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Research Paper

Exact Closed Form Characteristic Equations for Transverse Vibration of Timoshenko Beams

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ABSTRACT

The dimensionless equations of motion are derived based on the Timoshenko beam theory to study the transverse vibration of beams without further usage of any approximate method. The exact closed form characteristic equations are given within the validity of the Timoshenko beam theory for beams having various boundary conditions. Accurate Eigen frequency parameters are presented for a different length to height ratio for each case. The exact closed form mode shapes related to deflection, slope due to bending and stress resultants are also presented and illustrated for some cases. The modal tests are performed for beams with clamped-Free and Free-Free boundary conditions. Finally, the effect of boundary conditions, length to height ratio on the eigenvalues parameters and vibratory behavior of each distinct case are studied. Validity of the derived closed form characteristic equations are checked through comparison of numerical solutions with the available results. It is believed that in the present work, the exact closed form characteristic equations and their associated Eigen functions, except for the beams with simply supported ends, for the rest of considered cases are obtained for the first time.

Keywords : Bench mark equation; Transverse vibration; Timoshenko beam; Characteristic equation.

1 INTRODUCTION

BEAMS are known as extremely important structural elements due to their widely applications in many branches of modern technology pertaining to aerospace, mechanical, marine, nuclear and civil engineering. Moreover whole or part of structural components like ship hull, turbine blades, long span bridges, robot arms, and cantilever of atomic force microscopic (AFM) may be modeled with beam-like elements. Thus, the knowledge of their free vibrational behavior is very important to the structural designers. Beam vibrations described by the Euler-Bernoulli beam theory having various boundary conditions have been studied over the years by many researchers

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Rao [1] and Craig and Kurdila [2]. The well-known classical or Euler- Bernoulli beam theory disregards the effect of the transverse shear deformation and rotatory inertia. As a result the Euler- Bernoulli beam theory underestimates deflections and overestimates the natural frequencies. Improving on the elementary beam theory Timoshenko [3, 4] was the first to include effects of rotatory inertia and shear deformation in the beam theory. The first-order shear deformation beam theory of Timoshenko, however, requires a shear correction factor to compensate the error resulting from the approximation made on the non-uniform shear strain distribution. Recent investigation in this regard is given by Freund and Karakoc [5] by taking into account the warping of cross section for rectangle, open annular, and angle cross sections. Beam vibrations described by the Timoshenko model have been represented using several different approaches over the years. Cowper [6] studied the accuracy of Timoshenko beam theory for transverse vibrations of simply supported beam in respect of the fundamental frequency with a plane stress exact elasticity solution. Turgut Kocaturk and Mesut Simse [7] investigated the free vibration characteristics of Timoshenko rectangular beams, by using the Lagrange equations. They expressed trial functions denoting the deflection and the rotation of the cross-section of the beam in the polynomial form. Lee and Schultz [8] treated the eigenvalue analysis of Timoshenko beams and axisymmetric circular Mindlin plates by applying the Chebyshev pseudo spectral method. Clamped, Simply supported, free and sliding boundary condition of Timoshenko beams are treated. Civalek and Kiracioglu [9] used the discrete singular convolution method (DSC) for numerical solution of equation of motion of Timoshenko beam. They reported numerical results for first six frequency of clamped-clamped and simply supported- simply supported beam. Other works of interest are Avcar [10] and Sayyad [11]. The present paper is part of a large study of the transverse vibration analysis of Euler- Bernoulli and Timoshenko beams with classic and non-classic boundary conditions. Hosseini Hashemi [12]. The exact closed form characteristic equations for Euler- Bernoulli beam having classical boundary conditions namely Clamped-Clamped (C-C), Clamped-Simply (C-S), Simply-Simply (S-S), Clamped-Free (C-F), Simply-Free (S-F) and Free-Free (F-F) may be found in many books [5,6]. No such equations about Timoshenko beam except for the case of (S-S) are available in the literature. To fill this apparent void the present work is carried out to provide the exact characteristic equations for Timoshenko beam having classical boundary conditions. The integrated equations of motion in terms of the stress resultant are derived based on Timoshenko beam theory. The dimensionless frequency parameters calculated from the exact characteristic equations are tabulated for all considered cases, covering wide ranges of thickness to length ratio δ .

