Journal of Solid Mechanics Vol. 11, No. 1 (2019) pp. 166-180
DOI: 10.22034/JSM.2019.664227

On Static Bending, Elastic Buckling and Free Vibration Analysis of Symmetric Functionally Graded Sandwich Beams

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Received 1 December 2018; accepted 4 February 2019

ABSTRACT

This article presents Navier type closed-form solutions for static bending, elastic buckling and free vibration analysis of symmetric functionally graded (FG) sandwich beams using a hyperbolic shear deformation theory. The beam has FG skins and isotropic core. Material properties of FG skins are varied through the thickness according to the power law distribution. The present theory accounts for a hyperbolic distribution of axial displacement whereas transverse displacement is constant through the thickness i.e effects of thickness stretching are neglected. The present theory gives hyperbolic cosine distribution of transverse shear stress through the thickness of the beam and satisfies zero traction boundary conditions on the top and bottom surfaces of the beam. The equations of the motion are obtained by using the Hamilton's principle. Closed-form solutions for static, buckling and vibration analysis of simply supported FG sandwich beams are obtained using Navier's solution technique. The non-dimensional numerical results are obtained for various power law index and skin-core-skin thickness ratios. The present results are compared with previously published results and found in excellent agreement.

Keywords: Hyperbolic shear deformation theory; FG sandwich beam; Static bending; Elastic buckling; Free vibration.

1 INTRODUCTION

IN recent years, wide applications of sandwich structures in aerospace, automotive, marine, mechanical and civil engineering led to the development of sandwich structures due to their high strength-to-weight and stiffness-to-weight ratios. Laminated sandwich structures, composed of a soft core bonded to two thin and stiff skins, exhibit delamination problems at the layer interfaces. To overcome this problem, functionally graded (FG) sandwich structures are proposed due to the gradual variation of material properties through-the-thickness. Therefore, understanding bending, buckling and free vibration behaviour of FG sandwich structures becomes an important task for the researchers. A functionally graded material is formed by varying the microstructure from one material to

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another material with a specific gradient. Applications of FG materials in various fields are presented by Koizumi [1, 2], Muller et al. [3], Pompe et al. [4] and Schulz et al. [5]. In general, behavior of FG sandwich beams can be predicted using either elasticity theory or displacement based beam theories. Few researchers put their efforts towards development of elasticity solutions for the analysis of single layer FG beams such as Sankar [6], Zhong and Yu [7], Daouadji et al. [8], Ding et al. [9], Huang et al. [10], Ying et al. [11], Chu et al. [12] and Xu et al. [13]. Elasticity theory is mathematically difficult and computationally more cumbersome. Therefore, displacement based beam theories are widely used by various researchers to approximate the analysis of FG beam. These displacement-based beam theories are classified as classical beam theory (CBT) [14], Timoshenko beam theory (TBT) [15] and higher-order beam theories. Since the effect of transverse shear deformation is more pronounced in thick FG sandwich beams, CBT and TBT are not suitable for the analysis of thick beams. This led the use of higher-order beam theories for the accurate structural analysis of FG sandwich beams. Among several higher order beam theories, parabolic beam theory (PBT) of Reddy [16] is widely used by many researchers for the modeling and analysis of beams, plates and shells. Recently, Sayyad and Ghugal [17-19] presented a comprehensive literature survey on various displacement based beam theories available in the literature for the analysis of advanced composite beams/plates. Nguyen et al. [20, 21] and Nguyen and Nguyen [22] proposed a new hyperbolic shear deformation theory for the bending, buckling and free vibration analysis of FG sandwich beams. Thai and Vo [23] carried out bending and free vibration analysis of functionally graded beams using various higher order shear deformation theories. Osofero et al. [24, 25] developed 2D shear deformation theories for the bending behaviour of simply supported functionally graded sandwich beams. Bennai et al. [26] developed a new refined hyperbolic shear and normal deformation beam theory to study the free vibration and buckling response of functionally graded sandwich beams under various boundary conditions. Bouakkaz et al. [27] also developed a hyperbolic model for the free vibration analysis of functionally graded sandwich beams. Giunta et al. [28, 29] also developed refined theories for the analysis of mono-layer and sandwich functionally graded beams using unified formulation. Vo et al. [30-32] also developed a quasi-3D shear deformation theory for the static, buckling and vibration analysis of FG sandwich beams. Finite element model and Navier solutions are developed to determine the displacement, stresses, critical buckling loads and natural frequencies of FG sandwich beams. This theory includes both shear deformation and thickness stretching effects. Yarasca et al. [33] developed a finite element model based on hybrid shear and normal deformation theory for the static analysis of functionally graded single layer and sandwich beams. Amirani et al. [34] presented the natural frequencies of a sandwich beam with FG core using the element free Galerkin method. The penalty method is used for imposition of the essential boundary condition and material discontinuity condition. Tossapanon and Wattanasakulpong [35] applied Chebyshev collocation method for the buckling and vibration analysis of functionally graded sandwich beams resting on two-parameter elastic foundation using Timoshenko beam theory. Karamanli [36] analyzed the elastostatic behavior of the two directional simply supported FG sandwich beams based on a quasi-3D theory by using the symmetric smoothed particle hydrodynamics method. Mashat et al. [37] carried out free vibration of functionally graded layered beams by various theories and finite elements based on Carrera's unified formulation. Trinh et al. [38] obtained fundamental frequencies of functionally graded sandwich beams of various boundary conditions using the state space method based on the classical beam theory, first order and higher-order shear deformation theories. Wattanasakulpong et al. [39] studied vibration analysis of layered functionally graded beams using an improved third order shear deformation theory with experimental validation. Yang et al. [40] examined the influence of material composition, material gradient, the layer thickness proportion, thickness to length ratio and boundary conditions on the free vibration response of FG sandwich beams using mesh free boundary-domain integral equation method. Sayyad and Ghugal [41] have developed a unified shear deformation theory for the bending analysis of functionally graded beams and plates. Recently, Sayyad and Ghugal [42] have obtained analyrical solution for the bending, buckling and vibration analysis of functionally graded beams of various boundary conditions. Alipour and Shariyat [43-45] studied transient and forced dynamic responses of annular sandwich plates with functionally graded face sheets or cores by an analytical zigzag-elasticity approach. It was the first time that a global–local theory is combined with a layerwise analytical solution for analysis of the annular functionally graded sandwich plates. Shariyat et al. [46] and, Shariyat and Hosseini [47] proposed odd-even hyperbolic plate theory. This theory includes both odd and even functions and consequently, is especially adequate for description of the general asymmetric displacement fields. The important contributions of the present research are summarized as follows: 1) Despite of significant research available on analysis of single layer FG beams, studies on bending, buckling and free vibration analyses of functionally graded sandwich beams with FG skins and homogeneous core layer are limited in the literature, and it will be the main focus of this article. 2) Soldatos [48] have suggested the use of hyperbolic shearing strain function in the modelling and analysis of thick beams and plates. It is recommended by the Soldatos that the hyperbolic shearing strain function yields more accurate predictions of displacements, stresses, frequencies and the buckling loads of thick beams and plates. Hyperbolic shear deformation theory of Soldatos described the displacement field through odd functions to satisfy the zero shear traction condition on the top and bottom surfaces of the beam/plate and thus it is mainly suitable for symmetric lamination schemes or material properties distributions. Since then many researchers have used this function for the analysis isotropic, laminated and sandwich beams and plates. To the best of the authors' knowledge, in the whole variety of literature no one has applied this function to check global response of FG sandwich beams. 3) Natural frequencies for higher modes of vibration are the first time presented in this paper. 4) Accuracy in the numerical results is obtained without considering the effect of normal deformation i.e. thickness stretching which increases one additional unknown in the mathematical formulation if considered.

