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Study of Torsional Vibrations of Composite Poroelastic Spherical Shell-Biot's Extension Theory

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ABSTRACT

Torsional vibrations of composite poroelastic dissipative spherical shell are investigated in the framework of Biot's extension theory. Here composite poroelastic spherical shell consists of two spherical shells, one is placed on other, and both are made of different poroelastic materials. Consideration of the stress-free boundaries of outer surface and the perfect bonding between two shells leads to complex valued frequency equation. Limiting case when the ratio of thickness to inner radius is very small is investigated numerically. In this case, thick walled composite spherical shell reduces to thin composite spherical shell. For illustration purpose, four composite materials, namely, Berea sandstone saturated with water and kerosene, Shale rock saturated with water and kerosene are employed. The particular cases of a poroelastic solid spherical shell and poroelastic thick walled hollow spherical shell are discussed. If the shear viscosity of fluid is neglected, then the problem reduces to that of classical Biot's theory. Phase velocity and attenuation are computed and the results are presented graphically. Comparison is made between the results of Biot's extension theory and that of classical Biot's theory. It is conclude that shear viscosity of fluid is causing the discrepancy of the numerical results.

Keywords: Torsional vibrations; Composite spherical shell; Frequency equation; Phase velocity; Attenuation.

1 INTRODUCTION

THE analysis of torsional vibrations in poroelastic solids is important in several fields. In the Electrical Engineering, power needs to be transmitted using a rotating shaft or a coupling, which causes torsional vibrations. In the case of Automobile Engineering, the auto motives, truck and bus drivelines, recreation vehicles, and marine drivelines experience torsional vibrations. These vibrations are influenced by material properties and operating conditions. Thus, torsional vibrations find its vast applications in different branches of Engineering. On one hand, composite structures are combination of both load bearing as well as framed structures, and most useful materials. Because of their adaptability to different situations and the relative ease of combination with other

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materials, they serve specific purposes, and exhibit desirable properties. Hence, further investigation of composite materials is warranted. On other hand, spherical shell structures play vital role in various domains, particularly, in modern Structural Engineering. In this direction, the following papers are available in the literature. Frequency equations and mode shapes are presented in analytic form for solid prolate spheroids and thick prolate spheroidal shells [1]. Heyliger and Pan [2] presented a discrete-layer model and applied to the free vibration of layered anisotropic spheres with coupling among the elastic, electric, and magnetic fields. Employing Biot's theory [3], Shah and Tajuddin [4] discussed torsional vibrations of poroelastic spheroidal shells. In the paper [4], they derived frequency equations for poroelastic thin spherical shell, thick spherical shell, poroelastic solid sphere, and concluded that the wave frequency is same in all the three cases. For radial and rotatory vibrations, frequency equations of a poroelastic composite hollow sphere and a poroelastic composite hollow sphere with rigid core are obtained [5]. Vibrations in poroelastic elliptic cone against the angle made by the major axis of the cone in the spheroconal coordinate system are studied by Rajitha and Reddy [6, 7]. A comparative study is made between the modes of composite spherical shell and its ring modes [8]. In the paper [9], authors concluded that the torsional waves are non dispersive in thin coated hollow poroelastic sphere. In all the above studies, solid saturated with the non-viscous fluid is considered which may not be realistic in all the solids. In view of this limitation, Sahay [10] has developed constitutive relations for the case of viscous fluid. In the paper [10], the volume-average equations of motion for the angular displacement fields are derived. This approach introduces the missing fluid-strain rate term in the Biot's extended constitutive relations. Employing Biot's extension theory of poroelasticity, Solorza and Sahay [11] investigated the extensional wave in a poroelastic cylinder, wherein the axial motion is compressional in nature, that is, the direction of propagation is along the axial direction, and the radial motion is shear in nature, as it is being executed perpendicular to the direction of propagation. Reddy and Rajitha [12, 13] discussed the torsional and radial vibrations of thick walled hollow poroelastic cylinder in the framework of Biot's extension theory. To the best of authors' knowledge, torsional vibrations in composite poroelastic spherical shell in the framework of Biot's extension theory are not yet investigated. Hence in the present work, the same is taken up. Limiting case, when the ratio between thickness and inner radius is very small is investigated numerically.

In this case, the asymptotic expansions of Bessel functions are employed so that complex valued frequency equation can be separated into two real parts, which in turn give phase velocity and attenuation. In the particular case of thick walled hollow spherical shell, phase velocity and attenuation computed as a function of ratio of thickness to inner radius. In the case of classical Biot's theory, phase velocity is computed as a function of frequency. The rest of the paper is organized as follows. In section 2, formulation and solution of the problem are given. In section 3, the boundary conditions and frequency equation are discussed. Particular cases are discussed in section 4. Numerical results are presented graphically in section 5. Finally, the conclusion is given in section 6.

2 FORMULATION AND SOLUTION OF THE PROBLEM

Consider a composite poroelastic spherical shell with outer and inner radii r_2 and r_1 respectively, in the spherical coordinate system (r,θ,ϕ). The shell is made up of two different materials where in inner one is spherical shell of radius (r_1) and the outer one is spherical shell having uniform thickness $h=(r_2-r_1>0)$. For torsional vibrations, the volume-averaged equation of motion in terms of angular displacement field $\vec{u}$ expressed as follows [11, 12]:

$$\left(C+N\frac{\partial}{\partial t}\right)\left[\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right]\vec{u}=\Omega_i\ I_0\frac{\partial\vec{u}}{\partial t}+I\ \frac{\partial^2\vec{u}}{\partial t^2}. \tag{1}$$

In Eq. (1), $C=\begin{pmatrix}1 & m_f\\ d_f & d_f m_f\end{pmatrix}\beta_c^2$, $N=\begin{pmatrix}\alpha_\mu & -(\eta_0-m_f\alpha_\mu)\\ -d_s\alpha_\mu & d_s(\eta_0-m_f\alpha_\mu)\end{pmatrix}\beta_c^2\dfrac{1}{\Omega_\beta}$, $d_f=\dfrac{1}{S-m_f}$, and $d_s=\dfrac{\phi_0\rho_s}{\rho_i}$, ϕ_0 is unperturbed volume fraction of solid, S is tortuosity factor, $\beta_c^2=\dfrac{\mu_0}{\rho_m}$ is Gassmann S-wave velocity, μ_0 is dry solid frame shear modulus, α_μ is Biot shear coefficient, Ω_β is saturated frame shear relaxation frequency, $\Omega_i=d_f\Omega_\beta$ is

Biot relaxation frequency, $\Omega_b = \dfrac{\eta_0 \vartheta_f}{K}$ is the Biot critical frequency, η_0 is porosity, K is permeability, and $\vartheta_f = \dfrac{\mu_f}{\rho_f}$ is the shear viscosity, μ_f is the fluid shear viscosity. The matrix I_0 is 2×2 matrix whose element $(2\,2)$ is unity and rest of the elements are equal to zero, and I is the second order identity matrix. The vector $\vec{u} = \left(u_j^m \;\; u_j^i\right)^T = m^{-1}\left(u_j^s \;\; u_j^f\right)^T$. The notation u_j^m is the sum of mass-weighted angular displacement of the solid and fluid phases. The notation u_j^i is the difference of the angular displacements of two phases. The transformation matrix is defined by $m = \begin{pmatrix} 1 & m_f \\ 1 & -m_s \end{pmatrix}$, m_f , m_s are the solid and fluid mass fractions, respectively, and ρ_m , ρ_r are the total and reduced densities of the porous medium, respectively, given by $\rho_m = \phi_0\, \rho_s + \eta_0 \rho_f$, $\dfrac{1}{\rho_r} = \dfrac{1}{\phi_0\, \rho_s} + \dfrac{1}{\eta_0\, \rho_f}$, $\rho_i = \rho_r - \rho_{12}$ is the modified reduced density, and ρ_s , ρ_f are the solid and fluid densities. ρ_{12} is the induced mass coefficient which is linked to the tortuosity, as $\rho_{12} = -\,(S-1)\,\eta_0\,\rho_f$. In the frequency domain, Eq. (1) becomes

$$\left[\beta\left(\nabla^2 - \frac{2}{r^2}\right) + \omega^2 I\right]\vec{u} = 0, \tag{2}$$

where $\beta = \Omega^{-1}(C - \mathrm{i}\omega\, N) = \begin{pmatrix} \beta^{mm} & \beta^{mi} \\ \beta^{im} & \beta^{ii} \end{pmatrix}$, is a non-symmetric second order matrix associated with S motion, respectively, whose elements are dimensionally equal to velocity squared. The expressions of these parameters are given in [11, 12]. The notation $I + i\left(\dfrac{\Omega_i}{\omega}\right) I_0 = \Omega$ is a 2×2 diagonal matrix associated with the Biot relaxation frequency Ω_i . For torsional vibrations, the displacement components which are functions of r and t are introduced as follows:

$${}_y\vec{u} = R_\beta\; {}_yF(r)\,,\;\; y = 1,2. \tag{3}$$

here $R_\beta = [r_{\beta I}, r_{\beta II}]$ and $L_\beta = [I_{\beta I}, I_{\beta II}]$ are the right- and left- hand Eigen vector matrices of the β matrix, respectively, their explicit expressions are given in [11,12]. These Eigen vectors are orthonormal to each other, therefore $L_\beta^T R_\beta = I$. Here $F(r) = [F_1(r), F_2(r)]^T$, and k is the wave number. The subscript $y = 1$ refers to the inner shell, whereas $y = 2$ refers to the outer shell. Substituting Eq. (3) in Eq. (2), and then multiplying L_β^T on both sides, the equations for S motion are obtained as:

$$\left[\Lambda_\beta\left(\nabla^2 - \frac{2}{r^2}\right) + \omega^2 I\right]{}_yF(r) = 0,\;\; y = 1, 2, \tag{4}$$

here $\Lambda_\beta = L_\beta^T\,\beta\, R_\beta = \begin{pmatrix} {}_y\beta_I^2 & 0 \\ 0 & {}_y\beta_{II}^2 \end{pmatrix}$, $y = 1, 2.$ In the region below Biot relaxation frequency $(\omega \ll \Omega_i)$ ${}_y\beta_I$, ${}_y\beta_{II}$, are the fast and slow S wave velocities given by [11]

$$\begin{aligned} {}_y\beta_I^2 &\approx {}_y\beta_c^2\left[1 - i\left(\frac{\omega}{{}_y\Omega_i}\right){}_yd_f\; {}_ym_f\right],\;\; y = 1, 2, \\ {}_y\beta_{II}^2 &\approx -\omega\left(\frac{\omega}{{}_y\Omega_i}\right)\left[1 + i\left(\frac{\omega}{{}_y\Omega_i}\right)(1 + {}_yd_f\; {}_ym_f)\right]{}_yd_f\; {}_y\vartheta_f\,,\;\; y = 1, 2. \end{aligned} \tag{5}$$

