2)F_{k-l+2}+\sum_{l=0}^{k}\delta(l-3)(k-l+1)F_{k-l+1}-m^2\sum_{l=0}^{k}\delta(l-2)F_{k-l}\right]
\end{aligned} \tag{16}$$

Simplifying Eq. (16) and using the last rule listed in Table 1, the equation of motion can be transformed into the following recurrence equation:

Table 1
The transformation rules in the DTM

Original function	Transformed function
$f(r)=g(r)\pm h(r)$	$F_k=G_k\pm H_k$
$f(r)=\lambda g(r)$	$F_k=\lambda G_k$
$f(r)=g(r)\cdot h(r)$	$F_k=\sum_{l=0}^{k}G_l\cdot H_{k-l}$
$f(r)=\frac{g(r)}{h(r)}$	$F_k=\frac{G_k-\sum_{l=0}^{k-l}G_l\cdot H_{k-l}}{H_0}$
$f(r)=\frac{\mathrm{d}^n g(r)}{\mathrm{d}r^n}$	$F_k=\frac{(k+n)!}{k!}G_{k+n}$
$f(r)=r^n$	$F_k=\delta(k-n)=\begin{cases}1 & k=n\\ 0 & k\neq n\end{cases}$

$$F_{k+4}=\frac{(\Omega^2-K_w)}{A-B(k+4)(k+2)+(k+4)(k+3)^2(k+2)}F_k+\frac{((k+2)^2-m^2)K_s}{A-B(k+4)(k+2)+(k+4)(k+3)^2(k+2)}F_{k+2} \tag{17}$$

From Eq. (17), the following equations can be obtained for k = 0, 1, 2, ..., n:

$$\begin{aligned}
F_4&=\frac{(\Omega^2-K_w)}{A-B(4\cdot 2)+(4\cdot 3^2\cdot 2)}F_0+\frac{(2^2-m^2)K_s}{A-B(4\cdot 2)+(4\cdot 3^2\cdot 2)}F_2\\
F_5&=\frac{(\Omega^2-K_w)}{A-B(5\cdot 3)+(5\cdot 4^2\cdot 3)}F_1+\frac{(3^2-m^2)K_s}{A-B(5\cdot 3)+(5\cdot 4^2\cdot 3)}F_3\\
F_6&=\frac{(\Omega^2-K_w)}{A-B(6\cdot 4)+(6\cdot 5^2\cdot 4)}F_2+\frac{(4^2-m^2)K_s}{A-B(6\cdot 4)+(6\cdot 5^2\cdot 4)}F_4\\
&=\frac{(4^2-m^2)K_s}{A-B(6\cdot 4)+(6\cdot 5^2\cdot 4)}\cdot\frac{\Omega^2-K_w}{A-B(4\cdot 2)+(4\cdot 3^2\cdot 2)}F_0+\left[\frac{(\Omega^2-K_w)}{A-B(6\cdot 4)+(6\cdot 5^2\cdot 4)}+\frac{(4^2-m^2)K_s}{A-B(6\cdot 4)+(6\cdot 5^2\cdot 4)}\cdot\frac{(4^2-m^2)K_s}{A-B(4\cdot 2)+(4\cdot 3^2\cdot 2)}\right]F_2\\
F_7&=\frac{(\Omega^2-K_w)}{A-B(7\cdot 5)+(7\cdot 6^2\cdot 5)}F_3+\frac{(5^2-m^2)K_s}{A-B(7\cdot 5)+(7\cdot 6^2\cdot 5)}F_5\\
&=\frac{(5^2-m^2)K_s}{A-B(7\cdot 5)+(7\cdot 6^2\cdot 5)}\cdot\frac{(\Omega^2-K_w)}{A-B(5\cdot 3)+(5\cdot 4^2\cdot 3)}F_1+\left[\frac{\Omega^2-K_w}{A-B(7\cdot 5)+(7\cdot 6^2\cdot 5)}+\frac{5^2-m^2\ K_s}{A-B(7\cdot 5)+(7\cdot 6^2\cdot 5)}\cdot\frac{3^2-m^2\ K_s}{A-B(5\cdot 3)+(5\cdot 4^2\cdot 3)}\right]F_3\\
F_{2k}&=\frac{(\Omega^2-K_w)}{A-B(2k)(2k-2)+(2k)(2k-1)^2(2k-2)}F_{2(k-2)}+\frac{((2k-2)^2-m^2)K_s}{A-B(2k)(2k-2)+(2k)(2k-1)^2(2k-2)}F_{2(k-1)}\\
F_{2k-1}&=\frac{(\Omega^2-K_w)}{A-B(2k-1)(2k-3)+(2k-1)(2k-2)^2(2k-3)}F_{2(k-2)-1}+\frac{((2k-3)^2-m^2)K_s}{A-B(2k-1)(2k-3)+(2k-1)(2k-2)^2(2k-3)}F_{2(k-1)-1}
\end{aligned} \tag{18}$$

Thus, it can be deduced that the even terms (F_{2k}) depends finally on F_0 and F_2 and the odd terms (F_{2k+1}) depends lastly on F_1 and F_3.