These results may serve as benchmark solution for validating approximate methods and new computational techniques in future. The mode shapes related to transverse deflection, slope due to bending and stress resultants are also given in closed form equations and illustrated for some cases. Finally to validate the accuracy of numerical results comparison with the available results in literature are made. Modal tests are also carried out for C-F and F-F beams.

2 MATHEMATICAL FORMULATIONS

Consider a Timoshenko beam of length *l*, width *b*, and uniform thickness *h*, oriented so that its un-deformed middle surface contains the x_1 and x_2 axis of a Cartesian co-ordinate system (x_1, x_2, x_3), as shown in Fig. 1.

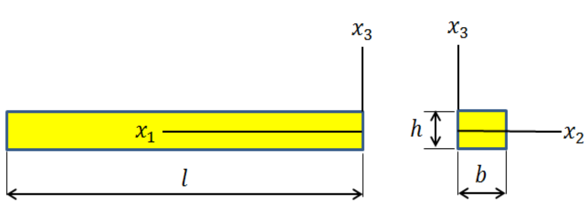

Fig.1
A Timoshenko beam with co-ordinate convention.

The displacements along the x_1 and x_2 axes are denoted by U_1 and U_2, respectively, while the displacement in the direction perpendicular to the un-deformed middle surface is denoted by U_3. In the Timoshenko beam theory, the displacement components are assumed to be given by

$$U_1 = -x_3 \psi(x_1, t), \; U_2 = 0, \; U_3 = w(x_1, t) \tag{1}$$

where t is the time, w is the transverse displacement, and ψ is the slope due to bending alone. Using the displacement field given in Eq. (1), the tonsorial components of the strains may be expressed as:

$$\epsilon_{11} = -x_3\psi_{,1}; \quad \epsilon_{22} = 0; \quad \epsilon_{33} = 0; \quad \epsilon_{12} = 0; \quad \epsilon_{13} = \frac{1}{2}\left(w_{,1} - \psi\right); \quad \epsilon_{23} = 0; \tag{2}$$

Based on the strain–displacement relations given in Eqs. (2) and assuming a stress distribution in accordance with Hook's law, the resultant bending moment and the transverse shear force, in terms of w, and ψ are obtained by integrating the shear stress and moment of the axial stress through the cross-section area of the beam. These are given by

$$\begin{aligned} Q &= \int_A \sigma_{13} dA = \kappa GA(w_{,1} - \psi) \\ M &= -\int_A \sigma_{11} x_3 dA = EI\psi_{,1} \end{aligned} \tag{3}$$

where $G = E/2(1+\nu)$ is the shear modulus, ν is the Poisson's ratio and k is the shear correction factor to account for the fact that the transverse shear strains are not truly independent of the thickness coordinate. The governing equations of motion may now be derived from the two-dimensional stress equations of motion which are written as the governing equations of motion may now be derived from the two-dimensional stress equations of motion which are written as:

$$\begin{aligned} \sigma_{11,1} + \sigma_{13,3} &= \rho\ddot{U}_1 \\ \sigma_{31,1} + \sigma_{33,3} &= \rho\ddot{U}_3 \end{aligned} \tag{4}$$

where ρ is the mass density per unit volume. The first equation is multiplied by $bx_3 dx_3$ and then integrated through the thickness of the beam, making use of Eqs. (3) and the fact that there is no shear force applied to the top and bottom of the beam, while the second equation is multiplied by bdx_3 and integrated through the thickness of the beam, making use of the fact that

$$b\sigma_{33}\Big|_{-h/2}^{h/2} = P(x_1, t) \tag{5}$$

where P is the applied load per unit length. Thus, the integrated equations of motion are given by

$$\begin{aligned} M_{,1} + Q &= \rho I\ddot{\psi} \\ Q_{,1} + P &= \rho A\ddot{w} \end{aligned} \tag{6}$$