The solutions are

$$
\begin{aligned}
&{}_2F_1(r)=\left(C_1J_0({}_2pr)+C_2Y_0({}_2pr)\right),\ {}_1F_1(r)=C_3J_0({}_1pr),\\
&{}_2F_2(r)=\left(C_4J_0({}_2qr)+C_5Y_0({}_2qr)\right),\ {}_1F_2(r)=C_6J_0({}_1qr).
\end{aligned}
\tag{6}
$$

In the above, C_1,C_2,C_3,C_4,C_5 and C_6 are constants, J_0 and Y_0 are the Bessel functions of the first and second kind of order zero, respectively, and

$$
{}_y p=\left(\frac{\omega^2}{{}_y\beta_I^2}-k^2\right)^{\frac{1}{2}},\ {}_y q=\left(\frac{\omega^2}{{}_y\beta_{II}^2}-k^2\right)^{\frac{1}{2}},\ y=1,2.
$$

The modified constitutive relations are expressed as [11]

$$
\begin{pmatrix}{}_y\sigma^m_{r\theta}\\ {}_y\sigma^i_{r\theta}\end{pmatrix}=2\left({}_y\mu_b+{}_y\vartheta_b\partial_t\right)\begin{pmatrix}{}_yu^m_{r\theta}\\ {}_yu^i_{r\theta}\end{pmatrix},\ \ y=1,2,
\tag{7}
$$

here $\begin{pmatrix}1&0\\0&0\end{pmatrix}{}_y\mu_0={}_y\mu_b,\ \begin{pmatrix}0&0\\{}_y(\alpha_\mu-\eta_0)&{}_y\eta_0\end{pmatrix}{}_y\mu_s={}_y\vartheta_b,\ y=1,2.$ Eq. (7) can also be expressed as:

$$
{}_y\sigma_{r\theta}=2\,{}_y\rho_y\,\Omega\,{}_y\beta\,{}_y\vec{u}_{r\theta},\ \ y=1,2,
\tag{8}
$$

here ${}_y\vec{u}_{r\theta}=\frac{1}{2}\left(\frac{\partial}{\partial r}-\frac{1}{r}\right){}_y\vec{u},\ \ y=1,2.$

3 BOUNDARY CONDITIONS AND FREQUENCY EQUATION

Consider the case when outer surface $r=r_2$ is stress free and perfect bonding prevails at the interface $r=r_1$. The boundary conditions in this case are mathematically expressed as follows:

$$
\begin{aligned}
&{}_2\sigma^m_{r\theta}={}_2\sigma^i_{r\theta}=0\ \text{at}\ r=r_2,\\
&{}_2\sigma^m_{r\theta}-{}_1\sigma^m_{r\theta}=0,\ {}_2\sigma^i_{r\theta}-{}_1\sigma^i_{r\theta}=0\ \text{at}\ r=r_1,\\
&{}_2u^m_{r\theta}={}_1u^m_{r\theta},\ {}_2u^i_{r\theta}={}_1u^i_{r\theta}\ \text{at}\ r=r_1.
\end{aligned}
\tag{9}
$$

These boundary conditions lead to the following system of homogeneous equations:

$$
[A_{lm}][C_l]=0,\ \ l,m=1,2,3,4,5,6.
\tag{10}
$$

Eq. (10) is complex valued and is transcendental in nature. Hence, the limiting case $\frac{h}{r_1}<<1$ is considered. In this case, composite thick walled hollow spherical shell reduces to composite thin spherical shell so that the following asymptotic approximations for Bessel's function [13, 14] can be used:

$$
J_n(x)\approx\sqrt{\frac{2}{\pi x}}\left[cos\left(x-\left(\frac{n}{2}+\frac{1}{4}\right)\pi\right)-\frac{15}{8x}sin\left(x-\left(\frac{n}{2}+\frac{1}{4}\right)\pi\right)\right],
$$

$$Y_n(x) \approx \sqrt{\frac{2}{\pi x}} \left[sin\left(x - \left(\frac{n}{2} + \frac{1}{4} \right)\pi \right) + \frac{15}{8x} cos\left(x - \left(\frac{n}{2} + \frac{1}{4} \right)\pi \right) \right].$$

In this case, the frequency equation is resolved as:

$$\left| c_{l\,m} \right| + i \left| d_{l\,m} \right| = 0 \quad (l, m = 1, 2, 3, 4, 5, 6). \tag{11}$$

In Eq. (11), the elements $c_{l\,m}$ and $d_{l\,m}$ are real valued and are given in the Appendix A.

4 PARTICULAR CASES

The composite poroelastic spherical shell will be reduced to the poroelastic solid spherical shell and the poroelastic thick walled hollow spherical shell as follows.

4.1 Poroelastic solid spherical shell

When the poroelastic parameters of both outer and inner shells are of same material, then the composite poroelastic spherical shell results in a solid spherical shell of one material. Then the frequency Eq. (11) reduces to

$$\left| e_{l\,m} \right| + i \left| f_{l\,m} \right| = 0 \quad (l, m = 1, 2). \tag{12}$$

4.2 Poroelastic thick walled hollow spherical shell

Consider the case where the inner shell material parameters are vanishes, the composite poroelastic spherical shell results in a thick walled hollow poroelastic spherical shell with thickness $h\,(= r_2 - r_1)$. Then the frequency Eq. (11) reduces to

$$\left| g_{l\,m} \right| + i \left| h_{l\,m} \right| = 0 \quad (l, m = 1, 2, 3, 4). \tag{13}$$

The expressions for Eq. (12) and (13) are similar expressions as Appendix A without left subscript.

5 NUMERICAL RESULTS

The real part of Eq. (11) or Eq. (13) gives the phase velocity as in the case of non dissipative solid for composite and thick walled hollow spherical shell. The attenuation (Q^{-1}) is computed by using the following equation [11].

$$Q^{-1} = \frac{2(\,\omega\, of\;\; Im aginary\;\; part\;\; in\;\; Eq.(11)\;\; or\;\; Eq.(13))}{\omega\, of\;\; real\;\; part\;\; in\;\; Eq.(11)\;\; or\;\; Eq.(13)}.$$

For the illustration purpose, four types of poroelastic solids namely, composite spherical shell-I, composite spherical shell-II, composite spherical shell-III and composite spherical shell-IV are employed. In composite spherical shell-1, outer shell is made up of Berea sandstone saturated with water [13] and inner shell is made up of Berea sandstone saturated with kerosene [13]. In composite spherical shell- II, the roles of materials are reversed. In composite spherical shell-III, outer shell is made up of Shale rock saturated with water [15] and inner shell is made up of Shale rock saturated with kerosene [15]. In composite spherical shell-IV, the roles of materials are reversed. The physical parameter values are given in Table 1 and Table 2.

Table 1
Parameter values of solids

Parameters	Berea Sandstone	Shale Rock
Solid density (ρ_s)	$2.65\times10^{3}\ (Kg/m^{3})$	$1398.72\ (Kg/m^{3})$
Solid frame shear modulus (μ_0)	$6.70\times10^{9}\ (Kg/m\sec^{2})$	$7.59\times10^{8}\ (Kg/m\sec^{2})$
Solid mineral shear modulus (μ_s)	$2.30\times10^{10}\ Pa$	$1.97\times10^{8}\ Pa$
Solid frame bulk modulus (K_0)	$5.20\times10^{9}\ Pa$	$37.91\times10^{9}\ Pa$
Solid mineral bulk modulus (K_0)	$35\times10^{9}\ Pa$	$44.6\times10^{9}\ Pa$
Porosity (η_0)	0.25	0.15
Tortuosity (s)	1.33	3.22
Permeability (K)	$1.00\times10^{-14}\ (m^{2})$	$1.973\times10^{-18}\ (m^{2})$

Table 2
Parameter values of fluids.

