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Axially Symmetric Vibrations of a Liquid-Filled Poroelastic Thin Cylinder Saturated with Two Immiscible Liquids Surrounded by a Liquid

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ABSTRACT

This paper studies axially symmetric vibrations of a liquid-filled poroelastic thin cylinder saturated with two immiscible liquids of infinite extent that is surrounded by an inviscid elastic liquid. By considering the stress free boundaries, the frequency equation is obtained. Particular case, namely, liquid-filled poroelastic cylinder saturated with single liquid is discussed. When the wavenumber is large, the frequency equation is reduced to that of Rayleigh-type surface wave at the plane boundary of a poroelastic half-space. In this case, the asymptotic expressions of Bessel functions and modified Bessel functions are used. In both general and particular cases, the case of the propagation of Rayleigh waves in a poroelastic half-space is obtained. The parameter values of Columbia fine sandy loam saturated with air-water mixture are used for the numerical evaluation. In all the cases, phase velocity as a function of wavenumber is computed and presented graphically. From the numerical results, some inferences are drawn.

Keywords : Axially symmetric vibrations; Thin cylinder; Liquid; Wavenumber; Phase velocity.

1 INTRODUCTION

IN general, wave propagation problems in liquid-saturated porous media is attracting many researchers due to its importance in various diversified areas of Science and Engineering. These problems have been studied extensively in Soil Mechanics, Seismology, Acoustics, Structural Engineering, Geophysics, and Biomechanics. Pipe lines in the oil exploration and water supply systems may be surrounded by some kind of liquids. These pipe lines are cylindrical and filled with liquids. Study of wave propagation in liquid-filled cylindrical pipes surrounded by liquid provides some information about the strength of pipe lines since wave characteristics depend on the strength of the material. Biot developed constitutive equations and the equations of motion in a poroelastic solid saturated with a single fluid [1]. This well established theory could not address the issues in important domains, namely, Petroleum Engineering and Bone Mechanics, wherein porous solid is saturated with two or more fluids. To this

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extent, Tuncay and Corapcioglu developed a theory of wave propagation in a poroelastic solid saturated by two immiscible Newtonian fluids in the Eulerian framework by employing the volume averaging technique [2]. A study of body waves in poroelastic solid saturated by two immiscible Newtonian fluids is presented [3]. In the paper [3], the existence of three compressional waves and one rotational wave in an infinite porous medium is shown analytically. Santos et al. proposed a method to determine the elastic constants for isotropic porous media saturated by two fluids [4]. Sahay et al. developed a set of equations to describe the low frequency seismic phenomenon in porous solid which is inhomogeneous and anisotropic by using the volume average technique [5]. Hunyga developed a general dynamical model for porous media saturated by two immiscible fluids [6]. Lo et al. derived a general set of coupled partial differential equations to describe dilatational wave propagation through an poroelastic medium saturated by two immiscible fluids [7]. Propagation of axisymmetric waves through compressible, inviscid fluid contained in a cylindrical, elastic shell is studied by Lin [8]. Using exact three dimensional equations of linear elasticity, the frequency equation for the vibrations of a fluid-filled circular cylindrical shell is derived by Ram Kumar[9]. Sharma and Gogna solved the problem of elastic wave propagation in a cylindrical bore in poroelastic solid and derived the frequency equation for empty and fluid-filled bores [10]. Axisymmetric wave propagation along fluid-loaded cylindrical shells is investigated both theoretically and experimentally by Plona et al. [11]. The longitudinal waves in homogeneous anisotropic cylindrical bars immersed in a fluid are studied by Dayal [12]. The laws of propagation of axisymmetric normal modes in a hollow cylinder filled with and surrounded by fluid media are investigated [13]. Vashishth and Khurana studied the propagation of wave along a cylindrical bore hole embedded in an anisotropic porous solid saturated by viscous fluid [14]. Harmonic radiation from a spherical surface vibrating at the center of a fluid-filled circular cylindrical cavity is investigated [15]. The propagation of elastic waves along a cylindrical bore hole filled with and without fluid embedded in an infinite porous medium saturated by two immiscible fluids is studied by Arora and Tomar [16]. In the paper [16], the dispersion equation of Rayleigh-type surface waves along the boundary of a poroelastic half space saturated by two immiscible fluids is obtained. Rayleigh waves in poroelastic half space and a porous solid lying over an elastic solid are examined by Tajuddin et. al. [17,18]. In both the papers, it is assumed that porous solid saturated with a single fluid. A comparative dispersive study between the bone without marrow and bone with marrow is made in the framework of Biot's theory of isotropic poroelasticity by Malla Reddy et al. [19]. Axially symmetric vibrations of fluid-filled and empty poroelastic circular cylindrical shell of infinite extent are investigated by Ahmed Shah [20]. Flexural vibrations of poroelastic cylinder of various thicknesses immersed in a fluid are studied in the paper [21]. Vibrations in a fluid loaded poroelastic hollow cylinder surrounded by a fluid under plane strain conditions are investigated by Shanker et al. [22]. wave propagation in composite hollow sphere and fluid-filled sphere surrounded by fluid are analyzed in the papers [23, 24]. Radial vibrations of poroelastic cylindrical shell immersed in an acoustic medium is studied in the paper [25].