3.4 Transformation of the boundary/regularity conditions

By applying rules of Table 1 to the boundary conditions of the outer edge (r=1), presented in Eqs. (7-9), the following equations are obtained:

Free edge:

$$\begin{aligned}
&\sum_{k=0}^{\infty}(k(k-1)+\nu k-m^2\nu)F_k=0\\
&\sum_{k=0}^{\infty}(k(k-1)(k-2)+k(k-1)+(m^2\nu-2m^2-1)k+(3m^2-m^2\nu))F_k=0
\end{aligned} \tag{19}$$

Simply-supported edge:

$$\sum_{k=0}^{\infty}F_k=0,\qquad \sum_{k=0}^{\infty}(k(k-1)+\nu k-m^2\nu)F_k=0 \tag{20}$$

Clamped edge:

$$\sum_{k=0}^{\infty}F_k=0,\qquad \sum_{k=0}^{\infty}kF_k=0 \tag{21}$$

At the center of the circular plate (r=0), the boundary conditions which are derived from the regularity conditions appeared in Eqs. (10) and (11) can be transformed as follows:
Antisymmetric case:

$$F_0 = F_2 = F_4 = \cdots F_{4k} = F_{4k+2} = 0, \qquad \text{for} \quad m = 1,3,5,\ldots \tag{22}$$

Symmetric case:

$$F_1 = F_3 = F_5 = \cdots F_{4k+1} = F_{4k+3} = 0, \qquad \text{for} \quad m = 2,4,6,\ldots \tag{23}$$

4 SOLUTION PROCEDURE

The frequency equations can be derived by imposing all the boundary conditions appeared in Eqs. (19-23). For a circular plate, the axisymmetric edge conditions of the outer edge may be one of the following types: simply-supported, clamped or free. It can also be assumed that the regularity conditions at the center are either symmetric or antisymmetric, depends on the number of nodal diameters (being even or odd). Hence, six combinations may be possible.

To clarify the procedure of obtaining the final frequency equation, the procedure is explained for one of the mentioned cases: a circular plate with free edges resting on a two-parameter elastic foundation experiencing a symmetric vibration (m = 0, 2, 4, …). However, all the cases have been treated in the present research. For plates with symmetric modes, based on Eq. (23), only even terms (F_{2k}) exist and the odd terms may be omitted. Then Eq. (19) may be rewritten as follows:

$$\sum_{k=0}^{\infty}\left[(2k)(2k-1)+\nu(2k)-m^2\nu\right]F_{2k} = 0 \tag{24}$$

$$\sum_{k=0}^{\infty} (2k)(2k-1)(2k-2)+(2k)(2k-1)+(m^2\nu-2m^2-1)(2k)+(3m^2-m^2\nu)\ F_{2k} = 0 \tag{25}$$

and by using Eq. (17) one may obtain the following expressions:

$$\begin{aligned}\Psi_{11}^{(n)}(\Omega)\, F_0 + \Psi_{12}^{(n)}(\Omega)\, F_2 &= 0\\ \Psi_{21}^{(n)}(\Omega)\, F_0 + \Psi_{22}^{(n)}(\Omega)\, F_2 &= 0\end{aligned} \tag{26}$$

where Ψ_{11}, Ψ_{12}, Ψ_{21}, and Ψ_{22} are polynomials of Ω corresponding to the nth term. It can be readily seen that the Ψ_{11}, Ψ_{12}, Ψ_{21}, and Ψ_{22} terms represent the closed form series expressions appeared in Eq. (26). Eq. (26) can be expressed in the following matrix form:

$$\begin{bmatrix}\Psi_{11}^{(n)}(\Omega) & \Psi_{12}^{(n)}(\Omega)\\ \Psi_{21}^{(n)}(\Omega) & \Psi_{22}^{(n)}(\Omega)\end{bmatrix}\begin{Bmatrix}F_0\\ F_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix} \tag{27}$$

Existence of a non-trivial answer requires that:

$$\begin{vmatrix}\Psi_{11}^{(n)}(\Omega) & \Psi_{12}^{(n)}(\Omega)\\ \Psi_{21}^{(n)}(\Omega) & \Psi_{22}^{(n)}(\Omega)\end{vmatrix} = 0 \tag{28}$$

and by solving Eq. (28) the dimensionless frequencies may be obtained. In determining the value of nth natural frequency, the following convergence criterion may be taken into account:

$$\left|\Omega_j^{n} - \Omega_j^{n-1}\right| / \left|\Omega_j^{n}\right| \le \xi, \qquad i = 1,2,...,n \tag{29}$$

where Ω_j is the jth estimated eigenvalue (jth natural frequency) and ξ is a sufficiently small number taken as ξ =0.00001 in the present study. The corresponding eigenfunction, $f(r)$, describing the instantaneous deflected shape of the circular plate can be obtained by substituting Eq. (26) into Eq. (13). Following a similar procedure, the final frequency equations of other types of the boundary and regularity conditions can be derived.