Assuming the free harmonic motion as:

$$\begin{aligned} M(x_1, t) &= \hat{M}(x_1)e^{i\omega t}; \\ Q(x_1, t) &= \hat{Q}(x_1)e^{i\omega t}; \\ w(x_1, t) &= \hat{w}(x_1)e^{i\omega t}; \\ \psi(x_1, t) &= \hat{\psi}(x_1)e^{i\omega t}. \end{aligned} \tag{7}$$

The integrated equations of motion in absence of the applied load and the stress resultants may be written as:

$$\hat{M}_{,1}+\hat{Q}=-\rho I\,\omega^2\hat{\psi}$$
$$\hat{Q}_{,1}=-\rho A\,\omega^2\hat{w} \tag{8}$$

It should be emphasize that in all above equations the comma-subscript convention represents the differentiation with respect to the x_1 coordinate. For generality and convenience, the x_1 coordinate is normalized with respect to the beam length and the following dimensionless terms are introduced

$$X=\frac{x_1}{l},\quad \bar{w}=\frac{\hat{w}}{l},\quad \bar{\psi}=\hat{\psi} \tag{9}$$

The stress resultants may then be written in dimensionless form as:

$$\bar{M}=\bar{\psi}_{,1}=\frac{\hat{M}l}{EI}$$
$$\bar{Q}=\bar{w}_{,1}-\bar{\psi}=\frac{\hat{Q}}{\kappa GA} \tag{10}$$

Substitution of the dimensionless stress resultants from expressions (10) into Eqs. (8) leads to

$$\eta\bar{\psi}_{,11}+\bar{w}_{,1}+\left(\eta\lambda\beta^2-1\right)\bar{\psi}=0 \tag{11a}$$

$$\bar{w}_{,11}-\bar{\psi}_{,1}+\eta\beta^2\bar{w}=0 \tag{11b}$$

where comma-subscript convention in Eqs. (10) and (11) represents the differentiation with respect to the normalized coordinate and

$$\eta=\frac{EI\,\delta^2}{12\kappa G},\quad \beta^2=\frac{\rho A\,\omega^2 l^4}{EI},\quad \lambda=\frac{1}{12}\delta^2,\quad \delta=\frac{h}{l} \tag{12}$$

are dimensionless parameters. The equations of motion are coupled in $\bar{w}$ and $\bar{\psi}$. In order to derive a single uncouple equation in term of $\bar{w}$, the first equation is differentiated with respect to normalized coordinate X and second one rearranged to give

$$\eta\bar{\psi}_{,111}+\bar{w}_{,11}+\left(\eta\lambda\beta^2-1\right)\bar{\psi}_{,1}=0 \tag{13a}$$

$$\bar{\psi}_{,1}=\bar{w}_{,11}+\eta\beta^2\bar{w} \tag{13b}$$

Using Eq. (13b) to eliminate $\bar{\psi}$ from Eq. (13a), one obtains the fourth order equation as:

$$\bar{w}_{,1111}+\beta^2(\eta+\lambda)\bar{w}_{,11}+\beta^2\left(\eta\lambda\beta^2-1\right)\bar{w}=0 \tag{14}$$

Also differentiating Eq. (11b) with respect to X and substituting into Eq. (11a) leads to

$$\bar{\psi}=\frac{1}{1-\eta\lambda\beta^2}\left[\eta\bar{w}_{,111}+(1+\eta^2\beta^2)\bar{w}_{,1}\right] \tag{15}$$

The solution to Eq. (14) may be expressed in the form

$$\bar{w}(X) = e^{\alpha X} \tag{16}$$

Substitution of Eq (16) into Eq. (14) yields the eigenvalue equation

$$\alpha^4 + (\eta + \lambda)\beta^2\alpha^2 + \left(\eta\lambda\beta^2 - 1\right)\beta^2 = 0. \tag{17}$$

The roots of this quadratic equation are

$$\alpha^2 = \frac{-(\eta+\lambda)\beta^2 \pm \sqrt{(\eta-\lambda)^2\beta^4 + 4\beta^2}}{2} \tag{18}$$