In this study, Navier type closed-form solutions are obtained for static, buckling and free vibration analysis of FG sandwich beams. The beam has functionally graded skins and isotropic core. Material properties of FG skins are varied through the thickness according to the power law distribution. The present theory accounts for a hyperbolic distribution of axial displacement whereas transverse displacement is constant through the thickness. The present theory gives hyperbolic cosine distribution of transverse shear stress through the thickness of the beam and satisfies zero traction boundary conditions on the top and bottom surfaces of the beam. The equations of the motion are obtained by using the Hamilton's principle. The non-dimensional numerical results are obtained for various power law index and skin-core-skin thickness ratio. This article is organized in six sections. The need of FG sandwich beams and research on its modeling and analysis is highlighted in section 1 i.e. Introduction. Geometry and coordinate system of FG sandwich beam, the power-law for layer wise material gradation and assumptions made in the theoretical formulation are discussed in section 2. In section 3, mathematical modeling of FG sandwich beam using hyperbolic shear deformation theory is presented. Section 4 covers the analytical solution for simply-supported FG sandwich beams using Navier's solution technique. Numerical results and discussion are presented in section 5. Concluding remarks on the present study are presented in section 6.

2 FUNCTIONALLY GRADED SANDWICH BEAMS

Consider a sandwich beam with functionally graded skins and isotropic core. The beam has length *L*, width *b* and thickness *h*; made of FG-metal-FG material as shown in Fig. 1. The beam occupies the region $0\le x \le L$; $-b/2\le y \le b/2$; $-h/2\le z \le h/2$ in Cartesian coordinate systems. The *x* -axis is coincident with the beam neutral axis and the origin is at the left support. The *z*-axis is assumed downward positive. It is assumed that the beam is deformed in the *x-z* plane only.

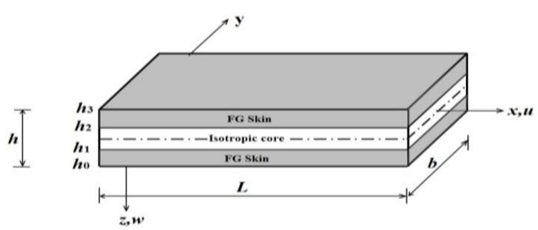


Fig.1
FG sandwich beam under bending in the *x-z* plane.

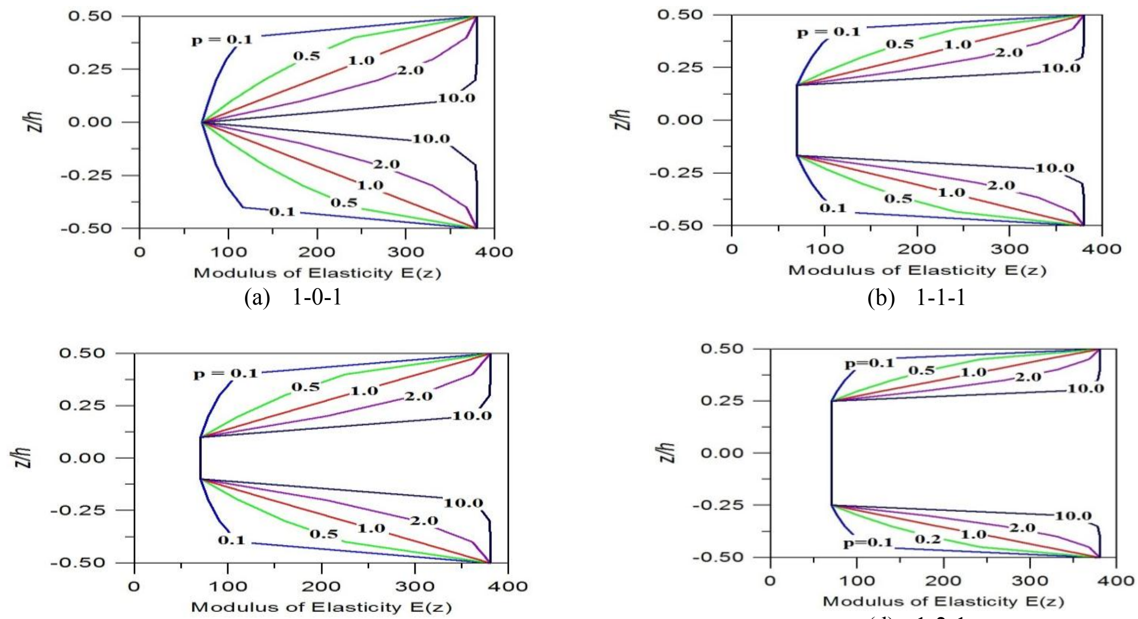
The power law for the material gradation introduced by Wakashima et al. [49] is the simplest rule which is widely used by many researchers for material property gradation. The law follows linear rule of mixture and properties are varying across the dimensions of FG beam.