5 RESULTS AND DISCUSSION

In this section, as a first stage, convergence and accuracy of the results of the employed method is investigated. In the second stage, results of free vibration analyses of isotropic plates with different edge conditions resting on two-parameter foundations are assessed. Finally, distributions of the dimensionless modal stress components are plotted and discussed. Poisson's ratio is assumed to be $\nu = 0.3$. Convergence and accuracy of the method is investigated by comparing results of the first three non-dimensional natural frequencies of circular plates without elastic foundations, with those of some well-known references. The natural frequencies results are derived for different number of terms of the series solution (N) and are given in Table 2. There is an excellent agreement with results of reference [11]. The high rate convergence of the method is quite evident and it is found that choosing few terms of the solution series (relatively small N) may yield accurate results. Furthermore, the high accuracy of the method may be noticed. To validate the results, the obtained first three dimensionless frequencies are compared with those reported by Refs. [1, 11] in Table 3.

Table 2
Convergence and accuracy of the first three non-dimensional natural frequencies of the free circular plate without elastic foundation, for various numbers of terms of the series solution (N)

		N=40	N=50	N=60	N=70	Ref. [11]
m=0	Ω_1	9.0031	9.0031	9.0031	9.0031	9.003
	Ω_2	38.4432	38.4432	38.4432	38.4432	38.443
	Ω_3	87.7502	87.7502	87.7502	87.7502	87.75
m=1	Ω_1	20.4746	20.4746	20.4745	20.4745	20.475
	Ω_2	59.8116	59.8116	59.8116	59.8116	59.812
	Ω_3	118.9573	118.9573	118.9573	118.9573	118.957
m=2	Ω_1	5.3583	5.3583	5.3583	5.3583	5.358
	Ω_2	35.2601	35.2601	35.2601	35.2601	35.26
	Ω_3	84.3662	84.3662	84.3662	84.3662	84.366

Table 3
A comparison among the first three dimensionless natural frequencies

Edge condition	Dimensionless frequency	Source	Number of nodal diameters, m			
			0	1	2	3
Simply- supported	Ω_1	Present	4.9351	13.8982	25.613	39.9573
		Ref. [1]	4.977	13.94	25.65	-
		Ref. [11]	4.935	13.898	25.613	39.957
	Ω_2	Present	29.72	48.4789	70.117	94.5489
		Ref. [1]	29.76	48.51	70.14	-
		Ref. [11]	29.72	48.479	70.117	94.549
	Ω_3	Present	74.1561	102.7733	134.2978	168.6749
		Ref. [1]	74.2	102.8	134.33	-
		Ref. [11]	74.156	102.772	134.298	168.675
Clamped	Ω_1	Present	10.2158	21.2604	34.877	51.03
		Ref. [1]	10.2158	21.26	34.88	51.04
		Ref. [11]	10.216	21.26	34.88	51.03
	Ω_2	Present	39.7711	60.8287	84.5826	111.021
		Ref. [1]	39.771	60.82	84.58	111.01
		Ref. [11]	39.771	60.829	84.583	111.021
	Ω_3	Present	89.1041	120.0792	153.815	190.3037
		Ref. [1]	89.104	120.08	153.81	190.3
		Ref. [11]	89.104	120.079	153.815	190.304

As it may be noticed form results of Tables 2 and 3, there is an excellent agreement among the present results and results of references [1] and [11]. Results of circular plates resting on elastic foundations are listed in Tables 4-6.

Table 4
First three dimensionless natural frequencies of circular plates with free edges, resting on elastic foundations