Parameters	Water	Kerosene
Density $(\rho_f, Kg/m^{3})$	1.00×10^{3}	820.1
Shear viscosity $(\mu_f, Kg/m)$	1.00×10^{-3}	1.64×10^{-3}
Bulk viscosity $(\xi_f, Kg/m^{3})$	2.8×10^{-3}	2.71×10^{-3}
Bulk modulus (K_f, Pa)	2.2×10^{9}	1.3×10^{9}

Employing these values in Eq. (11), phase velocity and attenuation are computed as a function of frequency, and the results are depicted in Figs. 1 and 2. Fig. 1 depicts variation of phase velocity with frequency for composite spherical shell-I, shell-II, shell-III and shell-IV respectively. From the Fig.1, it is clear that as frequency increases, phase velocity, in general increases for composite spherical shell-I and shell-IV. For composite spherical shell-II and shell-III, phase velocity, in general decreases. Fig. 2 depicts variation of attenuation with frequency for composite spherical shell-I, shell-II, shell-III, and shell-IV. From the Fig. 2, it is seen that as frequency increases, attenuation, in general increases for composite spherical shell-I, shell-II and shell-IV. In the case of composite spherical shell-III, as frequency increases, attenuation, in general decreases. Also, attenuation of composite spherical shell-I are less than that of shell-II and attenuation of composite spherical shell-III is less than that of shell-IV. If the shear viscosity of fluid is ignored, the problem reduces to that of classical Biot's theory. In this case phase velocity is computed as a function of frequency and the results are depicted in Fig. 3. From this figure, it is observed that, as frequency increases, phase velocity, in general increases for composite spherical shell-II and shell-III. For composite spherical shell-I and shell-IV, phase velocity, in general decreases. Moreover, phase velocity of composite spherical shell-I are greater than that of shell-II, and phase velocity of composite spherical shell-III are greater than that of shell-IV for both extension theory and classical theory. In the particular case of thick walled poroelastic spherical shell, phase velocity and attenuation are computed as a function of wavenumber, ratio of thickness to inner radius and the results are depicted in Figs. 4 and 5. From the Fig. 4, it is observed that, as wavenumber increases, phase velocity, in general increases for spherical shell-I and shell-III. From the Fig. 5, it is observed that, as ratio increases, attenuation, in general decreases for spherical shell-I and increases for shell-III. From these Figs. 4 and 5, it is clear that phase velocity and attenuation of spherical shell-I are less than that of shell-III.

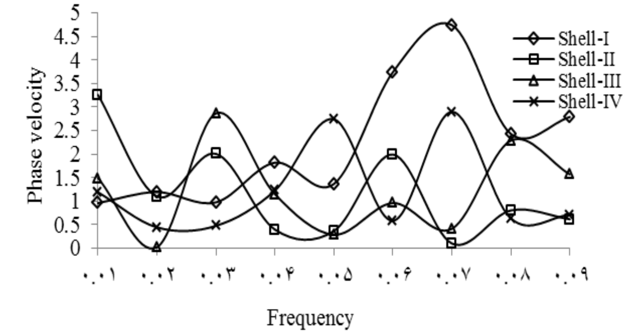

Fig.1
Variation of phase velocity with frequency in the case of Biot's extension theory.

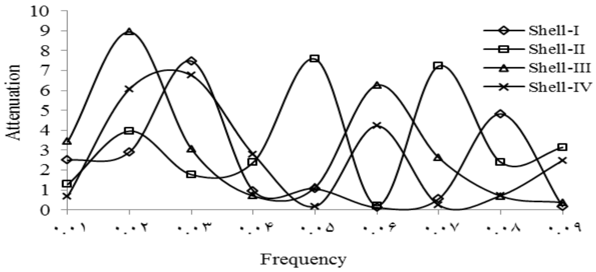

Fig.2
Variation of attenuation with frequency in the case of Biot's extension theory.

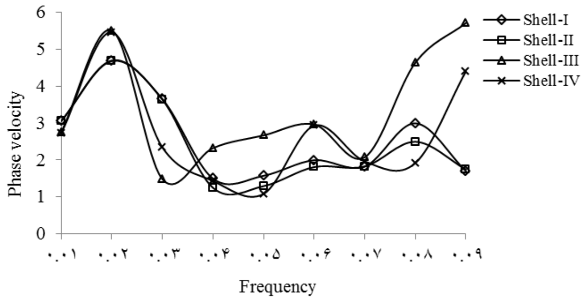

Fig.3
Variation of phase velocity with wavenumber in the case of classical Biot's theory.

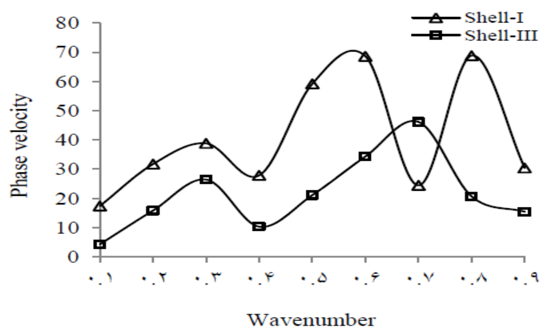

Fig.4
Variation of phase velocity with wavenumber in the case of thick walled poroelastic spherical shell.

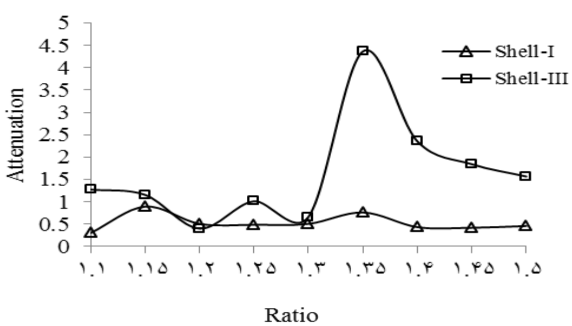

Fig.5
Variation of attenuation with ratio in the case of thick walled poroelastic spherical shell.

6 CONCLUSION

Employing the Biot's extension theory, torsional vibrations of composite poroelastic spherical shell are investigated. The frequency equation is investigated when the solids are saturated with viscous fluids. If the shear viscosity of fluid is ignored, the problem reduces to that of classical Biot's theory. Comparative study is made between the extension theory and classical theory. From the numerical results, it is concluded that phase velocity of shell-I, shell-III are higher than that of shell-II, shell-IV respectively for extension theory and classical theory. From this, it is clear that shear viscosity of fluid is causing the discrepancy. Particular cases, namely, solid spherical shell and thick walled hollow spherical shell are studied. Similar analysis can be made for any composite spherical shell which is made of two different poroelastic solids if the pertinent values are available.