The eigenvalue equation has one positive and one negative root under the condition

$$(\eta+\lambda)\beta^2 < \sqrt{(\eta-\lambda)^2\beta^4 + 4\beta^2}. \tag{19}$$

As a result the roots may be given in the form

$$\alpha = \pm\mu_1, \quad \mu_1 = \frac{1}{\sqrt{2}}\left[-(\eta+\lambda)\beta^2 + \sqrt{(\eta-\lambda)^2\beta^4 + 4\beta^2}\right]^{\frac{1}{2}}, \tag{20a}$$

$$\alpha = \pm i\,\mu_2, \quad \mu_2 = \frac{1}{\sqrt{2}}\left[(\eta+\lambda)\beta^2 + \sqrt{(\eta-\lambda)^2\beta^4 + 4\beta^2}\right]^{\frac{1}{2}}. \tag{20b}$$

$$\bar{\psi}(X) = \frac{\mu_1\left(\beta^2\eta^2 + \eta\mu_1^2 + 1\right)\left(C_1\sinh\mu_1 X + C_2\cosh\mu_1 X\right) - \mu_2\left(\beta^2\eta^2 - \eta\mu_2^2 + 1\right)\left(C_3\sin\mu_2 X - C_4\cos\mu_2 X\right)}{S} \tag{21a}$$

$$\bar{M}(X) = \frac{\mu_1^2\left(\beta^2\eta^2 + \eta\mu_1^2 + 1\right)\left(C_1\cosh\mu_1 X + C_2\sinh\mu_1 X\right) - \mu_2^2\left(\beta^2\eta^2 - \eta\mu_2^2 + 1\right)\left(C_3\cos\mu_2 X + C_4\sin\mu_2 X\right)}{S} \tag{21b}$$

$$\bar{Q}(X) = -\frac{\eta\mu_1\left(\beta^2\eta + \mu_1^2 + \beta^2\lambda\right)\left(C_1\sinh\mu_1 X + C_2\cosh\mu_1 X\right) - \eta\mu_2\left(\beta^2\eta - \mu_2^2 + \beta^2\lambda\right)\left(C_3\sin\mu_2 X - C_4\cos\mu_2 X\right)}{S} \tag{21c}$$

where

$$S = 1 - \eta\lambda\beta^2. \tag{22}$$

2.1 Boundary conditions

The boundary conditions are represented as follows:

Fixed or clamped end $(X = X_i = 0$ or $X = X_i = 1)$

$$\bar{w}(X_i) = 0, \qquad \bar{\psi}(X_i) = 0. \tag{23a}$$

Simply supported end $(X = X_i = 1 \, or \, X = X_i = 0)$

$$\bar{w}(X_i)=0, \qquad \bar{M}(X_i)=0. \tag{23b}$$

Free end $(X=X_i=0 \text{ or } X=X_i=1)$

$$\bar{M}(X_i)=0, \qquad \bar{Q}(X_i)=0. \tag{23c}$$

2.2 Relevant closed equations

Substituting appropriate Eqs. (21a-d) into the four appropriate boundary conditions at the ends $X=0$ and $X=1$, lead to a characteristic determinant of fourth order. Expanding the determinant and collecting terms yields a characteristic equation. The characteristic equations for all considered cases are listed below

Case 1. C–C

$$\left(R_1^2\mu_1^2-R_2^2\mu_2^2\right)\sin\mu_2\sinh\mu_1+2R_1R_2\mu_1\mu_2\left(1-\cos\mu_2\cosh\mu_1\right)=0 \tag{24}$$

where

$$\begin{aligned} R_1 &= 1+\beta^2\eta^2+\eta\mu_1^2 \\ R_2 &= 1+\beta^2\eta^2-\eta\mu_2^2 \end{aligned} \tag{25}$$

Case 2. C–S

$$R_1\mu_1-R_2\mu_2\tanh\mu_1\cot\mu_2=0 \tag{26}$$

Case 3. S–S

$$\sin\mu_2\sinh\mu_1=0. \tag{27}$$

Find these rigid body modes one may substitutes $\beta=0$ into Eq. (14) and then solve the resultant differential equations for each relevant boundary conditions. As a result the normalized rigid body modes for F-F beam may be written as the general solution to Eqs. (14) and (15) can now be written as:

Case 4. C–F

$$\begin{aligned} &\left[\left(R_2-S\right)R_1^2\mu_1^2-\left(R_1-S\right)R_2^2\mu_2^2\right]\cos\mu_2\cosh\mu_1+R_1R_2\left(R_1+R_2-2S\right)\mu_1\mu_2\sin\mu_2\sinh\mu_1 \\ &\quad -R_1R_2\left[\left(R_1-S\right)\mu_1^2-\left(R_2-S\right)\mu_2^2\right]=0 \end{aligned} \tag{28}$$

Case 5. S–F

$$\left(R_1-S\right)R_2\mu_2\tan\mu_2+\left(R_2-S\right)R_1\mu_1\tanh\mu_1=0. \tag{29}$$

Case 6. F–F

$$2\left(R_1-S\right)\left(R_2-S\right)R_1R_2\mu_1\mu_2\left(1-\cos\mu_2\cosh\mu_1\right)+\left(R_2^2\mu_2^2S\left(2R_1-S\right)-R_1^2\mu_1^2S\left(2R_2-S\right)+R_1^2R_2^2\left(\mu_1^2-\mu_2^2\right)\right)\sin\mu_2\sinh\mu_1=0 \tag{30}$$

3 NUMERICAL RESULTS

A Mathematica code is used to calculate the frequency parameters of characteristic equations given by Eqs. (24-30). The square roots of first three dimensionless frequency are listed in Tables 1-6 for C-C, C-S, S-S, C-F, S-F and F-F boundary conditions. The mode numbers are arranged according to the number of nodes, so that what is so called the nth mode, having n-1 nodes. The mode shapes may also be solved for each dimensionless natural frequencies through given closed form relations. The dimensionless natural frequencies are calculated for different thickness-to-length ratios $\delta = 0.01$ to 0.2 throughout the Tables 1-6, Poisson's ratio and the shear correction factor of the beam are $\nu = 0.3$ and $\kappa = 5/6$ respectively. The solutions based on the Euler-Bernoulli beam theory, Blevins [13], are also added to the tables for comparison. The results show that the Timoshenko beam results are very close to the Euler-Bernoulli results when δ is less than 0.01. This can also be observed in Fig.2. As δ increases, however, the computed dimensionless frequencies tend to show some quantitative differences from the Euler-Bernoulli results. These results decreases as δ increases which in turn states what is known over prediction of results by classical theory. Moreover the difference of the value of the eigenvalue of the classical beam theory and the Timoshenko beam theory increases for increasing mode numbers. Careful observation of Tables 1-6 revels that the lowest and highest values of frequency parameters correspond to F-F and C–C cases, respectively. Fig. 3 shows that for a fixed δ, the first six dimensionless frequencies for C-C beam are higher than those given for F-F beam. This observation remains an alter as boundary conditions changes from C-C to C-S, C-S to S-S, S-S to C-F, C-F to S-F and S-F to F-F. Thus higher constraints at the beam ends increase the flexural rigidity of the beam, resulting in a higher frequency response.

$$\bar{w}_{01} = 1, \qquad \bar{\psi}_{01} = 0$$
$$\bar{w}_{02} = \frac{\sqrt{3}(2X-1)}{\sqrt{1+12\lambda}}, \qquad \bar{\psi}_{02} = \frac{2\sqrt{3}}{\sqrt{1+12\lambda}} \tag{31}$$

It should be noted that for Timoshenko beam the modes may be normalized through relation

$$\int_0^1 \left(\bar{w}_n^{\,2} + \lambda \bar{\psi}_n^{\,2}\right) dX = 1 \tag{32}$$

The normalized transverse deflection and slope due to bending for C-F beam are given in Figs. 5 and 6.

Table 1
The first three dimensionless frequency parameters of the C-C Timoshenko beam for different thickness-to-length ratios; (I) Lee's results [8], (II) Civalek's results [9], (III) Kocaturk 's results [7].

δ	Method	$\sqrt{\beta_1}$	$\sqrt{\beta_2}$	$\sqrt{\beta_3}$
	Classical	4.73004	7.85320	10.9956
0.01	Present	4.72840	7.84690	10.9800
	(II)	4.72840	7.84690	10.9801
0.05	Present	4.68991	7.70352	10.6402
	(III)	4.68987	7.70351	10.6399
0.1	Present	4.57955	7.33122	9.85611
	(I)	4.57955	7.33122	9.85611
0.2	Present	4.24202	6.41794	8.28532
	(I)	4.24201	6.41794	8.28532

Table 2
The first three dimensionless frequency parameters of the C-S Timoshenko beam for different thickness-to-length ratios; (III) Kocaturk 's results [7].