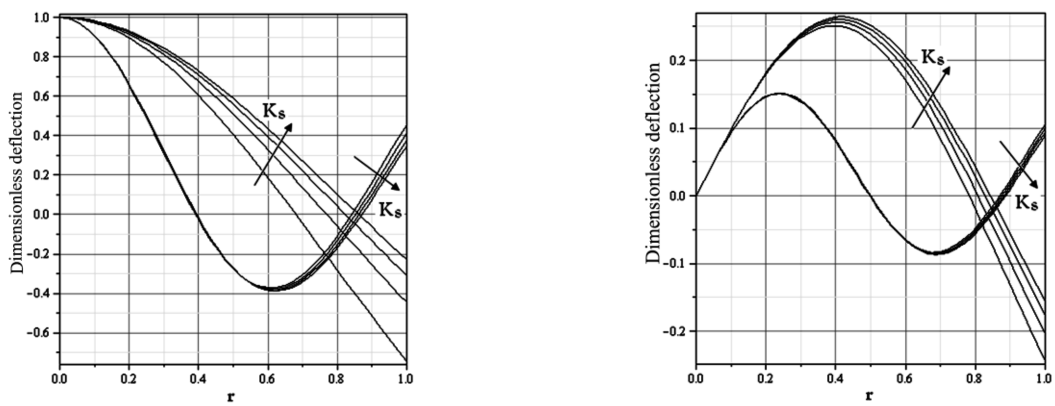
	m	Ω_1	Ω_2	Ω_3
K_w=10, K_s=0	0	9.5423	38.5730	87.8072
	1	20.7173	59.8951	118.9994
	2	6.2219	35.4016	84.4254
K_w=100, K_s=0	0	13.4557	39.7225	88.3182
	1	22.7861	60.6418	119.3769
	2	11.3451	36.6507	84.9567
K_w=1000, K_s=0	0	32.8794	49.7783	93.2743
	1	37.6724	67.6567	123.0888
	2	32.0735	47.3632	90.0980
K_w=0, K_s=5	0	10.3647	40.4694	89.9512
	1	22.2804	61.9350	121.2010
	2	6.2968	37.2121	86.5412
K_w=0, K_s=10	0	11.5491	42.3970	92.0992
	1	23.9399	63.9870	123.4036
	2	7.0940	39.0621	88.6624
K_w=0, K_s=15	0	12.6093	44.2385	94.1977
	1	25.4828	65.9741	125.5673
	2	7.7968	40.8241	90.7334
K_w=10, K_s=5	0	10.8364	40.5928	90.0068
	1	22.5037	62.0157	121.2422
	2	7.0463	37.3462	86.5989
K_w=100, K_s=10	0	15.2768	43.5604	92.6405
	1	25.9445	64.7637	123.8081
	2	8.7907	40.3218	89.2245

Table 5
First three dimensionless natural frequencies of circular plates with simply supported edges, resting on elastic foundations

	m	Ω_1	Ω_2	Ω_3
K_w=10, K_s=0	0	5.8614	29.8878	74.2234
	1	14.2534	48.5820	102.8220
	2	25.8078	70.1883	134.3350
K_w=100, K_s=0	0	11.1515	31.3573	74.8273
	1	17.1219	49.4996	103.2587
	2	27.4962	70.8265	134.6696
K_w=1000, K_s=0	0	32.0056	43.3968	80.6171
	1	34.5421	57.8890	107.5284
	2	40.6945	76.9181	137.9707
K_w=0, K_s=5	0	7.3039	32.1815	76.6392
	1	16.3293	50.9540	105.2609
	2	28.0711	72.5994	136.7882
K_w=0, K_s=10	0	9.0732	34.4679	79.0443
	1	18.4425	53.3144	107.6911
	2	30.3303	74.9996	139.2341
K_w=0, K_s=15	0	10.5493	36.6117	81.3785
	1	20.3371	55.5746	110.0676
	2	32.4324	77.3253	141.6377
K_w=10, K_s=5	0	7.9590	32.3365	76.7044
	1	16.6327	51.0521	105.3084
	2	28.2486	72.6682	136.8248
K_w=100, K_s=10	0	13.5027	35.8892	79.6744
	1	20.9791	54.2441	108.1544
	2	31.9363	75.6633	139.5927

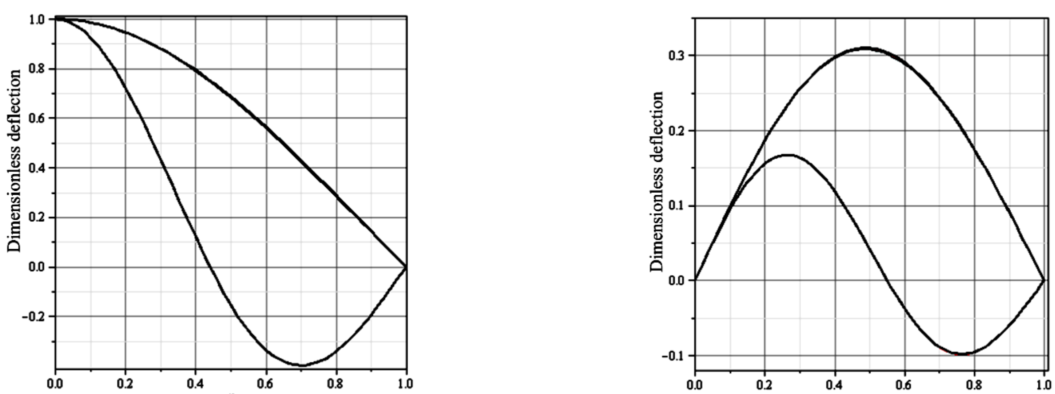
Table 6
First three dimensionless natural frequencies of circular plates with clamped edges, resting on elastic foundations