$$E(z) = E_m + (E_c - E_m)V_c(z) \qquad \rho(z) = \rho_m + (\rho_c - \rho_m)V_c(z) \tag{1}$$

where the volume fraction function (V_c) for sandwich beam with FG skins and isotropic core is as follows

$$\begin{aligned} V_c(z) &= \left(\frac{z-h_0}{h_1-h_0}\right)^p \quad \text{for} \quad z\in[h_0, h_1] \\ V_c(z) &= 1 \quad \text{for} \quad z\in[h_1, h_2] \\ V_c(z) &= \left(\frac{z-h_3}{h_2-h_3}\right)^p \quad \text{for} \quad z\in[h_2, h_3] \end{aligned} \tag{2}$$

where E represents the modulus of elasticity and ρ represents the mass density; subscripts m and c represent the metallic and ceramic constituents, respectively; and p is the power law index. The variation of the modulus of elasticity $E(z)$ through the thickness z/h of the beam for various values of the power law index is shown in Fig. 2.



(a) 1-0-1 (b) 1-1-1

(c) 2-1-2 (d) 1-2-1

Fig.2
Through the thickness variation of modulus of elasticity of FG sandwich beams for various skin-core-skin thickness ratios.

2.1 Assumptions in theoretical formulation

The theoretical formulation of the FG sandwich beams is based on the following assumptions.

1) The axial displacement u consists of the extension component, bending component and shear component.
2) The present theory accounts for a hyperbolic distribution of axial displacement whereas transverse displacement is constant through the thickness.
3) Effects of transverse normal deformations ($\varepsilon_z = 0$) are neglected.
4) Since there is no relative motion in the y-direction at any points in the cross section of the beam; one dimensional Hooke's law is used to obtain stresses.
5) The theory gives hyperbolic cosine distribution of transverse shear stress through the thickness of the beam and satisfies zero traction boundary conditions on the top and bottom surfaces of the beam.

3 A HYPERBOLIC SHEAR DEFORMATION THEORY

Based on the aforementioned assumptions, the displacement field of a hyperbolic shear deformation beam theory of Soldatos [41] is given by:

$$u(x,z) = u_0(x) - z\frac{dw_0}{dx} + f\,\phi(x)$$
$$w(x) = w_0(x) \tag{3}$$

where u_0 and w_0 are the x and z-directional displacements of a point on the neutral axis of the beam. Shape function f is a function of z and assumed according to transverse shearing stress distribution across the thickness of the beam i.e. zero at the top and bottom surfaces of the beam. For the classical beam theory, $f = 0$; and for the Timoshenko beam theory, $f = z$. The only nonzero normal and shear strains at any point of the beam are,

$$\varepsilon_x = \varepsilon_x^0 + z\, k_x^b + f\, k_x^s, \qquad \gamma_{zx} = g\, \gamma_{zx}^0 \tag{4}$$

where $\varepsilon_x^0, k_x^b, k_x^b, \gamma_{zx}^0$ are related with the unknown displacements variables as follows:

$$\varepsilon_x^0 = \frac{du_0}{dx}, \quad k_x^b = -\frac{d^2 w_0}{dx^2}, \quad k_x^b = \frac{d\phi}{dx}, \quad \gamma_{zx}^0 = \phi,$$
$$f = \left[z \cosh(1/2) - h \sinh(z/h)\right] \quad \text{and} \quad g = f' = \left[\cosh(1/2) - \cosh(z/h)\right] \tag{5}$$

3.1 Constitutive relations

The FG sandwich beam is made of aluminum (Al) and alumina (Al_2O_3) materials. The properties of material are varying continuously in the thickness direction according to power law distribution given by Eq. (1). The stress-strain relationship at any point of the k^{th} layer of the beam is given by one dimensional Hooke's law as follows:

$$\sigma_x^k = E^k(z)\varepsilon_x^k \quad \text{and} \quad \tau_{zx}^k = G^k(z)\gamma_{zx}^k, \qquad G^k(z) = \frac{E^k(z)}{2(1+\mu)} \tag{6}$$

3.2 Equations of motion

In order to derive the equations of motion of a hyperbolic shear deformation theory for FG sandwich beam, Hamilton's principle is used.

$$\int_{t_1}^{t_2} (\delta U - \delta V + \delta K)\,dt = 0 \tag{7}$$

where, $\delta U, \delta V$ and δK denotes the variations in total strain energy, potential energy and kinetic energy respectively; t_1 and t_2 are the initial and final time respectively.

The variation of the strain energy (δU) can be stated as:

$$\delta U = \int_0^L \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} \left(\sigma_x^k \delta\varepsilon_x + \tau_{zx}^k \delta\gamma_{zx}\right) dz\, dy\, dx = \int_0^L \left(N_x \frac{d\delta u_0}{dx} - M^c \frac{d^2 \delta w_0}{dx^2} + M^s \frac{d\delta\phi}{dx} + Q\delta\phi\right) dx \tag{8}$$

where N_x represents the resultant axial force, M^c represents the resultant bending moment analogous to classical beam theory, M^s represents the resultant higher order moment associated with shear deformation and Q represents the resultant shear force.

$$\begin{Bmatrix} N_x \\ M^c \\ M^s \\ Q \end{Bmatrix} = b\int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x^k \\ \sigma_x^k z \\ \sigma_x^k f \\ \tau_{zx}^k f' \end{Bmatrix} = \begin{bmatrix} A & B & C & 0 \\ B & D & E & 0 \\ C & E & F & 0 \\ 0 & 0 & 0 & H \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ k_x^b \\ k_x^s \\ \gamma_{zx}^0 \end{Bmatrix} \tag{9}$$

where,

$$\left(A,B,C,D,E,F\right)=b\int_{-h/2}^{h/2} E^k\left(z\right)\left(1,z,f,z^2,f\,z,f^2\right)dz,$$
$$H=b\int_{-h/2}^{h/2} G^k\left(z\right)g^2dz \tag{10}$$

The variation of the potential energy (δV) due to transverse load q and axial load N_0 can be written as:

$$\delta V=\int_0^L\left(q\,\delta w+N_0\frac{dw}{dx}\frac{d\,\delta w}{dx}\right)dx \tag{11}$$