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APPENDIX A

$$
\begin{aligned}
c_{11} &= (-{}_2\rho^m)[B_1(kr_2)(Q_1(D_{10}A_{10}-D_{20}A_{20})-Q_2(D_{10}A_{20}+D_{20}A_{10}))+B_1(A_{10}N_{10}-A_{20}N_{20}) \\
&\quad -B_2(kr_2)(Q_1(D_{10}A_{20}+D_{20}A_{10})+Q_2(D_{10}A_{10}-D_{20}A_{20}))+B_2(A_{10}N_{20}+A_{20}N_{10})], \\
c_{12} &= (-{}_2\rho^m)[B_1(kr_2)(Q_1(N_2A_1-N_1A_2)+Q_2(N_2A_1+A_2N_1))+B_1(A_1D_1-A_2D_2) \\
&\quad +B_2(kr_2)(Q_1(A_1N_2+A_2N_1)+Q_2(N_1A_1-N_2A_2))-B_2(A_1D_2+A_2D_1)], \\
c_{13} &= c_{16}=0, \quad c_{23}=c_{26}=0, \\
c_{14} &= (-{}_2\rho^m)[B_3Q_{10}(kr_2)(G_{10}F_{10}-G_{20}F_{20})-B_3Q_{20}(kr_2)(G_{10}F_{20}+G_{20}F_{10})+B_3(G_{10}H_{10}-G_{20}H_{20}) \\
&\quad -B_4Q_{10}(kr_2)(G_{10}F_{20}+G_{20}F_{10})-B_4Q_{20}(kr_2)(G_{10}F_{10}-G_{20}F_{20})-B_4(G_{10}H_{20}+G_{20}H_{10})], \\
c_{15} &= (-{}_2\rho^m)[B_3Q_{10}(kr_2)(G_2H_2-G_1H_1)-B_3Q_{20}(kr_2)(G_1H_2+G_2H_1)+B_3(E_1F_1-E_2F_2) \\
&\quad -B_4Q_{20}(kr_2)(G_2H_2-G_1H_1)+B_4Q_{10}(kr_2)(G_1H_2+G_2H_1)-B_4(E_1F_2+E_2F_1)], \\
c_{21} &= (-{}_2\rho^i)[B_5(kr_2)Q_1(A_1D_1-A_2D_2)-B_5Q_2(kr_2)(D_1A_2+D_2A_1)+B_5(A_1N_1-A_2N_2) \\
&\quad -B_6(kr_2)Q_1(D_1A_2+D_2A_1)-B_6Q_2(kr_2)(D_1A_1-D_2A_2)-B_6(A_1N_2+A_2N_1) \\
&\quad -(\frac{{}_2\Omega_i}{\omega})(B_5Q_1(kr_2)(D_1A_2+D_2A_1)+B_5Q_2(kr_2)(D_1A_1-D_2A_2)+B_5(A_1N_2+A_2N_1) \\
&\quad +B_6Q_1(kr_2)(D_1A_1-D_2A_2)-B_6Q_2(kr_2)(D_1A_2+D_2A_1)+B_6(A_1N_1-A_2N_2))], \\
c_{22} &= (-{}_2\rho^i)[(\frac{{}_2\Omega_i}{\omega})(-B_5)(kr_2)(Q_1(N_2A_1+N_1A_2)-Q_2(N_1A_1-A_2N_2))+B_5(A_1D_2+A_2D_1) \\
&\quad +B_6(kr_2)(Q_1(A_1N_2-A_2N_1)+Q_2(N_2A_1+N_1A_2))+B_6(A_1D_1-A_2D_2)+ \\
&\quad B_5(kr_2)(Q_1(N_2A_1-N_1A_2)+Q_2(N_2A_1+A_2N_1))+B_5(A_1D_1-A_2D_2) \\
&\quad +B_6(kr_2)(Q_1(A_1N_2+A_2N_1)+Q_2(N_1A_1-N_2A_2))-B_6(A_1D_2+A_2D_1)], \\
c_{24} &= (-{}_2\rho^i)[B_7Q_{10}(kr_2)(G_1F_1-G_2F_2)-B_7Q_{20}(kr_2)(G_1F_2+G_2F_1)+B_7(G_1H_1-G_2H_2) \\
&\quad -B_8Q_{10}(kr_2)(G_1F_2+G_2F_1)-B_8Q_{20}(kr_2)(G_1F_1-G_2F_2)+B_8(G_1H_2+G_2H_1) \\
&\quad -(\frac{{}_2\Omega_i}{\omega})(B_8Q_{10}(kr_2)(G_1F_2+G_2F_1)-B_8Q_{20}(kr_2)(G_1F_2+G_2F_1)+B_8(G_1H_1-G_2H_2) \\
&\quad +B_7Q_{10}(kr_2)(G_1F_2+G_2F_1)+B_7Q_{20}(kr_2)(G_1F_1-G_2F_2)+B_7(G_1H_2+G_2H_1)], \\
c_{25} &= (-{}_2\rho^i)[(-\frac{{}_2\Omega_i}{\omega})B_3(kr_2)(Q_{20}(G_2H_2-G_1H_1)-Q_{10}(G_1H_2+G_2H_1))+B_3(E_1F_2+E_2F_1) \\
&\quad +B_4(kr_2)(Q_{10}(G_2H_2-G_1H_1)-Q_{20}(G_1H_2+G_2H_1))+B_4(E_1F_1-E_2F_2) \\
&\quad +B_3(kr_2)(Q_{10}(G_2H_2-G_1H_1)-Q_{20}(G_1H_2+G_2H_1))+B_3(E_1F_1-E_2F_2) \\
&\quad -B_4(kr_2)(Q_{20}(G_2H_2-G_1H_1)+Q_{10}(G_1H_2+G_2H_1))-B_4(E_1F_2+E_2F_1)], \\
c_{31} &= (-{}_2\rho^m)[B_1(kr_1)(Q_1(D_1A_1-D_2A_2)-Q_2(D_1A_2+D_2A_1))+B_1(A_1N_1-A_2N_2) \\
&\quad -B_2(kr_1)(Q_1(D_1A_2+D_2A_1)+Q_2(D_1A_1-D_2A_2))+B_2(A_1N_2+A_2N_1)], \\
c_{32} &= (-{}_2\rho^m)[B_1(kr_1)(Q_{10}(N_{20}A_{10}-N_{10}A_{20})+Q_{20}(N_{20}A_{10}+A_{20}N_{10}))+B_1(A_{10}D_{10}-A_{20}D_{20}) \\
&\quad +B_2(kr_1)(Q_{10}(A_{10}N_{20}+A_{20}N_{10})+Q_{20}(N_{10}A_{10}-N_{20}A_{20}))-B_2(A_{10}D_{20}+A_{20}D_{10})], \\
c_{33} &= {}_1\rho^m[B_1Y_{11}(kr_1)(a_{10}d_{10}-a_{20}d_{20})-B_1Y_{22}(kr_1)(a_{10}d_{20}+a_{20}d_{10})+B_1(a_{10}n_{10}-a_{20}n_{20}) \\
&\quad -B_2Y_{11}(kr_1)(a_{10}d_{20}+a_{20}d_{10})-B_2Y_{22}(kr_1)(a_{10}d_{10}-a_{20}d_{20})-B_2(a_{10}n_{20}+a_{20}n_{10})],
\end{aligned}
$$

$$c_{34}=(-{}_2\rho^m)[B_3Q_{10}(kr_1)(G_{10}F_{10}-G_{20}F_{20})-B_3Q_{20}(kr_1)(G_{10}F_{20}+G_{20}F_{10})+B_3(G_{10}H_{10}-G_{20}H_{20})$$
$$-B_4Q_{10}(kr_1)(G_{10}F_{20}+G_{20}F_{10})-B_4Q_{20}(kr_1)(G_{10}F_{10}-G_{20}F_{20})-B_4(G_{10}H_{20}+G_{20}H_{10})],$$
$$c_{35}=(-{}_2\rho^m)[B_3Q_{10}(kr_1)(G_{20}H_{20}-G_{10}H_{10})-B_3Q_{20}(kr_1)(G_{10}H_{20}+G_{20}H_{10})+B_3(E_{10}F_{10}-E_{20}F_{20})$$
$$-B_4Q_{20}(kr_1)(G_{20}H_{20}-G_{10}H_{10})+B_4Q_{10}(kr_1)(G_{10}H_{20}+G_{20}H_{10})-B_4(E_{10}F_{20}+E_{20}F_{10})],$$
$$c_{36}={}_1\rho^m[B_3Y_{110}(kr_1)(g_{10}f_{10}-g_{20}f_{20})-B_3Y_{220}(kr_1)(g_{10}f_{20}+g_{20}f_{10})+B_3(g_{10}h_{10}-g_{20}h_{20})$$
$$-B_4Y_{110}(kr_1)(g_{10}f_{20}+g_{20}f_{10})-B_4Y_{220}(kr_1)(g_{10}f_{10}-g_{20}f_{20})-B_4(g_{10}h_{20}+g_{20}h_{10})],$$
$$c_{41}=(-{}_2\rho^i)[B_5(kr_1)Q_1(A_{10}D_{10}-A_{20}D_{20})-B_5Q_2(kr_1)(D_{10}A_{20}+D_{20}A_{10})+B_5(A_{10}N_{10}-A_{20}N_{20})$$
$$-B_6(kr_1)Q_1(D_{10}A_{20}+D_{20}A_{10})-B_6Q_2(kr_1)(D_{10}A_{10}-D_{20}A_{20})+B_5(A_{10}N_{20}+A_{20}N_{10})$$
$$-(\frac{{}_2\Omega_i}{\omega})(B_5Q_1(kr_1)(D_{10}A_{20}+D_{20}A_{10})+B_5Q_2(kr_1)(D_{10}A_{10}-D_{20}A_{20})+B_5(A_{10}N_{20}+A_{20}N_{10})$$
$$+B_6Q_1(kr_1)(D_{10}A_{10}-D_{20}A_{20})-B_6Q_2(kr_1)(D_{10}A_{20}+D_{20}A_{10})+B_6(A_{10}N_{10}-A_{20}N_{20}))],$$

c_{42} have similar expressions as those of c_{41} with Q_1,Q_2,D_{10},D_{20} replaced by $Q_{10},Q_{20},N_{10},N_{20}$

$$c_{43}={}_1\rho^i[B_5Y_{11}(kr_1)(a_{10}d_{10}-a_{20}d_{20})-B_5Y_{22}(kr_1)(a_{10}d_{20}+a_{20}d_{10})+B_5(a_{10}n_{10}-a_{20}n_{20})$$
$$-B_6Y_{11}(kr_1)(a_{10}d_{20}+a_{20}d_{10})-B_6Y_{22}(kr_1)(a_{10}d_{10}-a_{20}d_{20})-B_6(a_{10}n_{20}+a_{20}n_{10})$$
$$-(\frac{{}_1\Omega_i}{\omega})(B_5Y_{11}(kr_1)(a_{10}d_{20}+a_{20}d_{10})+B_5Y_{22}(kr_1)(a_{10}d_{10}-a_{20}d_{20})+B_5(a_{10}n_{20}+a_{20}n_{10})$$
$$+B_6Y_{11}(kr_1)(a_{10}d_{10}-a_{20}d_{20})-B_6Y_{22}(kr_1)(a_{10}d_{20}+a_{20}d_{10})+B_6(a_{10}n_{10}-a_{20}n_{20}))],$$
$$c_{44}=(-{}_2\rho^i)[B_7Q_{10}(kr_1)(G_1F_1-G_2F_2)-B_7Q_{20}(kr_1)(G_1F_2+G_2F_1)+B_7(G_1H_1-G_2H_2)$$
$$-B_8Q_{10}(kr_1)(G_1F_2+G_2F_1)-B_8Q_{20}(kr_1)(G_1F_1-G_2F_2)+B_8(G_1H_2+G_2H_1)$$
$$-(\frac{{}_2\Omega_i}{\omega})(B_8Q_{10}(kr_1)(G_1F_2+G_2F_1)-B_8Q_{20}(kr_1)(G_1F_2+G_2F_1)+B_8(G_1H_1-G_2H_2)$$
$$+B_7Q_{10}(kr_1)(G_1F_2+G_2F_1)+B_7Q_{20}(kr_1)(G_1F_1-G_2F_2)+B_7(G_1H_2+G_2H_1))],$$
$$c_{45}=(-{}_2\rho^i)[B_7Q_{10}(kr_1)(G_{20}H_{20}-G_{10}H_{10})-B_7Q_{20}(kr_1)(G_{10}H_{20}+G_{20}H_{10})+B_7(E_{10}F_{10}-E_{20}F_{20})$$
$$-B_8Q_{20}(kr_1)(G_{20}H_{20}-G_{10}H_{10})+B_8Q_{10}(kr_1)(G_{10}H_{20}+G_{20}H_{10})-B_8(E_{10}F_{20}+E_{20}F_{10})$$
$$-(\frac{{}_2\Omega_i}{\omega})(B_7Q_{20}(kr_1)(G_{20}H_{20}-G_{10}H_{10})-B_7Q_{10}(kr_1)(G_{10}H_{20}+G_{20}H_{10})+B_7(E_{10}F_{20}+E_{20}F_{10})$$
$$+B_8Q_{10}(kr_1)(G_{20}H_{20}-G_{10}H_{10})-B_8Q_{20}(kr_1)(G_{10}H_{20}+G_{20}H_{10})+B_8(E_{10}F_{10}-E_{20}F_{20}))],$$
$$c_{46}=(-{}_2\rho^i)[B_7Q_{10}(kr_1)(G_{10}F_{10}-G_{20}F_{20})-B_7Q_{20}(kr_1)(G_{10}F_{20}+G_{20}F_{10})+B_7(G_{10}H_{10}-G_{20}H_{20})$$
$$-B_8Q_{10}(kr_1)(G_{10}F_{20}+G_{20}F_{10})-B_8Q_{20}(kr_1)(G_{10}F_{10}-G_{20}F_{20})-B_8(G_{10}H_{20}+G_{20}H_{10})$$
$$-(\frac{{}_2\Omega_i}{\omega})(B_7Q_{10}(kr_1)(G_{10}F_{20}+G_{20}F_{10})+B_7Q_{20}(kr_1)(G_{10}F_{10}-G_{20}F_{20})+B_7(G_{10}H_{20}+G_{20}H_{10})$$
$$+B_8Q_{10}(kr_1)(G_{10}F_{10}-G_{20}F_{10})-B_8Q_{20}(kr_1)(G_{10}F_{20}+G_{20}F_{10})+B_8(G_{10}H_{10}-G_{20}H_{20})],$$
$$c_{51}=Z_{1000}(a_{20}n_{20}-a_{10}n_{10})+Z_{2000}(a_{10}n_{20}+a_{20}n_{10}),$$