	m	Ω_1	Ω_2	Ω_3
K_w=10, K_s=0	0	10.6941	39.8967	89.1602
	1	21.4943	60.9108	120.1208
K_w=100, K_s=0	0	14.2956	41.0091	89.6635
	1	23.4948	61.6452	120.4949
K_w=1000, K_s=0	0	33.2320	50.8109	94.5492
	1	38.1052	68.5575	124.1734
K_w=0, K_s=5	0	11.7805	41.8191	91.3113
	1	23.1465	62.9725	122.3307
K_w=0, K_s=10	0	13.1455	43.7694	93.4658
	1	24.8814	65.0445	124.5412
K_w=0, K_s=15	0	14.3703	45.6342	95.5713
	1	26.4955	67.0512	126.7133
K_w=10, K_s=5	0	12.1975	41.9385	91.3660
	1	23.3616	63.0518	122.3718
K_w=100, K_s=10	0	16.5168	44.8972	93.9993
	1	26.8157	65.8087	124.9415



(a) (b)

Fig. 2
Mode shapes of circular plates with free edges resting on elastic foundations: (a) m=0; (b) m=1.



(a) (b)

Fig. 3
Mode shapes of circular plates with simply-supported edges resting on elastic foundations: (a) m=0; (b) m=1.

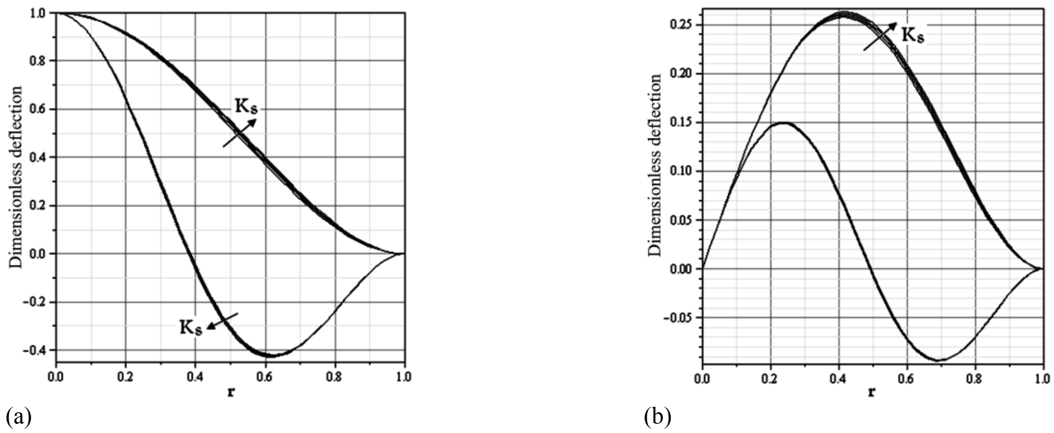


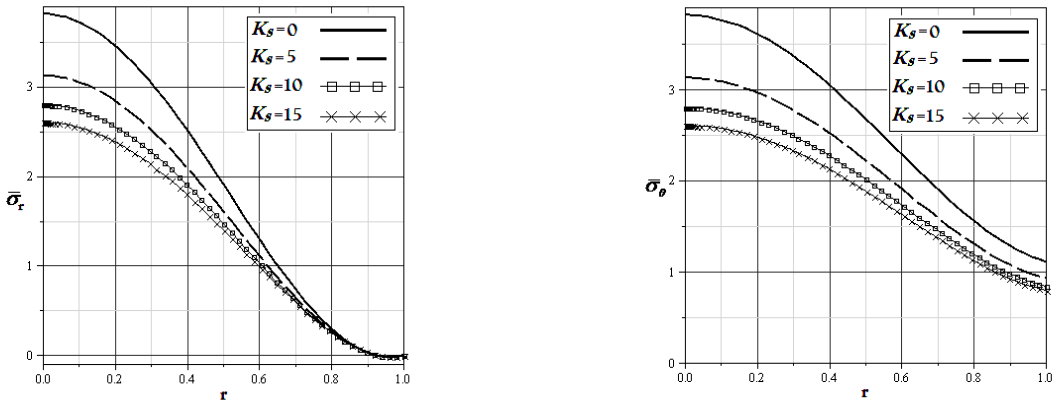
Fig. 4
Mode shapes of circular plates with clamped edges resting on elastic foundations: (a) m=0; (b) m=1.

The first three dimensionless natural frequencies are presented for various combinations of the elastic foundation coefficients (K_w and K_s). While one of the mentioned foundation coefficients is set equal to zero in the first cases, to evaluate influence of the coefficients individually, in the last two cases of each table, both Winkler (K_w) and elastic coefficient of the shearing layer (Pasternak coefficient, K_s) are assumed to be non-zero. It is obvious that both Winkler and elastic coefficient of the shearing layer have significant effect on the dimensionless natural frequency. For identical values, effect of Pasternak coefficient is more remarkable. Furthermore, results reveal that increasing each of the foundation coefficients leads to an increased plate stiffness and subsequently, higher Ω_i. Moreover, results show that for circular plates with free edges, in contrast to other boundary conditions, the smallest Ω_i is achieved at m=2.