The variation of kinetic energy (δK) can be written in following form,

$$\delta K=\int_0^L\int_{-b/2}^{b/2}\int_{-h/2}^{h/2}\rho\left(z\right)\left(\frac{d^2u}{dt^2}\delta u+\frac{d^2w}{dt^2}\delta w\right)dz\,dy\,dx$$
$$=\int_0^L\left(I_A\frac{d^2u_0}{dt^2}-I_B\frac{d^3w_0}{dxdt^2}+I_C\frac{d^2\phi}{dt^2}\right)\delta u_0dx+\int_0^L\left(-I_B\frac{d^2u_0}{dt^2}+I_D\frac{d^3w_0}{dxdt^2}-I_E\frac{d^2\phi}{dt^2}\right)\frac{d\,\delta w_0}{dx}dx$$
$$+\int_0^L\left(I_C\frac{d^2u_0}{dt^2}-I_E\frac{d^3w_0}{dxdt^2}+I_F\frac{d^2\phi}{dt^2}\right)\delta\phi dx+\int_0^L I_A\frac{d^2w_0}{dt^2}\delta w_0dx \tag{12}$$

where, $\rho\left(z\right)$ is the mass density of each layer. I_A, I_B, I_C, I_D, I_E, I_F are the inertia coefficients defined by:

$$\left(I_A,I_B,I_C,I_D,I_E,I_F\right)=b\int_{-h/2}^{h/2}\rho^k\left(z\right)\left(1,z,f,z^2,f\,z,f^2\right)dz \tag{13}$$

The equations of motion can be obtained by integrating Eq. (7) by parts; collecting the coefficients of $\delta u_0,\delta w_0,\delta\phi$ and equating with zero. The following equations of motion are obtained.

$$\frac{dN_x}{dx}=I_A\frac{d^2u_0}{dt^2}-I_B\frac{d^3w_0}{dxdt^2}+I_C\frac{d^2\phi}{dt^2}$$
$$\frac{d^2M^c}{dx^2}=-q+N_0\frac{d^2w_0}{dx^2}+I_B\frac{d^3u_0}{dxdt^2}-I_D\frac{d^4w_0}{dx^2dt^2}+I_A\frac{d^2w_0}{dt^2}+I_E\frac{d^3\phi}{dxdt^2} \tag{14}$$
$$\frac{dM^s}{dx}-Q=I_C\frac{d^2u_0}{dt^2}-I_E\frac{d^3w_0}{dxdt^2}+I_F\frac{d^2\phi}{dt^2}$$

By substituting the stress resultants (N_x,M^c,M^s,Q) from Eq. (9) into Eq. (14), the following equations of motion can be obtained in terms of unknown displacement variables (u_0,w_0,ϕ),

$$A\frac{d^2u_0}{dx^2}-B\frac{d^3w_0}{dx^3}+C\frac{d^2\phi}{dx^2}=I_A\frac{d^2u_0}{dt^2}-I_B\frac{d^3w_0}{dxdt^2}+I_C\frac{d^2\phi}{dt^2}$$
$$B\frac{d^3u_0}{dx^3}-D\frac{d^4w_0}{dx^4}+E\frac{d^3\phi}{dx^3}=-q+N_0\frac{d^2w_0}{dx^2}+I_B\frac{d^3u_0}{dxdt^2}-I_D\frac{d^4w_0}{dx^2dt^2}+I_A\frac{d^2w_0}{dt^2}+I_E\frac{d^3\phi}{dxdt^2} \tag{15}$$
$$C\frac{d^2u_0}{dx^2}-E\frac{d^3w_0}{dx^3}+F\frac{d^2\phi}{dx^2}-H\phi=I_C\frac{d^2u_0}{dt^2}-I_E\frac{d^3w_0}{dxdt^2}+I_F\frac{d^2\phi}{dt^2}$$

4 NAVIER'S SOLUTIONS

The Navier's solution technique is used to determine the closed-form solutions for static bending, elastic buckling and free vibration analysis of a simply-supported functionally graded sandwich beam. The boundary conditions of the simply-supported beam are as follows:

$$w_0 = N_x = M^c = M^s = 0 \text{ at } x = 0 \text{ and } x = L \ \left(\text{For buckling analysis } N_x \neq 0\right) \tag{16}$$

It is to be noted that the edges are movable simply supported ones for bending, buckling and vibration analyses. The solution is assumed to be of the form:

$$\begin{aligned} u_0(x,t) &= \sum_{m=1,3,5}^{\infty} u_m \cos\alpha x \ e^{i\omega t}, \\ w_0(x,t) &= \sum_{m=1,3,5}^{\infty} w_m \sin\alpha x \ e^{i\omega t}, \\ \phi(x,t) &= \sum_{m=1,3,5}^{\infty} \phi_m \cos\alpha x \ e^{i\omega t} \end{aligned} \tag{17}$$

where, $i = \sqrt{-1}$ the imaginary unit, ω is the natural frequency, $\alpha = m\pi / L$ and $\left(u_m, w_m, \phi_m\right)$ are the unknown coefficients to be determined. The uniform transverse load (q) acting on the top surface of the beam is assumed to be of the form

$$q(x) = \sum_{m=1,3,5}^{\infty} \frac{4q_0}{m\pi} \sin\alpha x \tag{18}$$

where q_0 is the maximum intensity of load at the centre of the length. The beam is subjected to an axial compressive force N_0. By substituting Eqs. (17) and (18) into Eq. (15), the analytical solution can be obtained from the following equations:

$$\begin{bmatrix} A\alpha^2 & -B\alpha^3 & C\alpha^2 \\ -B\alpha^3 & D\alpha^4 & -E\alpha^3 \\ C\alpha^2 & -E\alpha^3 & F\alpha^2 + H \end{bmatrix} \times \begin{Bmatrix} u_m \\ w_m \\ \phi_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \frac{4q_0}{m\pi} \tag{19}$$

$$\left\{ \begin{bmatrix} A\alpha^2 & -B\alpha^3 & C\alpha^2 \\ -B\alpha^3 & D\alpha^4 & -E\alpha^3 \\ C\alpha^2 & -E\alpha^3 & F\alpha^2 + H \end{bmatrix} - N_0 \begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \times \begin{Bmatrix} u_m \\ w_m \\ \phi_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{20}$$

$$\left\{ \begin{bmatrix} A\alpha^2 & -B\alpha^3 & C\alpha^2 \\ -B\alpha^3 & D\alpha^4 & -E\alpha^3 \\ C\alpha^2 & -E\alpha^3 & F\alpha^2 + H \end{bmatrix} - \omega^2 \begin{bmatrix} I_A & -I_B\alpha & I_C \\ -I_B\alpha & \left(I_D\alpha^2 + I_A\right) & -I_E\alpha \\ I_C & -I_E\alpha & I_F \end{bmatrix} \right\} \times \begin{Bmatrix} u_m \\ w_m \\ \phi_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{21}$$

The solution of the Eq. (19) allow to calculate the displacements and stresses, the solution of Eq. (20) allow to calculate critical buckling loads and the solution of Eq. (21) allow to calculate natural frequencies of FG sandwich beams.