In the present paper, axially symmetric vibrations of a liquid-filled poroelastic thin cylinder of infinite extent saturated with two immiscible liquids that is surrounded by an inviscid elastic fluid are investigated. Propagation of waves in a liquid-filled poroelastic thin cylinder saturated with a single liquid is recovered as a particular case. In both the cases, the frequency equation of Rayleigh wave is obtained. The rest of the paper is organized as follows. In section 2, governing equations and solution of the problem are presented. Boundary conditions and frequency equation are given in section 3. Particular cases are presented in section 3.1. Numerical results are discussed in section 4. Finally, conclusion is given in section 5.

2 GOVERNING EQUATIONS AND SOLUTION OF THE PROBLEM

Let (r,θ,z) be the cylindrical polar coordinates. Consider a homogeneous, isotropic, infinite poroelastic thin cylinder saturated with two immiscible liquids loaded internally and externally by inviscid liquids. The axis of the cylinder is in the direction of *z*- axis as shown in Fig. 1. The equations of motion in the absence of body forces for low frequency wave propagation in a poroelastic solid saturated with two immiscible fluids are given under [16]:

$$\begin{gathered}\nabla((a_{11}+\frac{1}{3}G_{fr})\nabla\cdot\vec{\mathbf{u}}_s+a_{12}\nabla\cdot\vec{\mathbf{u}}_1+a_{13}\nabla\cdot\vec{\mathbf{u}}_2)\nabla\cdot(G_{fr}\nabla\vec{\mathbf{u}}_s)+c_1(\vec{\mathbf{v}}_1-\vec{\mathbf{v}}_s)+c_2(\vec{\mathbf{v}}_2-\vec{\mathbf{v}}_s)=\rho_s\frac{\partial^2\vec{u}_s}{\partial t^2},\\ \nabla(a_{21}\nabla\cdot\vec{\mathbf{u}}_s+a_{22}\nabla\cdot\vec{\mathbf{u}}_1+a_{23}\nabla\cdot\vec{\mathbf{u}}_2)-c_1(\vec{\mathbf{v}}_1-\vec{\mathbf{v}}_s)=\rho_1\frac{\partial^2\vec{\mathbf{u}}_1}{\partial t^2},\ \nabla(a_{31}\nabla\cdot\vec{\mathbf{u}}_s+a_{32}\nabla\cdot\vec{\mathbf{u}}_1+a_{33}\nabla\cdot\vec{\mathbf{u}}_2)-c_2(\vec{\mathbf{v}}_2-\vec{\mathbf{v}}_s)=\rho_2\frac{\partial^2\vec{\mathbf{u}}_2}{\partial t^2},\end{gathered}\tag{1}$$

where,

$$a_{11}=K_{fr},\quad a_{12}=a_{21}=K_1\alpha_s S_1(A_2+K_2)/D,\ a_{13}=a_{31}=K_2\alpha_s(1-S_1)(A_2+K_1)/D,$$
$$a_{22}=K_1S_1^{\,2}(1-\alpha_s)(K_2+\frac{A_2}{S_1})/D,\quad a_{23}=a_{32}=K_1K_2S_1(1-S_1)(1-\alpha_s)/D,$$
$$a_{33}=K_2(1-S_1)^2(1-\alpha_s)(K_s+\frac{A_2}{(1-S_1)})/D,\quad D=K_1(1-S_1)+A_2+K_2S_1,$$
$$c_1=(1-\alpha_s)^2S_1^2\mu_1/KK_{r1},\quad c_2=(1-\alpha_s)^2(1-S_1)^2\mu_2/KK_{r2},\quad S_1=\alpha_1/(1-\alpha_s),$$
$$A_2=\Pr S_1(1-S_1),\quad \Pr=\frac{d}{dS_1}(p_1^*-p_2^*),$$