Figs. 2-4 show the first two mode shapes for circular plates with free, simply-supported and clamped edge conditions, resting on elastic foundations. Only two typical numbers of the nodal diameters, i.e. m=0 and m=1, are considered here to save space. These cases represent symmetric and antisymmetric regularity conditions, respectively. As it may be expected from results shown in Tables 4-6, effect of Pasternak's coefficient of the elastic foundation (K_s) on the mode shapes is more remarkable than influence of Winkler's coefficient (K_w). For this reason, curves indicating effect of Winkler's coefficient are not included. Meanwhile, effect of K_s on the first mode shape is more noticeable. From Fig. 3, it may be readily noticed that for the simply-supported circular plate, both the K_w and K_s have almost ignorable effects on the shape modes. Finally, effect of the foundation stiffness on the maximum values of the modal stress components of the radial sections is investigated. Variations of the dimensionless radial and tangential modal stress components in the radial direction are depicted in Figs. 5-7, for various edge conditions. Figures are drawn for the fundamental natural frequency (i.e., m=0, z=h/2 and Ω_1). The dimensionless stress components are defined as follows:

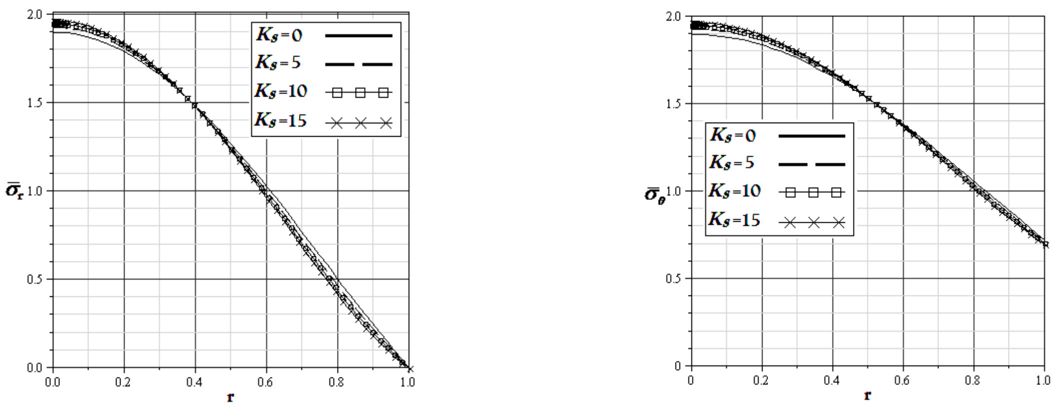
$$\bar{\sigma}_r = \frac{\sigma_{rr}\, a^2}{Eh^2}$$
$$\bar{\sigma}_\theta = \frac{\sigma_{\theta\theta}\, a^2}{Eh^2} \tag{30}$$
$$\bar{\tau}_{r\theta} = \frac{\tau_{r\theta}\, a^2}{Eh^2}$$

As its effect on the mode shapes, Winkler coefficient (K_w) has ignorable effect on the dimensionless tangential and radial stresses, but Pasternak coefficient (K_s) may decrease or increase $\bar{\sigma}_r$ and $\bar{\sigma}_\theta$, depending on the type of the edge condition. Fig. 8 shows the 3D plots of variations of the through-the-thickness dimensionless tangential and radial stresses in the radial direction, for (m=0, Ω_1) and various boundary conditions, for a better visualization. Besides, Fig. 9 shows the 3D plots of variations of the through-the-thickness dimensionless radial stress in the radial direction, for both (m=0, Ω_2) and (m=1, Ω_1), and various edge conditions.



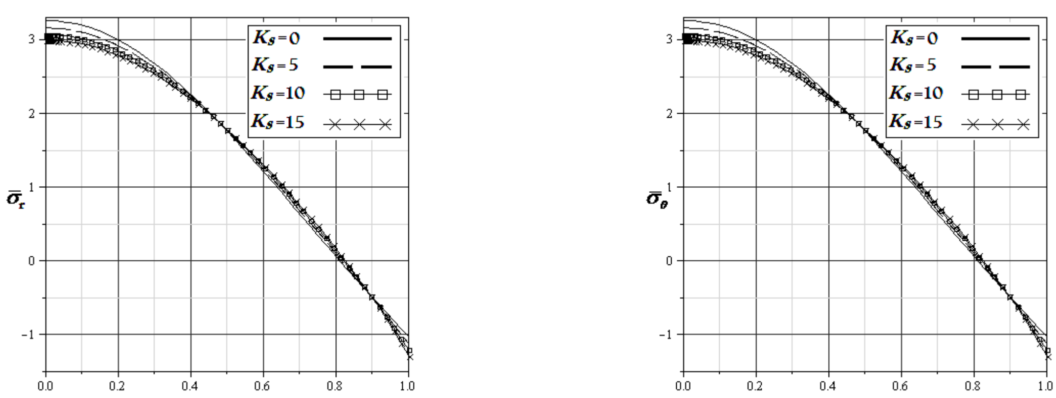
(a) (b)

Fig. 5
Effect of Pasternak coefficient of the elastic foundation on the dimensionless stress components, for circular plates with free edges: (a) radial stress; (b) tangential stress.