5 NUMERICAL RESULTS AND DISCUSSION

In this section, accuracy of a hyperbolic shear deformation theory is demonstrated by applying to bending, buckling and vibration analysis of simply supported functionally graded sandwich beams. The skins of the sandwich beams are made of FG (Al/Al_2O_3) material while the core is made of homogeneous isotropic (Al) material. Length of the beam is taken as 1*m* which is constant and thickness of the beam is varied according to aspect ratio (*L/h*). The material properties of alumina (Al_2O_3) are $E_c = 380 GPa, \rho_c = 3960 kg/m^3, \mu = 0.3$; and the material properties of aluminum (Al) are $E_m = 70 GPa, \rho_m = 2702 kg/m^3, \mu = 0.3$.

The numerical results are presented in the following non-dimensional form

Axial displacement (u) at $x = 0$ and $z = -h/2$: $\bar{u} = \dfrac{u\,100 E_m h^3}{q_0 L}$

Transverse displacement (w) at $x = L/2$ and $z = 0$: $\bar{w} = \dfrac{w\,100 E_m h^3}{q_0 L}$

Axial stress (σ_x) at $x = L/2$ and $z = -h/2$: $\bar{\sigma}_x = \dfrac{\sigma_x h}{q_0 L}$

Transverse shear stress (τ_{xz}) at $x = 0$ and $z = 0$: $\bar{\tau}_{xz} = \dfrac{\tau_{xz} h}{q_0 L}$

Critical buckling load: $\bar{N}_{cr} = \dfrac{12 N_0 a^2}{E_m h^3}$

Natural frequency: $\bar{\omega} = \dfrac{\omega L^2}{h}\sqrt{\dfrac{\rho_m}{E_m}}$

5.1 Static bending analysis

In this problem, static bending analysis of three types of symmetric (1-1-1, 1-2-1, 2-1-2) sandwich beams is carried out. The non-dimensional vertical displacements for various power law indexes are given in Table 1. The present results are compared with those obtained by using 1D beam theories ($\varepsilon_z = 0$) such as CBT [14], TBT [15], PBT [16] and 2D beam theory ($\varepsilon_z \neq 0$) of Vo et al. [29]. The examination of Table 1., reveals that the transverse displacements predicted by using the present theory are in excellent agreement with 1D and 2D higher order beam theories. As expected, due to ignoring the effect of transverse shear deformation, the CBT underestimates transverse displacement. The smallest and largest displacement correspond to the (1-2-1) and (2-1-2) sandwich beams since they have the highest and lowest portion of the ceramic phase. It can be seen that the increase in power law index increases transverse displacement. This is due to the fact that an increase of the power law index makes FG beams more flexible. Through the thickness distribution of axial displacement is plotted in Fig. 3 which shows that due to symmetric layup axial displacement of neutral axis is zero. Table 2., shows comparison of non-dimensional axial stress with those of CBT [14], TBT [15], PBT [16] and Vo et al. [29]. It is observed that as the power law index increases, axial stress decreases. Through the thickness distribution of axial stress is plotted in Fig. 4. The same maximum axial stress at the top and bottom surfaces of FG sandwich beam is observed due to symmetry in material gradation. Since core is made of isotropic/homogenous material (Metal: Aluminum) and elastic properties are constant through the thickness, all distributions for all volume fraction indices are observed linear. Whereas, skins are made of functionally graded materials i.e. nonhomogeneous which results in hyperbolic distributions of stresses. Also, in case of functionally graded skins, modulus of elasticity is varied across the thickness which results in reduction and growth trends within each individual skin for the higher values of the volume fraction indices. The axial stress is increases with increase in the power law index. The maximum axial stress is observed for configuration 2-1-2 whereas the minimum is observed in configuration 1-2-1. The non-dimensional values of transverse shear stress of simply-supported FG sandwich beams for various power law indexes are shown in Table 3., and plotted in Fig. 5. The present results of shear stresses are in excellent agreement with other theories. It can be seen that the maximum shear stress for symmetric FG sandwich beams occurs at the neutral axis of the beam.

Table 1
Non-dimensional vertical displacement of simply-supported FG sandwich beams.

p	Theory	*L*/*h*=5			*L*/*h*=20		
		1-1-1	1-2-1	2-1-2	1-1-1	1-2-1	2-1-2
0	1D [14]	2.8783	2.8783	-	2.8783	2.8783	-
	1D [15]	3.1657	3.1657	-	2.8963	2.8963	-
	1D [16]	3.1654	3.1654	-	2.8963	2.8963	-
	2D [29]	3.1397	3.1397	-	2.8947	2.8947	-
	Present	3.1241	3.1241	3.1241	2.8585	2.8585	2.8585
1	1D [14]	5.9181	5.0798	-	5.9181	5.0798	-
	1D [15]	6.3128	5.4408	-	5.9428	5.1024	-
	1D [16]	6.2693	5.4122	-	5.9401	5.1006	-
	2D [29]	6.2098	5.3612	-	5.9364	5.0975	-
	Present	6.3011	5.0341	6.8424	5.9561	5.3415	6.4978
2	1D [14]	8.0074	6.4056	-	8.0074	6.4056	-
	1D [15]	8.4582	6.8003	-	8.0356	6.4302	-
	1D [16]	8.3893	6.7579	-	8.0313	6.4276	-
	2D [29]	8.3893	6.6913	-	8.0262	6.4235	-
	Present	8.2734	6.3359	9.5104	7.9201	6.6697	9.1626
5	1D [14]	10.8117	8.1409	-	10.8117	8.1409	-
	1D [15]	11.3372	8.5762	-	10.8445	8.1681	-
	1D [16]	11.2274	8.5137	-	10.8376	8.1642	-
	2D [29]	11.1175	8.4276	-	10.8309	8.1589	-
	Present	11.0708	8.0576	13.0323	10.6766	8.4045	12.5898
10	1D [14]	12.1322	9.0232	-	12.1322	9.0232	-
	1D [15]	12.1322	9.4800	-	12.1677	9.0518	-
	1D [16]	12.5659	9.4050	-	12.1593	9.0471	-
	2D [29]	12.4453	9.3099	-	12.1519	9.0413	-
	Present	12.3910	8.9290	14.4346	11.9795	9.2824	13.9531

