

Moving Three Collinear Griffith Cracks at Orthotropic Interface

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ABSTRACT

This work deals with the interaction of P-waves between a moving central crack and a pair of outer cracks situated at the interface of an orthotropic layer and an elastic half-space. Initially, we considered a two-dimensional elastic wave equation in orthotropic medium. The Fourier transform has been applied to convert the basic problem to solve the set of four integral equations. These set of integral equations have been solved to get the analytical expressions for the stress intensity factor (SIF) and crack opening displacements (COD) by using the finite Hilbert transform technique and Cooke's result. The main objective of this work is to investigate the dynamic stress intensity factors and crack opening displacement at the tips of the cracks. The aims of the study of these physical quantities (SIF, COD) is the prediction of possible arrest of the cracks within a certain range of crack velocity by monitoring applied load. SIF and COD have been depicted graphically for various types of orthotropic materials. We presented a parametric study to explore the influence of crack growing and propagation. This result is very much applicable in bridges, roads, and buildings fractures.

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Keywords: Moving Griffith crack; Orthotropic media; P-wave; Stress intensity factor; Crack opening displacement.

1 INTRODUCTION

THE problems of interaction of elastic waves by cracks or inclusions in layered media are of considerable importance given their application in engineering mechanics. Cracks or inclusions are present essentially in all structural materials, either as natural defects or as a result of fabrication processes. With the increased usage of macroscopically anisotropic construction materials such as fiber-reinforced composites, the study of diffraction of elastic waves by cracks or inclusions in composite materials has gained much importance. The study becomes more relevant if the cracks or inclusions are located at the interface of layered media. The stress singularity near the edge of the finite crack is important because of its practical application. Solutions to problems of moving cracks at the interface of dissimilar materials are important since they can assist in the understanding of how composite can be

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best constructed to arrest running cracks. The study of crack problems is extremely important for the sake of safety and security of structural components. The works of Atkinson [3] focused attention on cracks propagating along with the interface of dissimilar materials behavior. Chen [4] considered the problem of the dynamic response of a central crack in a finite elastic strip. The crack was assumed to appear suddenly when the strip is being stretched at its two ends. The problem was solved by Laplace and Fourier transform techniques. Itou [5] also studied the dynamic problem for an infinite elastic medium weakened by two coplanar Griffith cracks in which a self equilibrated system of pressure is varied harmonically with time. To solve this problem, the author has expanded the surface displacement in a series of functions which is automatically zero outside the cracks and has used the Schmidt method. Itou [6] also solved two different problems involving two finite cracks. Srivastava *et al.* [7] studied the diffraction of elastic waves by one or more cracks moving along the interface of two elastic media. The transient response of two cracks at the interface of a layered half-space was investigated by Kundu [9]. In the presence of such flaws has been solved by many authors like Lowengrub [2], Erdogan [12], Shbeeb *et al.* [16]. Das *et al.* [11] considered two coplanar Griffith cracks moving along the interface of two dissimilar elastic medium. In this problem, the distribution of stress and displacement due to the propagation of two parallel and co-planar Griffith cracks with constant velocity under anti-plane shear stress at the interface of two dissimilar elastic media are presented. Analytical studies of crack interaction problems have been made by Lowengrub [1], Rose [8], Carpinteri *et al.* [15], Das *et al.* [14] and many authors. The problem of obtaining dynamic stress intensity factor for a semi-infinite crack in orthotropic materials with concentrated shear impact loads was investigated by Wang *et al.* [17]. Li [18] obtained a closed-form solution of stress intensity factor for mode-III interface crack between two bonded dissimilar elastic layers. E. Lira-Vergara and C. Rubio-Gonzalez [19] found the dynamic stress intensity factor of interfacial finite cracks in orthotropic materials. Matbuly [20] investigated the dynamical problem and obtained the analytical expression of the stress intensity factor. An effective approach for finding stress intensity factor in an elastic layered composite containing three moving collinear Griffith cracks under antiplane shear stress at the interface of an orthotropic layer and a half-space has been performed by Das [21]. The solution to determine the dynamic stress intensity factors for three collinear cracks in an orthotropic plate subjected to time-harmonic disturbance was given by Itou [22]. Palas *et al.* [23] considered the problem of interface crack at orthotropic media. Basak *et al.* [24] solved a semi-infinite moving crack at the orthotropic strip. Basu *et al.* [25] solved on the effect of a sudden impact of a torsional load on a penny-shaped crack sandwiched between two elastic layers embedded in an elastic medium. S. Das *et al.* [26] has been solved the interaction between Griffith cracks in a sandwiched orthotropic layer. Bagheri *et al.* [27] introduced the problem analytical solution of multiple moving cracks in functionally graded piezoelectric strip. Multiple interacting arbitrary shaped cracks in an FGM plane has been solved by Monfared *et al.* [28]. A. Habib *et al.* [30] suggested Several embedded cracks in a functionally graded piezoelectric strip under dynamic loading. The mixed mode analysis of arbitrary configuration of cracks in an orthotropic FGM strip using the distributed edge dislocations is considered by monfared *et al.* [31]. Dislocation technique to obtain the dynamic stress intensity factors for multiple cracks in a half-plane under impact load is found by A. Hijazi *et al.* [32]. H. Ershad *et al.* [33] studied the problem transient response of cracked nonhomogeneous substrate with piezoelectric coating by dislocation method and finally solved. R. Bhageri *et al.* [34] is proposed fracture analysis in an imperfect FGM orthotropic strip bonded between two magneto-electro-elastic layers.