(a) (b)

Fig. 6
Effect of Pasternak coefficient of the elastic foundation on the dimensionless stress components, for circular plates with simply-supported edges: (a) radial stress; (b) tangential stress.



(a) (b)

Fig. 7
Effect of Pasternak coefficient of the elastic foundation on the dimensionless stress components, for circular plates with clamped edges: (a) radial stress; (b) tangential stress.

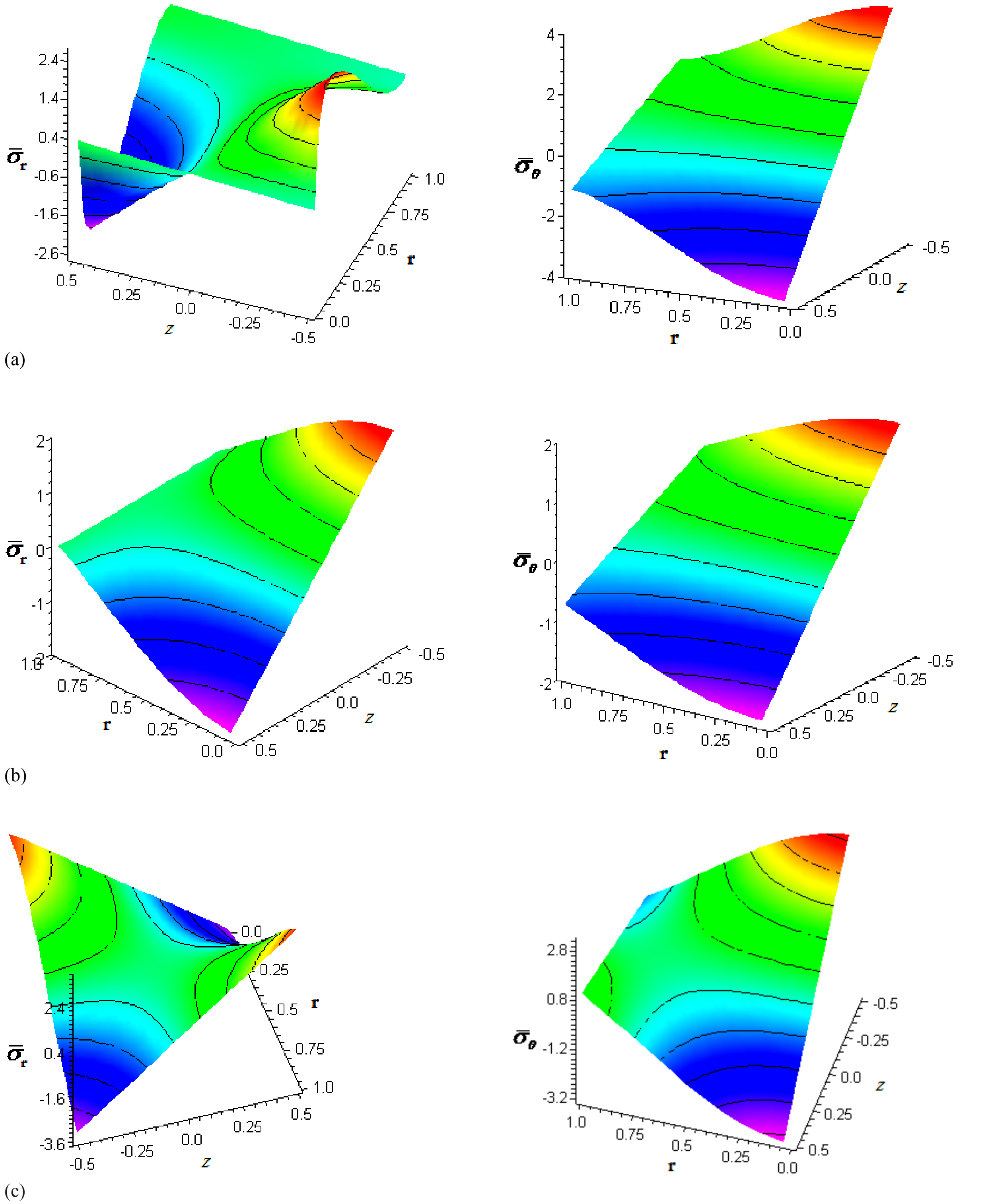


Fig. 8
Variations of the through-the-thickness non-dimensional stress components in the radial direction, for *m*=0 and Ω_1 for a circular plate with: (a) free edge; (b) simply-support edge; (c) clamped edge.