δ	Method	$\sqrt{\beta_1}$	$\sqrt{\beta_2}$	$\sqrt{\beta_3}$
	Classical	3.92700	7.06900	10.2101
0.01	Present	3.92581	7.06469	10.1993
	(III)	3.92580	7.06460	10.1992
0.05	Present	3.90713	6.97478	9.95631
	(III)	3.90710	6.97470	9.95620
0.1	Present	3.85177	6.73057	9.36591
	(III)	3.85170	6.73050	9.36580
0.2	Present	3.66561	6.07268	8.07437
	(III)	3.66560	6.07260	8.07430

The frequencies $\beta = 0$ listed in Tables 5 and 6 show there are rigid body modes for simply-free and free-free beams, respectively. For beam with F-F boundary conditions there are two rigid body modes corresponding to rigid translation and rotation. The mode shape related to rigid body modes for F-F beam are given by Eqs. (31). Variation of dimensionless frequency parameter with δ of C-C Timoshenko beam for different mode numbers is shown in Fig.4. It is shown from this figure that the effect of δ on the frequency parameter is almost insignificant for first mode. In other word, the results show that the thickness become more influence as mode numbers increase. Among six considered cases in the present work, there are rigid body modes for two cases namely beam with S-F and F-F boundary conditions.

Table 3
The first three dimensionless frequency parameters of the S-S Timoshenko beam for different thickness-to-length ratios; (I) Lee's results [8].

δ	Method	$\sqrt{\beta_1}$	$\sqrt{\beta_2}$	$\sqrt{\beta_3}$
	Classical	3.14159	6.28319	9.42478
0.01	Present	3.14133	6.28106	9.41761
	(I)	3.14133	6.28106	9.41761
0.05	Present	3.13498	6.23136	9.25537
	(I)	3.13498	6.23136	9.25537
0.1	Present	3.11568	6.09066	8.84051
	(I)	3.11568	6.09066	8.84052
0.2	Present	3.04533	5.67155	7.83952
	(I)	3.04533	5.67155	7.83952

Table 4
The first three dimensionless frequency parameters of the C-F Timoshenko beam for different thickness-to-length ratios.

δ	Method	$\sqrt{\beta_1}$	$\sqrt{\beta_2}$	$\sqrt{\beta_3}$
	Classical	1.87510	4.69410	7.85480
0.01	Present	1.87503	4.69278	7.84955
0.05	Present	1.87324	4.66204	7.73048
0.1	Present	1.86771	4.57241	7.41542
0.2	Present	1.84656	4.28529	6.61129

Table 5
The first three dimensionless frequency parameters of the S-F Timoshenko beam for different thickness-to-length ratios.

δ	Method	$\sqrt{\beta_1}$	$\sqrt{\beta_2}$	$\sqrt{\beta_3}$
	Classical	0	3.92660	7.06858
0.01	Present	0	3.92607	7.06557
0.05	Present	0	3.91382	6.99561
0.1	Present	0	3.87702	6.80198
0.2	Present	0	3.74860	6.25381

Table 6
The first three dimensionless frequency parameters of the F-F Timoshenko beam for different thickness-to-length ratios, (I) Lee's results [8].

δ	Method	$\sqrt{\beta_1}$	$\sqrt{\beta_2}$	$\sqrt{\beta_3}$
	Classical	0	0	4.73004
0.01	Present	0	0	4.72918
	(I)	0	0	4.72918
0.05	Present	0	0	4.70873
	(I)	0	0	4.70873
0.1	Present	0	0	4.64849
	(I)	0	0	4.64849
0.2	Present	0	0	4.44958
	(I)	0	0	4.44958

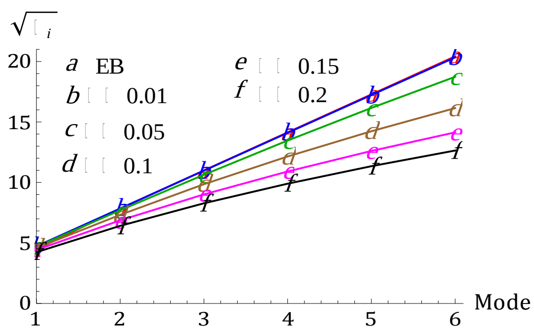

Fig.2
Variation of dimensionless frequency parameter against Mode number for C-C Timoshenko beam in different height to length ratio.