Table 2
Non-dimensional axial stress of simply-supported FG sandwich beams.

p	Theory	*L*/*h*=5			*L*/*h*=20		
		1-1-1	1-2-1	2-1-2	1-1-1	1-2-1	2-1-2
0	1D [15]	3.7500	3.7500	-	15.0000	15.0000	-
	1D [16]	3.8020	3.8020	-	15.0129	15.0129	-
	2D [29]	3.8005	3.8005	-	15.0125	15.0125	-
	Present	3.8025	3.8025	3.8025	15.0136	15.0136	15.0136
1	1D [15]	1.4203	1.2192	-	5.6814	4.8766	-
	1D [16]	1.4349	1.2339	-	5.6850	4.8801	-
	2D [29]	1.4330	1.2315	-	5.6845	4.8797	-
	Present	1.4614	1.2331	1.5900	5.7370	4.8802	6.3020
2	1D [15]	1.9218	1.5373	-	7.6871	6.1493	-
	1D [16]	1.9382	1.5527	-	7.6912	6.1532	-
	2D [29]	1.9352	1.5505	-	7.6904	6.1526	-
	Present	1.9369	1.5530	2.2384	7.6154	6.1534	8.8940
5	1D [15]	2.5948	1.9538	-	10.3792	7.8152	-
	1D [16]	2.6123	1.9705	-	10.3835	7.8194	-
	2D [29]	2.6079	1.9672	-	10.3824	7.8185	-
	Present	2.6101	1.9707	3.0733	10.2712	7.8196	12.2223
10	1D [15]	2.9117	2.1656	-	11.6469	8.6623	-
	1D [16]	2.9293	2.1826	-	11.6513	8.6665	-
	2D [29]	2.9245	2.1788	-	11.6500	8.6655	-
	Present	2.9268	2.1829	3.4047	11.5237	8.6667	13.5459

Table 3
Non-dimensional shear stress of simply-supported FG sandwich beams.

P	Theory	*L/h*=5			*L/h*=20		
		1-1-1	1-2-1	2-1-2	1-1-1	1-2-1	2-1-2
0	1D [15]	0.5976	0.5976	-	0.5976	0.5976	-
	1D [16]	0.7332	0.7332	-	0.7451	0.7451	-
	2D [29]	0.7233	0.7233	-	0.7432	0.7432	-
	Present	0.7285	0.7285	0.7285	0.7355	0.7355	0.7355
1	1D [15]	0.8208	0.7507	-	0.8208	0.7507	-
	1D [16]	0.8586	0.8123	-	0.8681	0.8215	-
	2D [29]	0.8444	0.7993	-	0.8657	0.8193	-
	Present	0.8767	0.8056	0.9050	0.8726	0.8106	0.9107
2	1D [15]	0.9375	0.8208	-	0.9375	0.8208	-
	1D [16]	0.9249	0.8493	-	0.9344	0.8581	-
	2D [29]	0.9084	0.8349	-	0.9316	0.8556	-
	Present	0.9170	0.8424	0.9103	0.9222	0.8486	0.9149
5	1D [15]	1.0929	0.9053	-	1.0929	0.9053	-
	1D [16]	1.0125	0.8925	-	1.0227	0.9014	-
	2D [29]	0.9931	0.8763	-	1.0194	0.8986	-
	Present	1.0048	0.8851	1.1766	1.0101	0.8897	1.1835
10	1D [15]	1.1819	0.9497	-	1.1819	0.9497	-
	1D [16]	1.0665	0.9151	-	1.0773	0.9243	-
	2D [29]	1.0458	0.8980	-	1.0736	0.9214	-
	Present	1.0586	0.9083	1.2982	1.0642	0.9128	1.3055

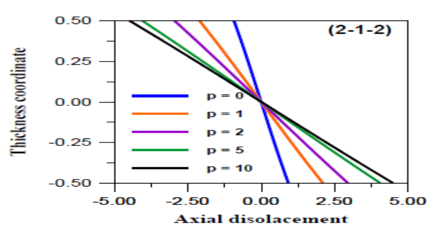


Fig.3
Through the thickness distribution of non-dimensional axial displacement ($\bar{u}$) in FG sandwich beams subjected to uniform load at *L/h* = 5.

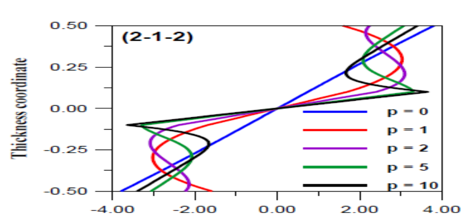
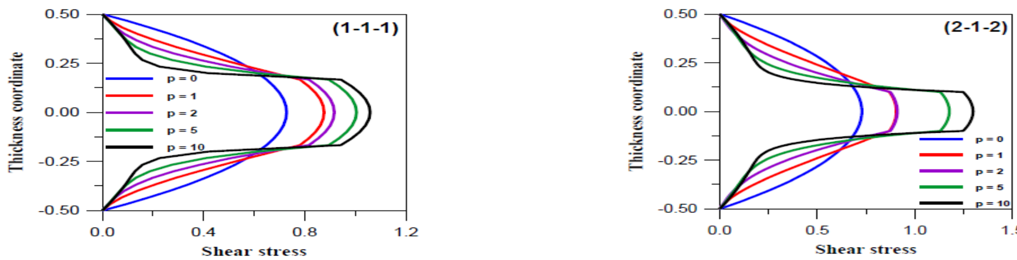


Fig.4
Through the thickness distribution of non-dimensional axial stress in FG sandwich beams subjected to uniform load at *L/h* = 5.