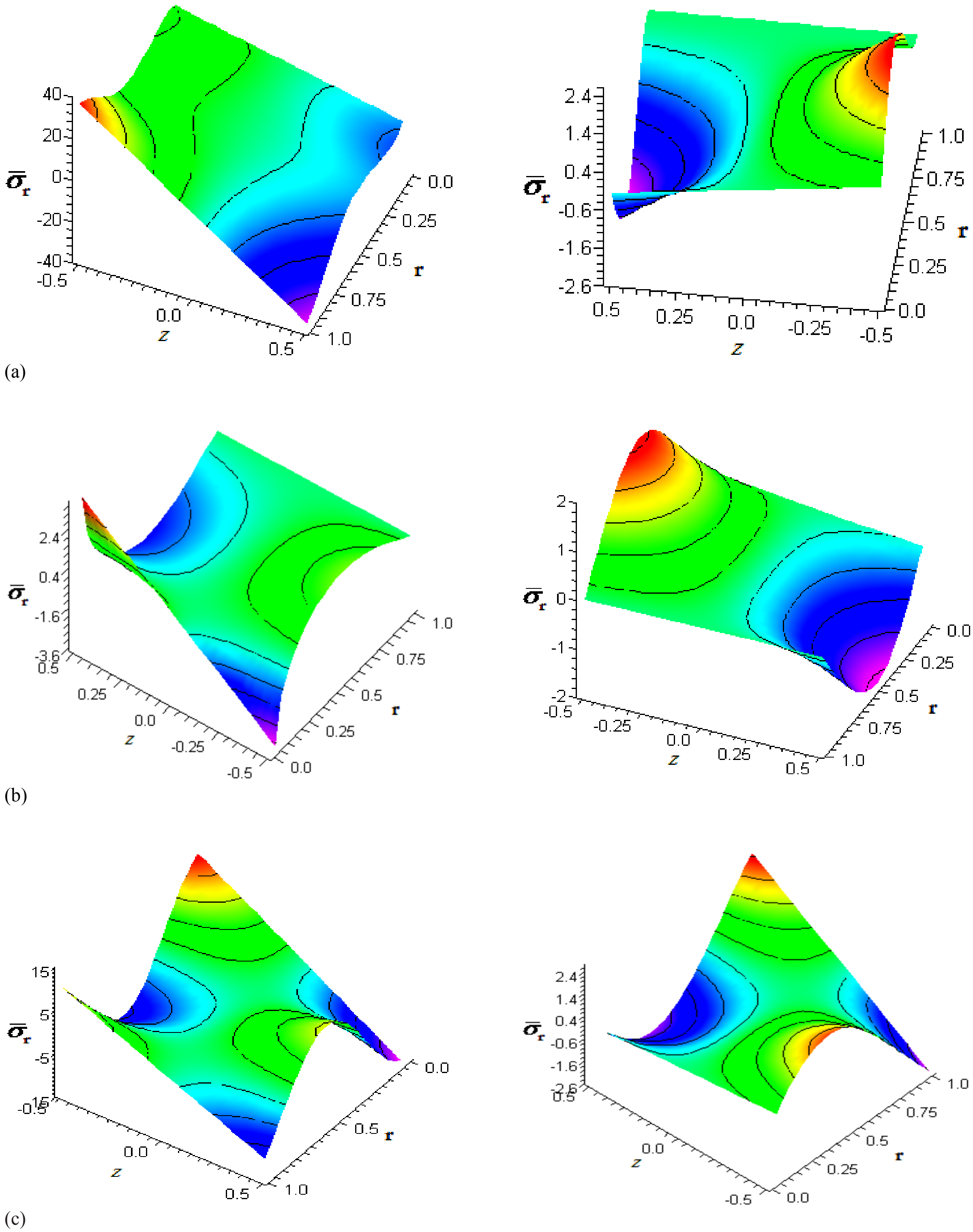


Fig. 9
Variations of the through-the-thickness non-dimensional radial stress in the radial direction for various edge conditions.

To obtain the maximum dimensionless radial stress, value of θ is considered to be zero for m=1. It is obvious that by increasing Ω_i, $\left|\bar{\sigma}_{r,\max}\right|$ and $\left|\bar{\sigma}_{r,\min}\right|$ will increase.

6 CONCLUSIONS

Free vibration and modal stress analyses of thin circular plates with arbitrary edge conditions resting on two-parameter elastic foundations is investigated in the present paper. The differential transform semi-analytical procedure is employed. Comparisons made with results of well-known references confirm accuracy of the present formulations. Some sensitivity analyses are performed to evaluate effect of the foundation parameters and edge conditions on the dimensionless natural frequencies, mode shapes, and modal stresses of the circular plates. Furthermore, effects of the foundation parameters and the edge conditions on the natural frequencies, mode shapes, and distribution of the maximum in-plane modal stresses are investigated.

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