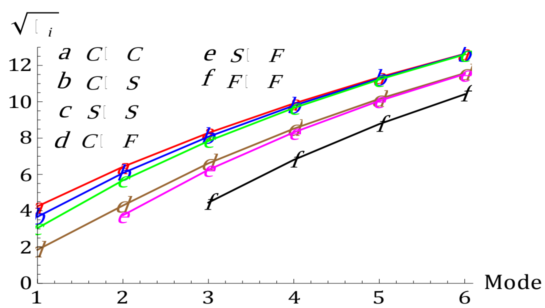

Fig.3
Variation of dimensionless frequency parameter against mode number for different boundary conditions $\delta = 0.2$.

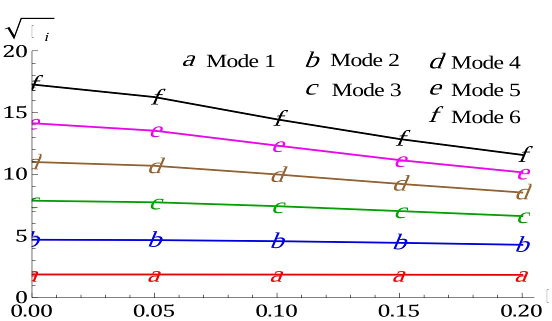

Fig.4
Variation of dimensionless frequency parameter against δ for C-F Timoshenko beam in different mode numbers.

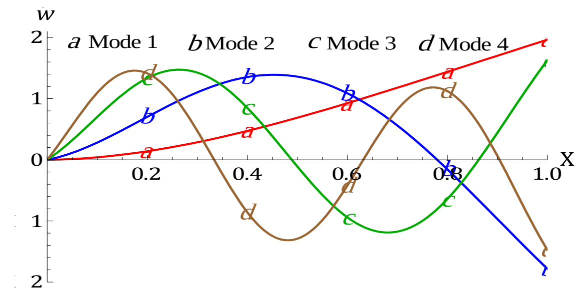

Fig.5
The first fourth mode shapes of normalized transverse displacement for C-F Timoshenko beam $\delta = 0.2$.

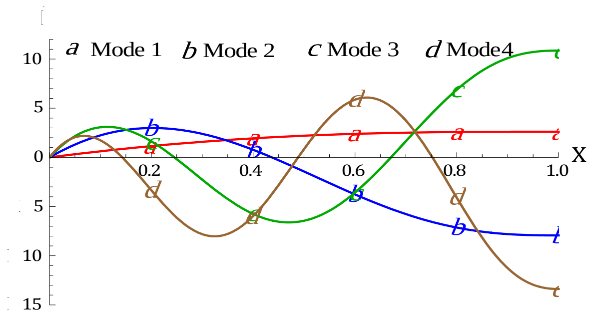

Fig.6
The first fourth mode shapes of normalized slope due to bending for C-F Timoshenko beam $\delta = 0.2$.

4 EXPERIMENTAL STUDY

In order to validate The results for F-F and C-F beams a modal test is carried out on a sample with dimension $199 \times 19 \times 20mm$, modulus of elasticity $E = 212GPa$ and Poisson's ratio $\nu = 0.3$. The measuring system consisted of an eight channels Econ analyzer and piezoelectric accelerometer covering frequency range up to 20 KHz also a hammer with covering frequency range up to 20 *kHz*. The results of performance tests on samples together with the finite element results are given in Table 7. The finite element results are obtained by using three dimensional element having eighteen nodes simulated in Ansys software. The sample under test for F-F and C-F boundary conditions is also illustrated in Fig. 7.