$\vec{\mathbf{u}}_s$ and $\vec{\mathbf{u}}_i$ are displacement vectors in the poroelastic solid and in the liquid of phase i respectively. $\vec{\mathbf{v}}_s$ and $\vec{\mathbf{v}}_i$ are velocity of solid phase and in the liquid of phase i respectively; ρ_s is the volume averaged density of porous solid, ρ_i , μ_i and K_i are the volume averaged density, viscosity, bulk modulus of fluid phase i respectively. α_s is the volume fraction of the solid, α_i is the volume fraction for the liquid phase i respectively; K is the intrinsic permeability of solid and K_{ri} is the relative permeability of liquid phase i. G_{fr} is the shear modulus of solid and K_{fr} is the frame or drained bulk modulus, p_1^* and p_2^* are the pressures of non-wetting liquid and wetting liquid.

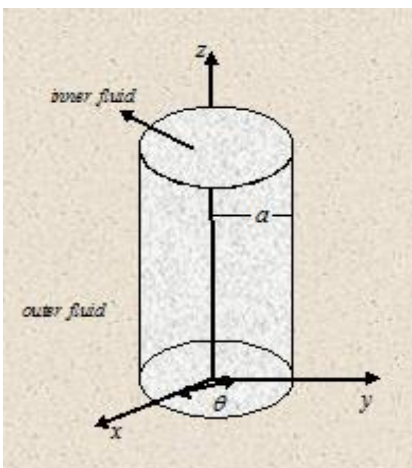

Fig.1
Liquid-filled thin cylinder surrounded by a liquid.

The stress in solid and the pressure in liquids are:

$$\begin{aligned}
\langle\tau_s\rangle &= (a_{11}\nabla\cdot\vec{\mathbf{u}}_s+a_{12}\nabla\cdot\vec{\mathbf{u}}_1+a_{13}\nabla\cdot\vec{\mathbf{u}}_2)I+G_{fr}(\nabla\vec{\mathbf{u}}_s+(\nabla\vec{\mathbf{u}}_s)^T-\frac{2}{3}\nabla\cdot\vec{\mathbf{u}}_s\,\mathbf{I}),\\
\langle\tau_1\rangle &= (a_{21}\nabla\cdot\vec{\mathbf{u}}_s+a_{22}\nabla\cdot\vec{\mathbf{u}}_1+a_{23}\nabla\cdot\vec{\mathbf{u}}_2)\mathbf{I},\\
\langle\tau_2\rangle &= (a_{31}\nabla\cdot\vec{\mathbf{u}}_s+a_{32}\nabla\cdot\vec{\mathbf{u}}_1+a_{33}\nabla\cdot\vec{\mathbf{u}}_2)\mathbf{I},
\end{aligned}\tag{2}$$

where I is unit tensor matrix. For axially symmetric vibrations, let $\vec{\mathbf{u}}_s=(u_s,0,w_s)$, $\vec{\mathbf{u}}_1=(u_1,0,w_1)$, and $\vec{\mathbf{u}}_2=(u_2,0,w_2)$ be the displacement vectors of solid, first liquid and second fluid, respectively; which are functions of r,z and t. The pertinent components are obtained from the field equations Eq. (1) and are given under:

$$
\begin{aligned}
u_s &= -(A_1 p_1 K_1(p_1 r) + A_2 p_2 K_1(p_2 r) + A_5 ik K_0(p_5 r)) e^{i(kz+\omega t)}, \\
w_s &= (A_1 ik K_0(p_1 r) + A_2 ik K_0(p_2 r) + A_5 (\frac{1}{r} K_0(p_5 r) - p_5 K_1(p_5 r)) e^{i(kz+\omega t)}, \\
u_1 &= -(A_1 \delta_1^2 p_1 K_1(p_1 r) + A_2 \delta_2^2 p_2 K_1(p_2 r) + A_3 \delta_3^2 p_3 K_1(p_3 r) + A_4 \delta_4^2 p_4 K_1(p_4 r) - m_{12} m_{22}^{-1} A_5 ik K_0(p_5 r)) e^{i(kz+\omega t)}, \\
w_1 &= (A_1 \delta_1^2 p_1 K_0(p_1 r) + A_2 \delta_2^2 p_2 K_0(p_2 r) + A_3 \delta_3^2 p_3 K_0(p_3 r) + A_4 \delta_4^2 p_4 K_0(p_4 r) - m_{12} m_{22}^{-1} A_5 (\frac{1}{r} K_0(p_5 r) - p_5 K_1(p_5 r)) e^{i(kz+\omega t)}, \\
u_2 &= -(A_3 p_3 K_1(p_3 r) + A_4 p_4 K_1(p_4 r) - m_{12} m_{22}^{-1} A_5 ik K_0(p_5 r)) e^{i(kz+\omega t)}, \\
w_2 &= (A_3 p_3 K_1(p_3 r) + A_4 p_4 K_1(p_4 r) - m_{12} m_{22}^{-1} A_5 (\frac{1}{r} K_0(p_5 r) - p_5 K_1(p_5 r)) e^{i(kz+\omega t)},
\end{aligned}
\tag{3}
$$

where ω is the frequency of the wave, *k* is the wavenumber, *t* is time, *i* is the complex unity, A_1, A_2, A_3, A_4 and A_5 are all constants, $K_n(x)$ is the modified Bessel functions of second kind of order *n*, and

$$
\begin{aligned}
& p_i = k(1-\xi_j^2)^{\frac{1}{2}}, \quad j = 1,2,3,4,5, \quad \xi_1^2 = c^2((a_{32}E_2 + a_{31}) + ((a_{32}E_2 + a_{31})^2 - 4\omega^2 m_{13} a_{32} E_1)^{\frac{1}{2}}) / 2a_{32}E_1, \\
& \xi_2^2 = c^2((a_{32}E_2 + a_{31}) - ((a_{32}E_2 + a_{31})^2 - 4\omega^2 m_{13} a_{32} E_1)^{\frac{1}{2}}) / 2a_{32}E_1, \quad \xi_3^2 = c^2(a_{32}E_4 + ((a_{32}E_4)^2 - 4\omega^2 m_{23} a_{32} E_3)^{\frac{1}{2}}) / 2a_{32}E_3, \\
& \xi_4^2 = c^2(a_{32}E_4 - ((a_{32}E_4)^2 - 4\omega^2 m_{23} a_{32} E_3)^{\frac{1}{2}}) / 2a_{32}E_3, \quad \xi_5^2 = c^2(m_{11} - \frac{m_{12}^2}{m_{22}} - \frac{m_{13}^2}{m_{23}}), \\
& E_1 = a_{22}(a_{11} + \frac{4}{3}G_{fr}) - a_{12}^2 / \omega^2 (m_{22}a_{12} - m_{12}a_{22}), \quad E_2 = a_{22}m_{11} - a_{12}m_{12} / (m_{22}a_{12} - m_{12}a_{22}), \\
& E_3 = a_{12}a_{13} - a_{23}a_{12} / \omega^2 (m_{22}a_{12} - m_{12}a_{22}), \quad E_4 = a_{22}m_{13} / (m_{22}a_{12} - m_{12}a_{22}), \\
& \delta_l^2 = E_1 \xi_l + E_2, \quad l = 1,2, \quad \delta_l^2 = E_3 \xi_l + E_4, \quad l = 3,4, \\
& m_{11} = \rho_s - \frac{i(c_1 + c_2)}{\omega}, \quad m_{12} = \frac{ic_1}{\omega}, \quad m_{12} = \frac{ic_2}{\omega}, \quad m_{22} = \rho_1 - \frac{ic_1}{\omega}, \quad m_{23} = \rho_2 - \frac{ic_2}{\omega}.
\end{aligned}
\tag{4}
$$

In Eq. (4), *c* is the phase velocity. Here poroelastic thin cylinder under consideration is loaded and surrounded by two different liquids. Therefore, we shall proceed to find the stresses of a liquid. For axially symmetric vibrations, the displacement potential function ϕ of a liquid satisfies the following equation:

$$
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{V_f^2}\frac{\partial^2 \phi}{\partial t^2}. \tag{5}
$$

The displacement components of inner liquid $\vec{\mathbf{u}}_{if} = (u_{if}, 0, w_{if})$ are obtained from the solution of Eq. (5) and the equations $u_{if} = \frac{\partial \phi}{\partial r}$ and $w_{if} = \frac{\partial \phi}{\partial z}$.