c_{52} have similar expressions as those of c_{51} with n_{10},n_{20} replaced by d_{10},d_{20},

$$c_{53}=Z_{100}(A_{10}N_{10}-A_{20}N_{20})-Z_{200}(A_{10}N_{20}+A_{20}N_{10}),$$
$$c_{54}=(g_{20}h_{20}-g_{10}h_{10})\left(\frac{Z_{3000}Z_{7000}+Z_{8000}Z_{4000}}{(Z_{3000})^2-(Z_{4000})^2}\right)+(g_{10}h_{20}+g_{20}h_{10})\left(\frac{Z_{3000}Z_{8000}-Z_{7000}Z_{4000}}{(Z_{3000})^2-(Z_{4000})^2}\right),$$

c_{55} have similar expressions as those of c_{54} with h_{10},h_{20} replaced by f_{10},f_{20},

$$c_{56}=(G_{10}H_{10}-G_{20}H_{20})\left(\frac{Z_{300}Z_{700}+Z_{800}Z_{400}}{(Z_{300})^2-(Z_{400})^2}\right)-(G_{10}H_{20}+G_{20}H_{10})\left(\frac{Z_{300}Z_{800}-Z_{700}Z_{400}}{(Z_{300})^2-(Z_{400})^2}\right),$$

c_{61} have similar expressions as those of c_{51} with Z_{1000},Z_{2000} replaced by H_{100},H_{200},

c_{62} have similar expressions as those of c_{61} with n_{10}, n_{20} replaced by d_{10}, d_{20},

$$c_{63} = H_{10}(A_{10}N_{10} - A_{20}N_{20}) - H_{20}(A_{10}N_{20} + A_{20}N_{10}),$$
$$c_{64} = Z_{1000}(g_{10}h_{10} - g_{20}h_{20}) - Z_{2000}(g_{10}h_{20} + g_{20}h_{10}),$$

c_{65} have similar expressions as those of c_{64} with h_{10}, h_{20} replaced by f_{10}, f_{20},

$$c_{66} = Z_{300}(G_{10}H_{10} - G_{20}H_{20}) - Z_{400}(G_{10}H_{20} + G_{20}H_{10}),$$
$$d_{11} = d_{12} = d_{14} = d_{15} = 0,$$
$$\begin{aligned} d_{11} = (-{}_2\rho^m)[&B_1(kr_2)(Q_1(D_{10}A_{20} + D_{20}A_{10}) + Q_2(D_{10}A_{10} - D_{20}A_{20})) + B_1(A_{10}N_{20} + A_{20}N_{10}) \\ &+ B_2(kr_2)(Q_1(D_{10}A_{10} - D_{20}A_{20}) - Q_2(D_{10}A_{20} + D_{20}A_{10})) + B_2(A_{10}N_{10} - A_{20}N_{20})], \end{aligned}$$
$$\begin{aligned} d_{12} = (-{}_2\rho^m)[&(-B_1)(kr_2)(Q_1(N_2A_1 + N_1A_2) + Q_2(N_1A_1 - A_2N_2)) + B_1(A_1D_2 + A_2D_1) \\ &+ B_2(kr_2)(Q_1(A_1N_2 - A_2N_1) + Q_2(N_2A_1 + N_1A_2)) + B_2(A_1D_1 - A_2D_2)], \end{aligned}$$
$$d_{13} = d_{16} = 0,\ d_{23} = d_{26} = 0,$$
$$\begin{aligned} d_{14} = (-{}_2\rho^m)[&B_3Q_{10}(kr_2)(G_{10}F_{20} + G_{20}F_{10}) + B_3Q_{20}(kr_2)(G_{10}F_{10} - G_{20}F_{20}) + B_3(G_{10}H_{20} + G_{20}H_{10}) \\ &+ B_4Q_{10}(kr_2)(G_{10}F_{10} - G_{20}F_{20}) - B_4Q_{20}(kr_2)(G_{10}F_{20} + G_{20}F_{10}) + B_4(G_{10}H_{10} - G_{20}H_{10})], \end{aligned}$$
$$\begin{aligned} d_{15} = (-{}_2\rho^m)[&B_3Q_{20}(kr_2)(G_2H_2 - G_1H_1) - B_3Q_{10}(kr_2)(G_1H_2 + G_2H_1) + B_3(E_1F_2 + E_2F_1) \\ &+ B_4Q_{10}(kr_2)(G_2H_2 - G_1H_1) - B_4Q_{20}(kr_2)(G_1H_2 + G_2H_1) + B_4(E_1F_1 - E_2F_2)], \end{aligned}$$
$$\begin{aligned} d_{21} = (-{}_2\rho^i)[&(\frac{{}_2\Omega_i}{\omega})(B_5(kr_2)Q_1(A_1D_1 - A_2D_2) - B_5Q_2(kr_2)(D_1A_2 + D_2A_1) + B_5(A_1N_1 - A_2N_2) \\ &- B_6(kr_2)Q_1(D_1A_2 + D_2A_1) - B_6Q_2(kr_2)(D_1A_1 - D_2A_2) - B_6(A_1N_2 + A_2N_1)) \\ &+ (B_5Q_1(kr_2)(D_1A_2 + D_2A_1) + B_5Q_2(kr_2)(D_1A_1 - D_2A_2) + B_5(A_1N_2 + A_2N_1) \\ &+ B_6Q_1(kr_2)(D_1A_1 - D_2A_2) - B_6Q_2(kr_2)(D_1A_2 + D_2A_1) + B_6(A_1N_1 - A_2N_2))], \end{aligned}$$
$$\begin{aligned} d_{22} = (-{}_2\rho^i)[&(-B_1)(kr_2)(Q_1(N_2A_1 + N_1A_2) + Q_2(N_1A_1 - A_2N_2)) + B_1(A_1D_2 + A_2D_1) \\ &+ B_2(kr_2)(Q_1(A_1N_2 - A_2N_1) + Q_2(N_2A_1 + N_1A_2)) + B_2(A_1D_1 - A_2D_2) - \\ &(\frac{{}_i\Omega_2}{\omega})(B_1)(kr_2)(Q_1(N_2A_1 - N_1A_2) + Q_2(N_2A_1 + A_2N_1)) + B_1(A_1D_1 - A_2D_2) \\ &+ B_2(kr_2)(Q_1(A_1N_2 + A_2N_1) + Q_2(N_1A_1 - N_2A_2)) - B_2(A_1D_2 + A_2D_1)], \end{aligned}$$
$$\begin{aligned} d_{24} = (-{}_2\rho^i)[&(\frac{{}_2\Omega_i}{\omega})(B_5(kr_2)Q_1(A_1D_1 - A_2D_2) - B_5Q_2(kr_2)(D_1A_2 + D_2A_1) + B_5(A_1N_1 - A_2N_2) \\ &- B_6(kr_2)Q_1(D_1A_2 + D_2A_1) - B_6Q_2(kr_2)(D_1A_1 - D_2A_2) - B_6(A_1N_2 + A_2N_1)) \\ &+ (B_5Q_1(kr_2)(D_1A_2 + D_2A_1) + B_5Q_2(kr_2)(D_1A_1 - D_2A_2) + B_5(A_1N_2 + A_2N_1) \\ &+ B_6Q_1(kr_2)(D_1A_1 - D_2A_2) - B_6Q_2(kr_2)(D_1A_2 + D_2A_1) + B_6(A_1N_1 - A_2N_2))], \end{aligned}$$
$$\begin{aligned} d_{25} = (-{}_2\rho^i)[&(-\frac{{}_2\Omega_i}{\omega})B_3(kr_2)(Q_{10}(G_2H_2 - G_1H_1) - Q_{20}(G_1H_2 + G_2H_1)) + B_3(E_1F_1 - E_2F_2) \\ &- B_4(kr_2)(Q_{20}(G_2H_2 - G_1H_1) + Q_{10}(G_1H_2 + G_2H_1)) - B_4(E_1F_2 + E_2F_1) + \\ &(B_3(kr_2)(Q_{20}(G_2H_2 - G_1H_1) - Q_{10}(G_1H_2 + G_2H_1)) + B_3(E_1F_2 + E_2F_1) \\ &+ B_4(kr_2)(Q_{10}(G_2H_2 - G_1H_1) - Q_{20}(G_1H_2 + G_2H_1)) + B_4(E_1F_1 - E_2F_2))], \end{aligned}$$
$$\begin{aligned} d_{31} = {}_2\rho^m[&B_1(kr_1)(Q_1(D_1A_2 + D_2A_1) + Q_2(D_1A_1 - D_2A_2)) + B_1(A_1N_2 + A_2N_1) \\ &+ B_2(kr_1)(Q_1(D_1A_1 - D_2A_2) - Q_2(D_1A_2 + D_2A_1)) + B_2(A_1N_1 - A_2N_2)], \end{aligned}$$
$$\begin{aligned} d_{32} = (-{}_2\rho^m)[&(-B_1)(kr_1)(Q_{10}(N_{20}A_{10} + N_{10}A_{20}) + Q_{20}(N_{10}A_{10} - A_{20}N_{20})) + B_1(A_{10}D_{20} + A_{20}D_{10}) \\ &+ B_2(kr_1)(Q_{10}(A_{10}N_{20} - A_{20}N_{10}) + Q_{20}(N_{20}A_{10} + N_{10}A_{20})) + B_2(A_{10}D_{10} - A_{20}D_{20})], \end{aligned}$$
$$\begin{aligned} d_{33} = {}_1\rho^m[&B_1Y_{11}(kr_1)(a_{10}d_{20} + a_{20}d_{10}) + B_1Y_{22}(kr_1)(a_{10}d_{10} - a_{20}d_{20}) + B_1(a_{10}n_{20} + a_{20}n_{10}) \\ &+ B_2Y_{11}(kr_1)(a_{10}d_{10} - a_{20}d_{20}) - B_2Y_{22}(kr_1)(a_{10}d_{20} + a_{20}d_{10}) + B_2(a_{10}n_{10} - a_{20}n_{20})], \end{aligned}$$
$$\begin{aligned} d_{34} = (-{}_2\rho^m)[&B_3Q_{10}(kr_1)(G_{10}F_{20} + G_{20}F_{10}) + B_3Q_{20}(kr_1)(G_{10}F_{10} - G_{20}F_{20}) + B_3(G_{10}H_{20} + G_{20}H_{10}) \\ &+ B_4Q_{10}(kr_1)(G_{10}F_{10} - G_{20}F_{20}) - B_4Q_{20}(kr_1)(G_{10}F_{20} + G_{20}F_{10}) + B_4(G_{10}H_{10} - G_{20}H_{10})], \end{aligned}$$