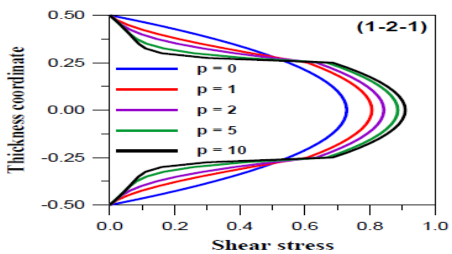
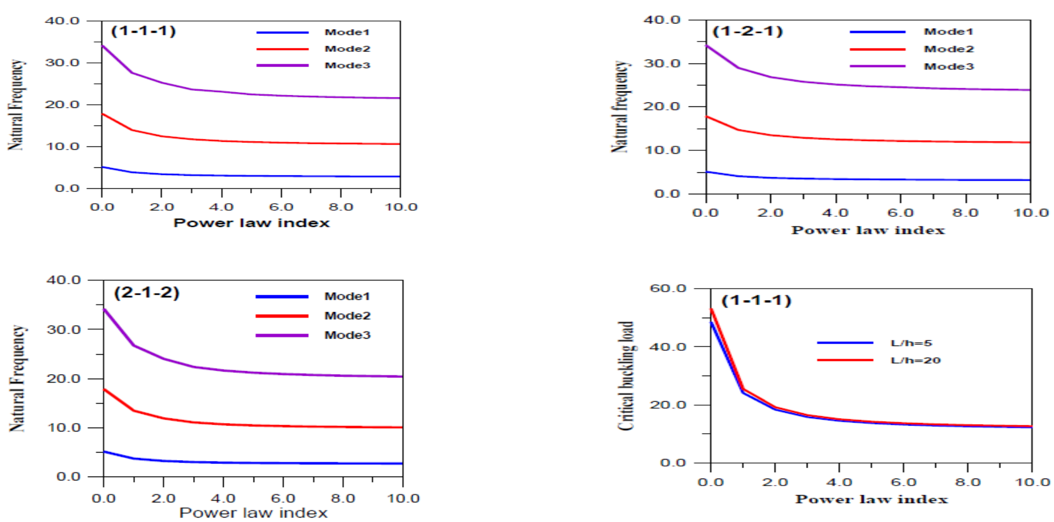


Fig.5
Through the thickness distribution of non-dimensional shear stress in FG sandwich beams subjected to uniform load at *L/h* =5.

5.2 Elastic buckling analysis

Tables 4., shows the comparison of non-dimensional critical buckling loads of FG sandwich beams for different values of the power law index and skin–core–skin thickness ratio. The critical buckling loads are obtained for *L/h* = 5 and 20. The present results are compared with those obtained from other higher order 1D and 2D beam theories available in the literature. It can be observed from Table 4., that the present results agree very well with the existing 1D beam theories [20, 31] and 2D beam theories [19, 30]. It is worthy of note that the increase in thickness of FG skin results in a decrease in the critical buckling loads. The maximum critical buckling load is observed for configuration 1-2-1 where thickness of skin is 0.25*h* whereas the minimum is observed for configuration 2-1-2 where thickness of skin is 0.4*h*. It is also important to note that the critical buckling load decreases with the increase in the power law index. This is in fact due to an increase in the power law index results in a decrease in the modulus of elasticity. The beam therefore becomes more flexible and buckles at a much lower load. As expected, when $p = 0$ i.e. for fully ceramic beam, critical buckling loads are the same irrespective of the beam configuration. Fig. 6 shows the effect of power law index on the critical buckling loads and natural frequencies for varying *L/h* values. It can be seen that increase in *L/h* results in an increase in the critical buckling loads.



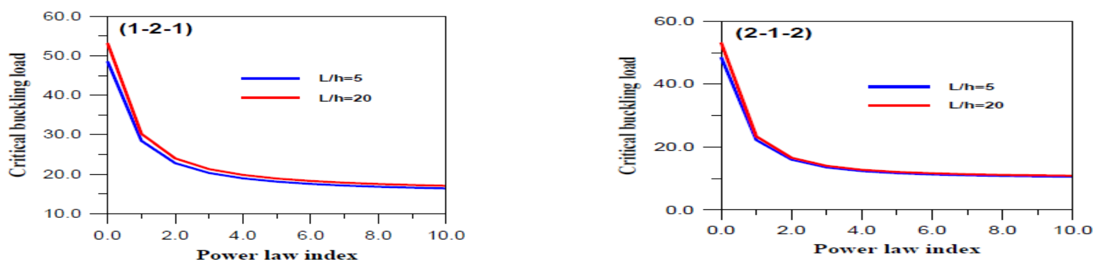


Fig.6
Variation of natural frequencies and critical buckling loads with respect to power law index for FG sandwich beams.

Table 4
Nondimensional critical buckling load $\left(\bar{N}_{cr}\right)$ of simply-supported FG sandwich beams.

p	Theory	*L/h*=5			*L/h*=20		
		1-1-1	1-2-1	2-1-2	1-1-1	1-2-1	2-1-2
0	2D [19]	49.5970	49.5970	49.5970	53.3175	53.3175	53.3175
	1D [20]	48.5964	48.5964	48.5964	53.2364	53.2364	53.2364
	1D [31]	48.5959	48.5959	48.5959	53.2364	53.2364	53.2364
	2D [30]	49.5906	49.5906	49.5906	53.3145	53.3145	53.3145
	Present	48.5960	48.5960	48.5960	53.2367	53.2367	53.2367
1	2D [19]	25.1060	29.0723	22.7061	26.0001	30.2785	23.4584
	1D [20]	24.5602	28.4440	22.2121	25.9588	30.2706	23.4212
	1D [31]	24.5596	28.4447	22.2108	25.9588	30.2307	23.4212
	2D [30]	25.1075	29.0755	22.7065	25.9989	30.2374	23.4572
	Present	24.5116	28.4451	22.2112	25.9517	30.2550	23.4211
2	2D [19]	18.7750	23.3002	16.2761	19.2309	24.0284	16.6317
	1D [20]	18.3596	22.7859	15.9167	19.2000	23.9899	16.6051
	1D [31]	18.3587	22.7863	15.9152	19.3116	23.9900	16.6050
	2D [30]	18.7772	23.3042	16.2761	19.2299	24.0276	16.6307
	Present	18.3733	22.7870	15.9860	19.2160	23.9900	16.6100
5	2D [19]	14.0358	18.5058	11.9320	14.2515	18.9180	12.1078
	1D [20]	13.7226	18.0915	11.6697	14.2285	18.8874	12.0886
	1D [31]	13.7212	18.0914	11.6676	14.2284	18.8874	12.0883
	2D [30]	14.0353	18.5092	11.9301	14.2505	18.9172	12.1086
	Present	13.7340	18.0921	11.6671	14.2421	18.8880	12.088
10	2D [19]	12.5402	16.7550	10.7715	12.7023	17.0723	10.9246
	1D [20]	12.2621	16.3789	105370	12.6820	17.0445	10.9075
	1D [31]	12.2605	16.3783	10.5348	12.6819	17.0443	10.9075
	2D [30]	12.5393	16.7574	10.7689	12.7014	17.0712	10.9239
	Present	12.2710	16.3787	10.5345	12.6940	17.0455	10.9075