The displacement components are

$$
u_{if} = B_1 q_{if} I_1(q_{if} r) e^{i(kz+\omega t)}, w_{if} = B_1 ik I_0(q_{if} r) e^{i(kz+\omega t)}, \tag{6}
$$

where B_1 is constant, $I_n(x)$ is the modified Bessel function of first kind of order *n*, and $q_{if} = k(1 - \frac{\omega^2}{V_{if}^2})^{\frac{1}{2}}$. The inner liquid pressure is

$$
p_{if} = -\rho_{if} \frac{\partial^2 \phi}{\partial t^2} == \rho_{if} \omega^2 B_1 I_0(q_{if} r) e^{i(kz+\omega t)}. \tag{7}
$$

The displacement components of outer liquid $\vec{\mathbf{u}}_{of} = (u_{of},0,w_{of})$ are obtained on similar lines, which are given under:

$$u_{of} = B_2 q_{of} H_1^{(1)}(q_{of} r)e^{i(kz+\omega t)}, w_{of} = B_2 ikH_0^{(1)}(q_{of} r)e^{i(kz+\omega t)}, \tag{8}$$

where B_2 is constant, $H_n^{(1)}(x)$ is the Hankel function of first kind of order *n*, and $q_{of} = k(1-\frac{\omega^2}{V_{of}^2})^{\frac{1}{2}}$. The outer liquid pressure is

$$p_{of} = \rho_{of}\omega^2 B_2 J_0(q_{of} r)e^{i(kz+\omega t)}. \tag{9}$$

By substituting the displacement components from Eq. (3) into Eq. (2), the solid stresses and liquid pressures are obtained, which are

$$\begin{aligned}
&\tau_{rr}+\tau_1+\tau_2 = (A_1M_{11}(r)+A_2M_{12}(r)+A_3M_{13}(r)+A_4M_{14}(r)+A_5M_{15}(r))e^{i(kz+\omega t)},\\
&\tau_{rz} = (A_1M_{21}(r)+A_2M_{22}(r)+A_3M_{23}(r)+A_4M_{24}(r)+A_5M_{25}(r))e^{i(kz+\omega t)},\\
&\tau_1 = (A_1M_{31}(r)+A_2M_{32}(r)+A_3M_{33}(r)+A_4M_{34}(r))e^{i(kz+\omega t)},\\
&\tau_2 = (A_1M_{41}(r)+A_2M_{42}(r)+A_3M_{43}(r)+A_4M_{44}(r))e^{i(kz+\omega t)},
\end{aligned} \tag{10}$$

where,

$$\begin{aligned}
&M_{11}(r) = (2G_{fr}+(a_{11}-\frac{2}{3}G_{fr}+a_{21}+a_{31}+(a_{12}+a_{22}+a_{32})\delta_1^2)(p_1^2-k^2))K_0(p_1r)+\frac{2p_1}{r}G_{fr}K_1(p_1r),\\
&M_{13}(r) = (a_{13}+a_{23}+a_{33}+(a_{12}+a_{22}+a_{32})\delta_3^2)(p_3^2-k^2))K_0(p_3r), \quad M_{15}(r) = 2G_{fr}ikK_1(p_5r),\\
&M_{21}(r) = -2G_{fr}p_1ikK_1(p_1r), \quad M_{23}(r) = M_{24}(r) = 0, \quad M_{25}(r) = (k^2-p_5^2)K_0(p_5r),\\
&M_{31}(r) = (a_{21}+a_{22}\delta_1^2)(p_1^2-k^2))K_0(p_1r), \quad M_{33}(r) = (a_{23}+a_{22}\delta_3^2)(p_3^2-k^2))K_0(p_3r),\\
&M_{41}(r) = (a_{31}+a_{32}\delta_1^2)(p_1^2-k^2))K_0(p_1r), \quad M_{43}(r) = (a_{33}+a_{32}\delta_3^2)(p_3^2-k^2))K_0(p_3r),
\end{aligned}$$

$M_{12}(r), M_{14}(r)$ are similar expressions as $M_{11}(r), M_{13}(r)$ with p_1,δ_1^2 and p_3,δ_3^2 are replaced by p_2,δ_2^2 and p_4,δ_4^2, respectively.

$M_{22}(r)$ is similar expressions as $M_{21}(r)$ with p_1,δ_1^2 are replaced by p_2,δ_2^2, respectively.