$$d_{35}=(-{}_2\rho^{m})[B_3Q_{20}(kr_1)(G_{20}H_{20}-G_{10}H_{10})-B_3Q_{10}(kr_1)(G_{10}H_{20}+G_{20}H_{10})+B_3(E_{10}F_{20}+E_{20}F_{10})$$
$$+B_4Q_{10}(kr_1)(G_{20}H_{20}-G_{10}H_{10})-B_4Q_{20}(kr_1)(G_{10}H_{20}+G_{20}H_{10})+B_4(E_{10}F_{10}-E_{20}F_{20})],$$
$$d_{36}={}_1\rho^{m}[B_3Y_{110}(kr_1)(g_{10}f_{20}+g_{20}f_{10})+B_3Y_{220}(kr_1)(g_{10}f_{10}-g_{20}f_{20})+B_3(g_{10}h_{20}+g_{20}h_{10})$$
$$+B_4Y_{110}(kr_1)(g_{10}f_{10}-g_{20}f_{20})-B_4Y_{220}(kr_1)(g_{10}f_{20}+g_{20}f_{10})+B_4(g_{10}h_{10}-g_{20}h_{20})],$$
$$d_{41}=(-{}_2\rho^{i})[(\frac{{}_2\Omega_i}{\omega})(B_5(kr_1)Q_1(A_{10}D_{10}-A_{20}D_{20})-B_5Q_2(kr_1)(D_{10}A_{20}+D_{20}A_{10})+B_5(A_{10}N_{10}-A_{20}N_{20})$$
$$-B_6(kr_1)Q_1(D_{10}A_{20}+D_{20}A_{10})-B_6Q_2(kr_1)(D_{10}A_{10}-D_{20}A_{20})+B_5(A_{10}N_{20}+A_{20}N_{10}))$$
$$+(B_5Q_1(kr_1)(D_{10}A_{20}+D_{20}A_{10})+B_5Q_2(kr_1)(D_{10}A_{10}-D_{20}A_{20})+B_5(A_{10}N_{20}+A_{20}N_{10})$$
$$+B_6Q_1(kr_1)(D_{10}A_{10}-D_{20}A_{20})-B_6Q_2(kr_1)(D_{10}A_{20}+D_{20}A_{10})+B_6(A_{10}N_{10}-A_{20}N_{20}))],$$

d_{42} have similar expressions as those of d_{41} with Q_1,Q_2,D_{10},D_{20} replaced by $Q_{10},Q_{20},N_{10},N_{20}$,

$$d_{43}={}_1\rho^{i}[(\frac{{}_1\Omega_i}{\omega})(B_5Y_{11}(kr_1)(a_{10}d_{10}-a_{20}d_{20})-B_5Y_{22}(kr_1)(a_{10}d_{20}+a_{20}d_{10})+B_5(a_{10}n_{10}-a_{20}n_{20})$$
$$-B_6Y_{11}(kr_1)(a_{10}d_{20}+a_{20}d_{10})-B_6Y_{22}(kr_1)(a_{10}d_{10}-a_{20}d_{20})-B_6(a_{10}n_{20}+a_{20}n_{10}))$$
$$+B_5Y_{11}(kr_1)(a_{10}d_{20}+a_{20}d_{10})+B_5Y_{22}(kr_1)(a_{10}d_{10}-a_{20}d_{20})+B_5(a_{10}n_{20}+a_{20}n_{10})$$
$$+B_6Y_{11}(kr_1)(a_{10}d_{10}-a_{20}d_{20})-B_6Y_{22}(kr_1)(a_{10}d_{20}+a_{20}d_{10})+B_6(a_{10}n_{10}-a_{20}n_{20})],$$
$$d_{44}=(-{}_2\rho^{i})[(\frac{{}_2\Omega_i}{\omega})(B_5(kr_1)Q_1(A_1D_1-A_2D_2)-B_5Q_2(kr_1)(D_1A_2+D_2A_1)+B_5(A_1N_1-A_2N_2)$$
$$-B_6(kr_1)Q_1(D_1A_2+D_2A_1)-B_6Q_2(kr_1)(D_1A_1-D_2A_2)-B_6(A_1N_2+A_2N_1))$$
$$+(B_5Q_1(kr_1)(D_1A_2+D_2A_1)+B_5Q_2(kr_1)(D_1A_1-D_2A_2)+B_5(A_1N_2+A_2N_1)$$
$$+B_6Q_1(kr_1)(D_1A_1-D_2A_2)-B_6Q_2(kr_1)(D_1A_2+D_2A_1)+B_6(A_1N_1-A_2N_2))],$$
$$d_{45}=(-{}_2\rho^{i})[(\frac{{}_2\Omega_i}{\omega})B_7Q_{10}(kr_1)(G_{20}H_{20}-G_{10}H_{10})-B_7Q_{20}(kr_1)(G_{10}H_{20}+G_{20}H_{10})+B_7(E_{10}F_{10}-E_{20}F_{20})$$
$$-B_8Q_{20}(kr_1)(G_{20}H_{20}-G_{10}H_{10})+B_8Q_{10}(kr_1)(G_{10}H_{20}+G_{20}H_{10})-B_8(E_{10}F_{20}+E_{20}F_{10})$$
$$+(B_7Q_{20}(kr_1)(G_{20}H_{20}-G_{10}H_{10})-B_7Q_{10}(kr_1)(G_{10}H_{20}+G_{20}H_{10})+B_7(E_{10}F_{20}+E_{20}F_{10})$$
$$+B_8Q_{10}(kr_1)(G_{20}H_{20}-G_{10}H_{10})-B_8Q_{20}(kr_1)(G_{10}H_{20}+G_{20}H_{10})+B_8(E_{10}F_{10}-E_{20}F_{20}))],$$
$$d_{46}={}_1\rho^{i}[(\frac{{}_1\Omega_i}{\omega})(B_7Y_{110}(kr_1)(g_{10}f_{10}-g_{20}f_{20})-B_7Y_{220}(kr_1)(g_{10}f_{20}+g_{20}f_{10})+B_7(g_{10}h_{10}-g_{20}h_{20})$$
$$-B_8Y_{110}(kr_1)(g_{10}f_{20}+g_{20}f_{10})-B_8Y_{220}(kr_1)(g_{10}f_{10}-g_{20}f_{20})-B_8(g_{10}h_{20}+g_{20}h_{10}))$$
$$+(B_7Y_{110}(kr_1)(g_{10}f_{20}+g_{20}f_{10})+B_7Y_{220}(kr_1)(g_{10}f_{10}-g_{20}f_{12})+B_7(g_{10}h_{20}+g_{20}h_{10})$$
$$+B_8Y_{110}(kr_1)(g_{10}f_{10}-g_{20}f_{20})-B_8Y_{220}(kr_1)(g_{10}f_{20}+g_{20}f_{10})+B_8(g_{10}h_{10}-g_{20}h_{20}))],$$
$$d_{51}=(-Z_{1000})(a_{10}n_{20}+a_{20}n_{10})-Z_{2000}(a_{10}n_{10}-a_{20}n_{20}),$$