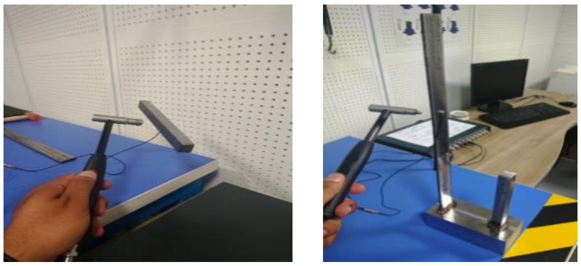

Fig.7
Performance of test on sample.

Table 7
Comparison of the first three dimensionless frequency parameters with experimental results.

BC	Method	$\sqrt{\beta_1}$	$\sqrt{\beta_2}$	$\sqrt{\beta_3}$
C-F	Present	1.86771	4.57241	7.41542
	Test	1.76899	4.48768	7.24060
	FEM	1.82160	4.47302	7.27511
F-F	Present	4.64849	7.49719	10.1255
	Test	4.53502	7.31677	9.90709
	FEM	4.53321	7.33303	9.93689

Numerical calculations have been carried out to clarify the effects of the thickness to length ratio on the eigenvalues of the beams. It is observed from the investigations that the results of the classical and the Timoshenko beam theory are very close to each other for small values of δ However, as the thickness to length ratio becomes larger, the results of the classical theory and the Timoshenko beam theory differ from each other significantly. All of the obtained results are very accurate and may be useful to other researchers as a bench mark eqns. to compare their results.

5 CONCLUSION

The free vibration of the Timoshenko beams have been investigated for different thickness-to-length ratios. The obtained eigenvalues for the Timoshenko beams having various boundary conditions are compared with the previously published results. Using the closed form characteristic equations is a very good way for studying the free vibration characteristics of the beams.

REFERENCES

[1] Rao S.S., 2011, *Mechanical Vibrations*, Prentice-Hall Inc, New Jersey.
[2] Craig R.R., Cordial A.J., 2006, *Fundamentals of Structural Dynamics*, John Wiley & Sons, New Jersey.
[3] Timoshenko S.P., 1921, On the correction for shear of the differential equation for transverse vibrations of prismatic bars, *Philosophical Magazine* **41**(6): 742-746.
[4] Timoshenko S. P., 1922, On the transverse vibration of bars of uniform cross section, *Philosophical Magazine* **43**(6): 379-384.
[5] Freund J., Karaoke A., 2016, Shear and torsion correction factors of Timoshenko beam model for generic cross sections, *Research on Engineering Structures & Materials* **2**: 19-27.
[6] Cowper G.R., 1968, On the accuracy of Timoshenko's beam theory, *ASCE Journal of Engineering Mechanics Division* **94**(6): 1447-1453.
[7] Kocaturk T., Simsek M., 2005, Free vibration analysis of Timoshenko beams under various boundary conditions, *Sigma Journal of Engineering and Natural Sciences* **2**(1): 30-44.
[8] Lee J., Schultz W.W., 2004, Eigenvalue analysis of Timoshenko beams and axisymmetric Mindlin plates by the pseudospectral method, *Journal of Sound and Vibration* **269**: 609-621.
[9] Civalek O., Kiracioglu O., 2010, Free vibration analysis of Timoshenko beams by DSC method, *International Journal for Numerical Methods in Biomedical Engineering* **26**: 1890-1898.
[10] Avcar M., 2014, Free vibration analysis of beams considering different geometric characteristic and boundary conditions, *International Journal of Mechanical and Applications* **4**(3): 94-100.
[11] Sayyad A.S., 2011, Comparison of various refined beam theories for the bending and free vibration analysis of thick beams, *Applied and Computational Mechanics* **5**: 217-230.
[12] Hosseini Hashemi Sh., 2017, *Transverse Vibration Analysis of Euler-Bernoulli and Timoshenko Beams with Classic and Non-Classic Boundary Conditions*, BSc. Dissertation, School of Mechanical and Aerospace Engineering, Islamic Azad University, Science and Research Branch.
[13] Blevins R.D., 1979, *Formula for Natural Frequency and Mode Shape*, Van Nostrand Reinhold, Newyork.