$M_{32}(r), M_{34}(r)$ are similar expressions as $M_{31}(r), M_{33}(r)$ with p_1,δ_1^2 and p_3,δ_3^2 are replaced by p_2,δ_2^2 and p_4,δ_4^2, respectively.

$M_{42}(r), M_{44}(r)$ are similar expressions as $M_{41}(r), M_{43}(r)$ with p_1,δ_1^2 and p_3,δ_3^2 are replaced by p_2,δ_2^2 and p_4,δ_4^2, respectively.

3 BOUNDARY CONDITIONS AND FREQUENCY EQUATION

For perfect contact of solid and liquid at the interface, the displacements and stresses must be continuous. Thus the boundary conditions to be satisfied on the curved surface $r = a$ to be stress free are turned to be as follows:

$$\begin{aligned}
&\tau_{rr}+\tau_1+\tau_2 = -p_{if}, \quad \tau_{rr}+\tau_1+\tau_2 = -p_{of}, \quad \tau_{rz} = 0,\\
&\tau_1 = 0, \quad \tau_2 = 0, \quad u_s - u_{if} = 0, \quad u_s - u_{of} = 0.
\end{aligned} \tag{11}$$

Substitution of Eqs. (6)-(10) into Eq. (11) result in a system of seven homogeneous equations in seven unknowns $A_1, A_2, A_3, A_4, A_5, B_1$ and B_2. In order to obtain a non-trivial solution of this system, the coefficient matrix must be singular. Accordingly, the frequency equation is obtained, which is given by

$$\left|A_{ij}\right| = 0, \quad i, j = 1,2,3,...7, \tag{12}$$

where,

$$A_{1j} = M_{1j}, j = 1,2,3,4,5, \quad A_{16} = \rho_{if}\,\omega^2 I_1(q_{if}a), \quad A_{17} = 0, \quad A_{2j} = M_{2j}, j = 1,2,3,4,5, \quad A_{26} = 0, \quad A_{27} = \rho_{of}\,\omega^2 H_0^{(1)}(q_{if}a),$$
$$A_{3j} = M_{3j}, j = 1,2,3,4,5, \quad A_{36} = 0, \quad A_{37} = 0, \quad A_{4j} = M_{4j}, j = 1,2,3,4, \quad A_{45} = A_{46} = A_{47} = 0,$$
$$A_{5j} = M_{5j}, j = 1,2,3,4, \quad A_{55} = A_{56} = A_{57} = 0, \quad A_{61} = -p_1 K_1(p_1 a), \quad A_{62} = -p_2 K_1(p_2 a), \quad A_{63} = A_{64} = 0,$$
$$A_{65} = ikK_0(p_5 a), \quad A_{66} = -q_{if} I_1(q_{if}a), \quad A_{67} = 0, \quad A_{71} = A_{61}, \quad A_{72} = A_{62}, \quad A_{73} = A_{74} = 0,$$
$$A_{75}(r) = A_{65}, \quad A_{77} = -q_{of} H_1^1(q_{of}a).$$

3.1 Particular cases

In this subsection, two particular cases when ka is large are considered. The first case is solid saturated with two liquids and the other case is solid saturated with a single liquid. Both the cases result in the frequency equations of Rayleigh waves in a poroelastic half space.

3.1.1 Half space saturated with two liquids

When the wave length is very small or the wavenumber is large (i. e., when $ka \to \infty$); the frequency equation Eq. (12) is reduced to that of Rayleigh-type surface wave at the plane boundary of a poroelastic half-space saturated with two immiscible liquids. In this case, the asymptotic expressions of Bessel functions and modified Bessel functions [26] are employed and the pertinent frequency equation Eq. (12) is obtained, which is given under:

$$\left|B_{ij}\right| = 0, \quad i, j = 1,2,3,...7, \tag{13}$$

where,

$$B_{11} = 2G_{fr} + (a_{11} - \frac{2}{3}G_{fr} + a_{21} + a_{31} + (a_{12} + a_{22} + a_{32})\delta_1^2)\xi_1^2, \; B_{13}(r) = (a_{13} + a_{23} + a_{33} + (a_{12} + a_{22} + a_{32})\delta_3^2)\xi_3^2,$$
$$B_{15} = 2iG_{fr}\xi_5^2, \quad B_{16} = c^2\rho_{if}, \quad B_{17} = 0, B_{2j} = A_{1j}, j = 1,2,3,4,5, \quad B_{26} = 0, \quad B_{27} = \rho_{of}c^2,$$
$$B_{31} = -2iG_{fr}\xi_1^2, \quad B_{32} = -2iG_{fr}\xi_2^2, \quad B_{33} = B_{34} = 0, \quad B_{35} = G_{fr}\xi_5^2, \quad B_{36} = B_{37} = 0,$$
$$B_{41} = (a_{21} + a_{22}\delta_1^2)\xi_1^2, \quad B_{43} = (a_{23} + a_{22}\delta_3^2)\xi_3^2, \quad B_{45} = B_{46} = B_{47} = 0,$$
$$B_{51} = (a_{31} + a_{33}\delta_1^2)\xi_1^2, \quad B_{53} = (a_{31} + a_{33}\delta_3^2)\xi_3^2, \quad B_{55} = B_{56} = B_{57} = 0,$$
$$B_{61} = (1 - \xi_1^2)^{\frac{1}{2}}, \quad B_{62} = (1 - \xi_3^2)^{\frac{1}{2}}, \quad B_{63} = B_{64} = 0, \; B_{65} = i, \quad B_{66} = (1 - \frac{c^2}{V_{if}^2})^{\frac{1}{2}}, B_{67} = 0,$$
$$B_{7j} = B_{6j}\;(1 - \xi_1^2)^{\frac{1}{2}}, j = 1,2,3,4,5, \quad B_{76} = 0, \; B_{77} = (1 - \frac{c^2}{V_{0f}^2})^{\frac{1}{2}},$$

B_{12}, B_{14} are similar expressions as B_{11}, B_{13} with ξ_1, δ_1^2 and ξ_3, δ_3^2 are replaced by ξ_2, δ_2^2 and ξ_4, δ_4^2, respectively,
B_{42}, B_{44} are similar expressions as B_{41}, B_{43} with ξ_1, δ_1^2 and ξ_3, δ_3^2 are replaced by ξ_2, δ_2^2 and ξ_4, δ_4^2, respectively,
B_{52}, B_{54} are similar expressions as B_{51}, B_{53} with ξ_1, δ_1^2 and ξ_3, δ_3^2 are replaced by ξ_2, δ_2^2 and ξ_4, δ_4^2, respectively.

3.1.2 Half space saturated with a single liquid

When the presence of one liquid, out of the two immiscible liquids and outer fluid are neglected, so that $a_{12} = a_{22} = a_{32} = 0, \quad \rho_{of} = V_{of} = 0$. Then the frequency equation Eq. (12) reduces to

$$\left|C_{ij}\right| = 0, \quad i, j = 1,2,3,4, \tag{14}$$

where,

$$C_{11} = (2G_{fr} + (a_{11} - \frac{2}{3}G_{fr} + a_{12}\delta_1^2)(p_1^2 - k^2))K_0(p_1 r) + \frac{2p_1}{r}G_{fr}K_1(p_1 r),$$
$$C_{13} = A_{15}, \quad C_{14} = A_{16}, \quad C_{21} = A_{31}, \quad C_{22} = A_{22}, \quad C_{23} = A_{25}, \quad C_{24} = 0, \quad C_{31} = A_{51},$$
$$C_{32} = A_{52}, \quad C_{33} = C_{34} = 0, \quad C_{41} = A_{61}, \quad C_{42} = A_{62}, \quad C_{43} = 0, \quad C_{44} = A_{66}.$$

C_{12} are similar expressions as C_{11} with p_1, δ_1^2 and p_3, δ_3^2 are replaced by p_2, δ_2^2 and p_4, δ_4^2, respectively. The frequency equation Eq. (14) is analogous to the Eq. (3.8) of the paper [19]. when $ka \to \infty$, the Eq. (14) reduces to the frequency equation of Rayleigh wave in a fluid-filled poroelastic half space saturated with single liquid, which is given under

$$\begin{gathered}\left|D_{ij}\right| = 0, \quad i, j = 1,2,3,4, \\ D_{11} = B_{11}, \quad D_{12} = B_{12}, \quad D_{13} = B_{15}, \quad D_{14} = B_{16}, \quad D_{21} = B_{31}, \quad D_{22} = B_{22}, \quad D_{23} = B_{25}, \quad D_{24} = 0, \\ D_{31} = B_{51}, \quad D_{32} = B_{52}, \quad D_{33} = B_{34} = 0, \quad D_{41} = B_{61}, \quad D_{42} = B_{62}, \quad D_{43} = 0, \quad D_{44} = B_{66}.\end{gathered} \tag{15}$$