5.3 Free vibration analysis

Table 5., presents the comparison of the fundamental frequencies of simply-supported FG sandwich beams calculated for various values of the power law index and skin-core-skin thickness ratios. The present results are compared with the 1D [20, 31] and 2D [19, 30] beam theories. It is seen that the solutions obtained from the proposed theory are in excellent agreement with those obtained from other 1D and 2D beam theories. It can be seen from the table that the fundamental frequencies decrease with the increase of the power law index. The lowest values of the fundamental frequency are observed for configuration 2-1-2 whereas the highest values of frequency are observed for configuration 1-2-1. It is due to the fact that these beams correspond to the lowest and highest volume fractions of the ceramic phase. Furthermore, it can be seen that fundamental frequencies are decrease with increase in power law index. As expected, it is observed that when p = 0, fundamental frequencies are same for all

configurations. Table 6 shows natural frequencies of simply-supported FG sandwich beams calculated for first two modes at L/h = (5, 20) and p = (0, 1, 2, 5, 10). These frequencies are presented for the first time in literature.

Table 5
Non-dimensional fundamental frequencies of simply-supported FG sandwich beams.

p	Theory	*L/h*=5			*L/h*=20		
		1-1-1	1-2-1	2-1-2	1-1-1	1-2-1	2-1-2
0	2D [19]	5.1620	5.1620	5.1620	5.4611	5.4611	5.4611
	1D [20]	5.1528	5.1528	5.1528	5.4603	5.4603	5.4603
	1D [31]	5.1528	5.1528	5.1528	5.4603	5.4603	5.4603
	2D [30]	5.1618	5.1618	5.1618	5.4610	5.4610	5.4610
	Present	5.1527	5.1527	5.1527	5.4603	5.4603	5.4603
1	2D [19]	3.8830	4.1185	3.7369	40334	4.2896	3.1753
	1D [20]	3.8756	4.1105	3.7298	4.0328	4.2889	3.7147
	1D [31]	3.8755	4.1105	3.7298	4.0328	4.2889	3.7147
	2D [30]	3.8830	4.0005	3.7369	4.033	4.2895	3.7152
	Present	3.8763	4.1105	3.7297	4.0336	4.2889	3.8767
2	2D [19]	3.4258	3.7410	3.2428	3.5395	3.8775	3.1769
	1D [20]	3.4190	3.7334	3.2366	3.5389	3.8769	3.1764
	1D [31]	3.4190	3.7334	3.2365	3.5389	3.8769	3.1764
	2D [30]	3.4257	3.7410	3.2427	3.5394	3.8774	3.1768
	Present	3.4200	3.7334	3.2433	3.5400	3.8769	3.3470
5	2D [19]	3.0239	3.3840	2.8491	3.1116	3.4927	2.8444
	1D [20]	3.0182	3.3771	2.8441	3.1111	3.4921	2.8440
	1D [31]	3.0181	3.3771	2.8439	3.1111	3.4921	2.8439
	2D [30]	3.0238	3.3840	2.8489	3.1115	3.4926	2.8443
	Present	3.0191	3.3770	2.8438	3.1122	3.4921	2.9310
10	2D [19]	2.8862	3.2423	2.7402	2.99666	3.3412	2.8046
	1D [20]	2.8810	3.2357	2.7357	2.9662	3.3406	2.8042
	1D [31]	2.8808	3.2356	2.7355	2.9662	3.3406	2.8041
	2D [30]	2.8860	3.2422	2.7400	2.9786	3.3411	2.8045
	Present	2.8817	3.2356	2.7353	2.9672	3.3406	2.8188

Table 6
Non-dimensional frequencies of simply-supported FG sandwich beams.

		Mode 1			Mode 2		
L/h	*p*	1-1-1	1-2-1	2-1-2	1-1-1	1-2-1	2-1-2
5	0	17.8818	17.8818	17.8818	34.2018	34.2018	34.2018
	1	13.9691	14.7252	13.4749	27.6430	28.9711	26.7339
	2	12.4725	13.5151	11.9127	25.2792	26.8563	24.0305
	5	11.1175	12.3431	10.4775	22.4844	24.7642	21.2090
	10	10.6411	11.8692	10.0825	21.5866	23.9034	20.4234
20	0	21.5333	21.5333	21.5733	47.5933	47.5933	47.5933
	1	16.0011	17.0036	15.3823	35.5215	37.7124	34.1611
	2	14.0589	15.3861	13.3010	31.2654	34.1806	29.6114
	5	12.3710	13.8717	11.6510	27.5512	30.8606	25.9491
	10	11.9338	13.2743	11.2051	26.2840	29.5472	24.9574

6 CONCLUSIONS

Navier type closed-form solutions for static bending, elastic buckling and free vibration analysis of simply supported functionally graded sandwich beams using a hyperbolic shear deformation theory is proposed in this paper. The proposed theory accounts for hyperbolic variation of axial displacement and transverse shear strains. Hamilton's principle is employed to derive the equations of motion. Effects of power law index, span-to-depth ratio and skin-core-skin thickness ratios on the displacements, stresses, critical buckling loads and natural frequencies are discussed. It is concluded that the displacements, stresses, critical buckling loads and natural frequencies obtained by using the present theory are accurate as compared to those obtained by using other 1D and 2D refined shear deformation theories. Increasing the power law index reduces the stiffness of FG sandwich beam and consequently leads to an increase in displacements and a reduction of frequencies and buckling loads. The proposed theory is accurate and efficient in solving the static bending, elastic buckling and free vibration problems of the FG sandwich beams.

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