d_{52} have similar expressions as those of d_{51} with n_{10},n_{20} replaced by d_{10},d_{20},

$$d_{53}=Z_{100}(A_{10}N_{20}+A_{20}N_{10})+Z_{200}(A_{10}N_{10}-A_{20}N_{20}),$$
$$d_{54}=(-g_{10}h_{20}-g_{20}h_{10})\left(\frac{Z_{3000}Z_{7000}+Z_{8000}Z_{4000}}{(Z_{3000})^2-(Z_{4000})^2}\right)+(g_{20}h_{20}-g_{10}h_{10})\left(\frac{Z_{3000}Z_{8000}-Z_{7000}Z_{4000}}{(Z_{3000})^2-(Z_{4000})^2}\right),$$

d_{55} have similar expressions as those of d_{54} with h_{10},h_{20} replaced by f_{10},f_{20},

$$d_{56}=(G_{10}H_{20}+G_{20}H_{10})\left(\frac{Z_{300}Z_{700}+Z_{800}Z_{400}}{(Z_{300})^2-(Z_{400})^2}\right)+(G_{10}H_{10}-G_{20}H_{20})\left(\frac{Z_{300}Z_{800}-Z_{700}Z_{400}}{(Z_{300})^2-(Z_{400})^2}\right),$$

d_{61} have similar expressions as those of d_{51} with Z_{1000},Z_{2000} replaced by H_{100},H_{200},
d_{62} have similar expressions as those of d_{61} with n_{10},n_{20} replaced by

$$d_{10}, d_{20},\ d_{63} = H_{10}(A_{10}N_{20} + A_{20}N_{10}) + H_{20}(A_{10}N_{10} - A_{20}N_{20}),$$

d_{64} have similar expressions as those of c_{64} with Z_{1000}, Z_{2000} replaced by $Z_{2000}, -Z_{1000}$,
d_{65} have similar expressions as those of c_{65} with Z_{1000}, Z_{2000} replaced by $Z_{2000}, -Z_{1000}$,
d_{66} have similar expressions as those of c_{66} with Z_{300}, Z_{400} replaced by $Z_{400}, -Z_{300}$.
In all the above equations, the notation expressions are as follows:

$$B_1 = {}_1\beta_c^2,\ B_2 = \frac{{}_1\beta_c^2\,\omega\,{}_1\alpha_\mu}{{}_1\Omega_\beta},\ B_3 = {}_1\beta_c^2\,{}_1m_f,\ B_4 = \frac{{}_1\beta_c^2\,{}_1m_f\,\omega}{{}_1\Omega_\beta}\left(\frac{{}_1\eta_0}{{}_1m_f} - {}_1\alpha_\mu\right),$$

$$B_5 = {}_1\beta_c^2\left(\frac{\omega}{{}_1\Omega_i}\right)^2 {}_1d_f + {}_1\beta_c^2\left(\frac{\omega}{{}_1\Omega_i}\right)\left(\frac{\omega}{{}_1\Omega_\beta}\right){}_1d_s\,{}_1\alpha_\mu,\ B_6 = {}_1\beta_c^2\left(\frac{\omega}{{}_1\Omega_i}\right)^2\left(\frac{\omega}{{}_1\Omega_\beta}\right){}_1d_s\,{}_1\alpha_\mu - {}_1d_f\,{}_1\beta_c^2\left(\frac{\omega}{{}_1\Omega_i}\right),$$

$$B_7 = {}_1\beta_c^2\,{}_1m_f\left[\left(\frac{\omega}{{}_1\Omega_i}\right)^2 {}_1d_f - \left(\frac{\omega}{{}_1\Omega_i}\right)\left(\frac{\omega}{{}_1\Omega_\beta}\right){}_1d_s\left(\frac{{}_1\eta_0}{{}_1m_f} - {}_1\alpha_\mu\right)\right],$$

$$B_8 = {}_1\beta_c^2\,{}_1m_f\left[\left(\frac{\omega}{{}_1\Omega_i}\right)^2\left(\frac{\omega}{{}_1\Omega_\beta}\right){}_1d_s\left({}_1\alpha_\mu - \frac{{}_1\eta_0}{{}_1m_f}\right) - \left(\frac{\omega}{{}_1\Omega_i}\right){}_1d_f\right],$$

$B_{10}, B_{20}, B_{30}, B_{40}, B_{50}, B_{60}, B_{70}, B_{80}$ are similar expressions as $B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8$ with left subscript 1 replaced by 2,

$$L_1 = {}_2\beta_c^2,\ L_2 = \frac{-{}_2\beta_c^2\,\omega\,{}_2d_f\,{}_2m_f}{{}_2\Omega_i},\ L_3 = \left(-\frac{\omega^2}{{}_2\Omega_i}\right),\ L_4 = -\omega\left(\frac{\omega}{{}_2\Omega_i}\right)^2(1 + {}_2d_f\,{}_2m_f)\,{}_2d_f\,{}_2\vartheta_f,$$

$$L_{10} = {}_2\beta_c,\ L_{20} = \frac{-{}_2\beta_c\,\omega\,{}_2d_f\,{}_2m_f}{2\,{}_2\Omega_i},\ L_{30} = \sqrt{\frac{{}_2d_f\,{}_2\vartheta_f}{{}_2\Omega_i}}\left(-\frac{\omega^2}{{}_2\Omega_i}\right)(1 + {}_2d_f\,{}_2m_f),\quad L_{40} = (-\omega)\sqrt{\frac{{}_2d_f\,{}_2\vartheta_f}{{}_2\Omega_i}},$$

$$W_1 = \left(V_1^2 + V_2^2\right)^{1/2} cos\left[\frac{1}{2}tan^{-1}\left(\frac{V_2}{V_1}\right)\right],\ W_2 = \left(V_1^2 + V_2^2\right)^{1/2} sin\left[\frac{1}{2}tan^{-1}\left(\frac{V_2}{V_1}\right)\right],$$

W_{11}, W_{22} are similar expressions as W_1, W_2 with V_1, V_2 replaced by V_{11}, V_{22},

$$V_1 = (V^2 - {}_2\beta_c^2),\ V_2 = \frac{{}_2\beta_c^2\,\omega\,{}_2d_f\,{}_2m_f}{{}_2\Omega_i},\ V_{11} = V^2 + \frac{\omega^2\,{}_2d_f\,{}_2\vartheta_f}{{}_2\Omega_i},\ V_{22} = \left(\frac{\omega}{{}_2\Omega_i}\right)^2(1 + {}_2d_f\,{}_2m_f)\,{}_2d_f\,{}_2\vartheta_f,$$

$$Q_1 = \frac{W_1L_{10} + W_2L_{20}}{L_{10}^2 + L_{20}^2},\ Q_2 = \frac{W_2L_{10} - W_1L_{20}}{L_{10}^2 + L_{20}^2},\ Q_{10} = \frac{W_{11}L_{30} + W_{22}L_{40}}{L_{30}^2 + L_{40}^2},\ Q_{20} = \frac{W_{22}L_{30} - W_{11}L_{40}}{L_{30}^2 + L_{40}^2},$$

R_1, R_2, R_3, R_4 are similar expressions as L_1, L_2, L_3, L_4 with left subscript 2 replaced by 1,
$R_{10}, R_{20}, R_{30}, R_{40}$ are similar expressions as $L_{10}, L_{20}, L_{30}, L_{40}$ with left subscript 2 replaced by 1,

$$c_1 = \left(o_1^2 + o_2^2\right)^{1/2} cos\left[\frac{1}{2}tan^{-1}\left(\frac{o_2}{o_1}\right)\right],\ c_2 = \left(o_1^2 + o_2^2\right)^{1/2} sin\left[\frac{1}{2}tan^{-1}\left(\frac{o_2}{o_1}\right)\right],$$

$c_{11},\ c_{22}$ are similar expressions as $c_1,\ c_2$ with o_1, o_2 replaced by o_{11}, o_{22},
o_1, o_2, o_{11}, o_{22}, are similar expressions as V_1, V_2, V_{11}, V_{22}, with left subscript 2 replaced by 1,

$$Y_{11} = \frac{c_1R_{10} + c_2R_{20}}{R_{10}^2 + R_{20}^2},\ Y_{22} = \frac{c_2R_{10} - c_1R_{20}}{R_{10}^2 + R_{20}^2},\ Y_{110} = \frac{W_{110}R_{30} - W_{220}R_{40}}{R_{30}^2 + R_{40}^2},\ Y_{220} = \frac{W_{220}R_{30} - W_{110}R_{40}}{R_{30}^2 + R_{40}^2},$$