4 NUMERICAL RESULTS

For numerical work, we consider poroelastic thin cylinder saturated with non-viscous liquids. In this case, $\mu_j = 0$, which implies $c_j = 0, j = 1,2.$ Frequency equations are investigated for a cylinder made up of Columbia fine sandy loam saturated with air-water mixture [16]. The physical parameters for the above solid are given in Table 1. Oil is mostly available in sandstone reservoirs. Cylindrical casing pipes containing perforations are used to pump the oil to the surface. These casing pipes sometimes made of sandstone itself. The velocity of sound in water (V_{if}) and kerosene (V_{of}) are taken to be $1.432\times10^3 m/s$ and $1.3251\times10^3 m/s$, respectively. Density of water (ρ_{if}) and kerosene (ρ_{of}) are taken to be $1000 kg/m^3$ and $820.1 kg/m^3$, respectively. Employing these values in frequency equations, we obtain a implicit relation between phase velocity and wavenumber.

Table 1
Parameter values.

Material parameter	value
K_{fr}	$1.44\times10^9 N/m^2$
K_s	$35\times10^9 N/m^2$
K_1	$0.145\times10^6 N/m^2$
K_2	$2.25\times10^9 N/m^2$
G_{fr}	$1.02\times10^9 N/m^2$
ρ_s	$2650 kg/m^3$
ρ_1	$1.1 kg/m^3$
ρ_2	$997 kg/m^3$

Bisection method is used to solve the frequency equations and numerical process is performed in MATLAB. Numerical results are depicted graphically in Figs. 2-3. Fig. 2 depicts variation of phase velocity against wavenumber in the case of liquid -filled poroelastic thin cylinder saturated with two liquids surrounded by a liquid. From Fig. 2, it is seen that the phase velocity is periodic in nature. Fig. 3 depicts variation of phase velocity against wavenumber in the case of liquid-filled poroelastic thin cylinder saturated with a single liquid. From Fig. 3, it is observed that the phase velocity is periodic in nature. when $ka \to \infty$, liquid-filled thin cylinder immersed in a liquid reduces to half-space separating two liquids. The phase velocity of the Rayleigh mode at the half space is computed and are depicted in the figures. From figures, it is seen that the phase velocity of Rayleigh wave in the case of half space saturated with two liquids is more than that of half space saturated with a single liquid.

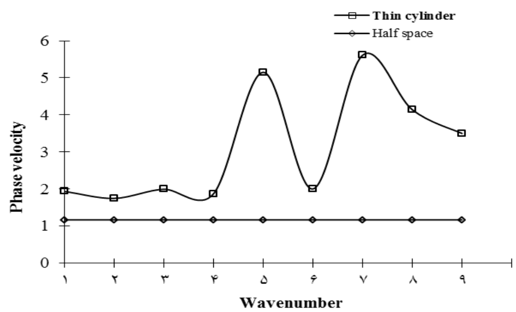

Fig.2
Variation of phase velocity against wavenumber in the case of liquid-filled poroelastic solid saturated with two liquids surrounded by a liquid.

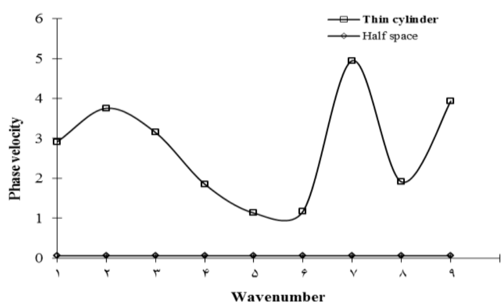

Fig.3
Variation of phase velocity against wavenumber in the case of liquid-filled poroelastic solid saturated with a single liquid.

5 CONCLUSIONS

Axially symmetric vibrations in a liquid-filled poroelastic thin cylinder saturated with two immiscible liquids surrounded by a liquid are investigated in the framework of Tuncay and Corapcioglu. Frequency equation in the case of poroelastic cylinder saturated with a single liquid is obtained as a particular case. In both the cases, Rayleigh waves in the poroelastic half space are investigated. Phase velocity against wavenumber is computed and depicted graphically. From numerical results, it is observed that the phase velocity of Rayleigh mode in the case of half space saturated with two liquids is more than that of half space saturated with a single liquid.

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