W_{110}, W_{220} are similar expressions as W_{11}, W_{22} with left subscript 2 replaced by 1,

$$a_{10} = \frac{1}{\sqrt{\left(Y_{11}^{\,2} + Y_{22}^{\,2}\right)}\, kr_1 \sqrt{\pi}} cos\left[\frac{1}{2} tan^{-1}\left(\frac{Y_{22}}{Y_{11}}\right)\right], \;\; a_{20} = \frac{1}{\sqrt{\left(Y_{11}^{\,2} + Y_{22}^{\,2}\right)}\, kr_1 \sqrt{\pi}} sin\left[\frac{1}{2} tan^{-1}\left(\frac{Y_{22}}{Y_{11}}\right)\right],$$

$$n_{10} = cos(Y_{11}kr_1)\, cosh(Y_{22}kr_1) + sin(Y_{11}kr_1)\, cosh(Y_{22}kr_1),$$

$$n_{20} = cos(Y_{11}kr_1)\, sinh(Y_{22}kr_1) - sin(Y_{11}kr_1)\, sinh(Y_{22}kr_1),$$

a_1, a_2, n_1, n_2 are similar expressions as $a_{10}, a_{20}, n_{10}, n_{20}$ with r_1 replaced by r_2,
A_{10}, A_{20}, N_{10}, N_{20} are similar expressions as a_{10}, a_{20}, n_{10}, n_{20} with Y_{11}, Y_{22} replaced by Q_1, Q_2,
A_1, A_2, N_1, N_2 are similar expressions as $A_{10}, A_{20}, N_{10}, N_{20}$ with r_1 replaced by r_2,
$g_{10}, g_{20}, h_{10}, h_{20}$ are similar expressions as $a_{10}, a_{20}, n_{10}, n_{20}$ with Y_{11}, Y_{22} replaced by Y_{110}, Y_{220},
g_1, g_2, h_1, h_2 are similar expressions as $g_{10}, g_{20}, h_{10}, h_{20}$ with r_1 replaced by r_2,
$G_{10}, G_{20}, H_{10}, H_{20}$ are similar expressions as $g_{10}, g_{20}, h_{10}, h_{20}$ with Y_{110}, Y_{220} replaced by Q_{10}, Q_{20},
G_1, G_2, H_1, H_2 are similar expressions as $G_{10}, G_{20}, H_{10}, H_{20}$ with r_1 replaced by r_2,

$$d_{10} = sin(Y_{11}kr_1)\, cosh(Y_{22}kr_1) - cos(Y_{11}kr_1)\, cosh(Y_{22}kr_1),$$

$$d_{20} = cos(Y_{11}kr_1)\, sinh(Y_{22}kr_1) + sin(Y_{11}kr_1)\, sinh(Y_{22}kr_1),$$

d_1, d_2, are similar expressions as d_{10}, d_{20} with r_1 replaced by r_2,
D_{10}, D_{20}, D_1, D_2 are similar expressions as d_{10}, d_{20}, d_1, d_2 with Y_{11}, Y_{22} replaced by Q_1, Q_2,
f_{10}, f_{20} are similar expressions as d_1, d_2 with Y_{11}, Y_{22} replaced by Y_{110}, Y_{220},
f_1, f_2 are similar expressions as f_{10}, f_{20} with r_1 replaced by r_2,
F_{10}, F_{20}, F_1, F_2 are similar expressions as f_{10}, f_{20}, f_1, f_2 with Y_{110}, Y_{220} replaced by Q_{10}, Q_{20},

$$X_{10} = -cos(Y_{110}kr_1)\, cosh(Y_{220}kr_1) - sin(Y_{110}kr_1)\, cosh(Y_{220}kr_1),$$

$$X_{20} = -cos(Y_{110}kr_1)\, sinh(Y_{220}kr_1) + sin(Y_{110}kr_1)\, sinh(Y_{220}kr_1),$$

X_1, X_2 are similar expressions as X_{10}, X_{20} with r_1 replaced by r_2,

$$Z_{100} = \left(l_1^2 + l_2^2\right)^{1/2} cos\left[\frac{1}{2} tan^{-1}\left(\frac{l_2}{l_1}\right)\right], \;\; Z_{200} = \left(l_1^2 + l_2^2\right)^{1/2} sin\left[\frac{1}{2} tan^{-1}\left(\frac{l_2}{l_1}\right)\right],$$

$$l_1 = \frac{(L_1 - B_{70})(L_1 - L_3) + (L_2 - B_{80})(L_2 - L_4)}{(L_1 - L_3)^2 + (L_2 - L_4)^2}, \;\; l_2 = \frac{(L_2 - B_{80})(L_1 - L_3) - (L_1 - B_{70})(L_2 - L_4)}{(L_1 - L_3)^2 + (L_2 - L_4)^2},$$

$$Z_{1000} = \left(l_3^2 + l_4^2\right)^{1/2} cos\left[\frac{1}{2} tan^{-1}\left(\frac{l_4}{l_3}\right)\right], \;\; Z_{2000} = \left(l_3^2 + l_4^2\right)^{1/2} sin\left[\frac{1}{2} tan^{-1}\left(\frac{l_4}{l_3}\right)\right],$$

$$l_3 = \frac{(R_1 - B_7)(R_1 - R_3) + (R_2 - B_8)(R_2 - R_4)}{(R_1 - R_3)^2 + (R_2 - R_4)^2}, \;\; l_4 = \frac{(R_2 - B_8)(R_1 - R_3) - (R_1 - B_7)(R_2 - R_4)}{(R_1 - R_3)^2 + (R_2 - R_4)^2},$$

$$Z_{300} = \left(p_1^2 + p_2^2\right)^{1/2} cos\left[\frac{1}{2} tan^{-1}\left(\frac{p_2}{p_1}\right)\right], \;\; Z_{400} = \left(p_1^2 + p_2^2\right)^{1/2} sin\left[\frac{1}{2} tan^{-1}\left(\frac{p_2}{p_1}\right)\right],$$

$$p_1 = \frac{(B_{10} - L_3)(L_1 - L_3) + (B_{20} - L_4)(L_2 - L_4)}{(L_1 - L_3)^2 + (L_2 - L_4)^2}, \;\; P_2 = \frac{(B_{20} - L_4)(L_1 - L_3) + (B_{10} - L_3)(L_4 - L_2)}{(L_1 - L_3)^2 + (L_2 - L_4)^2},$$

$$Z_{3000} = \left(l_{100}^2 + l_{200}^2\right)^{1/2} cos\left[\frac{1}{2} tan^{-1}\left(\frac{l_{200}}{l_{100}}\right)\right], \;\; Z_{4000} = \left(l_{100}^2 + l_{200}^2\right)^{1/2} sin\left[\frac{1}{2} tan^{-1}\left(\frac{l_{200}}{l_{100}}\right)\right],$$

$$l_{100}=\frac{m_{100}(R_3-R_1)+m_{200}(R_4-R_2)}{(R_1-R_3)^2+(R_2-R_4)^2},\ l_{200}=\frac{m_{100}(R_2-R_4)+m_{200}(R_3-R_1)}{(R_1-R_3)^2+(R_2-R_4)^2},$$
$$m_{100}=-\left(\frac{\omega^2\,{}_1d_f\,{}_1\nu_f}{{}_1\Omega_i}+{}_1\beta_c^2\right),\ m_{200}=-\left(\frac{\omega^3}{{}_1\Omega_i^2}\right)\left(1+{}_1d_f\,{}_1m_f\right){}_1d_f\,{}_1\nu_f+\frac{{}_1\beta_c^2\,\omega\alpha_\mu}{{}_1\Omega_\beta},$$
$$Z_{7000}=\frac{B_3m_{100}+B_4m_{200}}{m_{100}^2+m_{200}^2},\ Z_{8000}=\frac{B_4m_{100}-B_3m_{200}}{m_{100}^2+m_{200}^2},\ Z_{700}=\frac{B_{30}m_{10}+B_{40}m_{20}}{m_{10}^2+m_{20}^2},\ Z_{800}=\frac{B_{40}m_{10}-B_{30}m_{20}}{m_{10}^2+m_{20}^2},$$

m_{10}, m_{20} are similar expressions as m_{100}, m_{200} with left subscript 1 replaced by 2,

$$H_{100}=\left(\frac{Z_{5000}Z_{1000}+Z_{6000}Z_{2000}}{(Z_{1000})^2+(Z_{2000})^2}\right),\ H_{200}=\left(\frac{Z_{6000}Z_{1000}-Z_{5000}Z_{2000}}{(Z_{1000})^2+(Z_{2000})^2}\right),\ H_{10}=\left(\frac{Z_{500}Z_{100}+Z_{600}Z_{200}}{(Z_{100})^2+(Z_{200})^2}\right),$$
$$H_{20}=\left(\frac{Z_{600}Z_{100}-Z_{500}Z_{200}}{(Z_{100})^2+(Z_{200})^2}\right),\ Z_{5000}=\left(\frac{a_{500}B_4}{B_3^2+B_4^2}\right),\ Z_{6000}=\left(\frac{a_{500}B_3}{B_3^2+B_4^2}\right),$$
$$a_{500}={}_1\beta_c^2\left(\frac{\omega}{{}_1\Omega_\beta}\right){}_1\alpha_\mu-{}_1\beta_c^2\left(\frac{\omega}{{}_1\Omega_i}\right){}_1d_f\,{}_1m_f,\ Z_{500}=\left(\frac{a_{50}B_{40}}{B_{30}^2+B_{40}^2}\right),\ Z_{600}=\left(\frac{a_{50}B_{30}}{B_{30}^2+B_{40}^2}\right),$$

a_{50} are similar expressions as a_{500} with left subscript 1 replaced by 2.

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