Although a fair amount of research has been done on the interaction of waves due to cracks between dissimilar media, most of the problems considered so far have been either the interaction of shear waves or the diffraction due to finite cracks or interaction in infinite cracked media. As per the best of our knowledge the problem involving moving collinear cracks between two dissimilar media due to normally applied displacements at the boundaries has not been considered earlier.

In this work, we have considered the interaction of P-waves in a bonded dissimilar orthotropic strip and a semi-infinite medium containing three collinear Griffith cracks at the interface with the action of constant normal displacement applied at the boundary. The Fourier transform is used to reduce the problem to a system of dual integral equations. These sets of equations have been solved by using the finite Hilbert transform technique and Cooke's result. An iterative procedure is adopted to obtain the solution of the problem. This procedure leads to obtain the analytical expressions of the stress intensity factor (SIF) and crack opening displacement (COD). Finally the effects of material constants, the width of the upper-medium and crack length on stress intensity factor and crack opening displacement have been shown by the graphs.

2 FORMULATION OF THE PROBLEM

Let us consider three collinear Griffith cracks moving with constant velocity c along with the interface of an orthotropic layer of width h_1 and an orthotropic half-space. Let x_1, y_1, z_1 be the cartesian co-ordinate axes which are the axes of symmetry of the orthotropic materials. The cracks are assumed to occupy the region $|x_1| \leq l_1, l_2 < |x_1| \leq l, -\infty < z_1 < \infty, y_1 = 0$. Normalizing all the lengths by l and setting $\frac{x_1}{l} = x, \frac{y_1}{l} = y, \frac{z_1}{l} = a, \frac{l_2}{l} = b, \frac{h_1}{l} = h$ the new location of the crack at the interface becomes $|x| \leq a, b \leq |x| \leq 1, -\infty < z < \infty, y = 0$. The displacements $U_x^{(k)}(x, y, t)$ and $U_y^{(k)}(x, y, t); k = 1, 2$ along the X and Y axes respectively are given by the following equations

$$C_{11}^{(k)} \frac{\partial^2 U_x^{(k)}}{\partial X^2} + \mu_{12}^{(k)} \frac{\partial^2 U_x^{(k)}}{\partial Y^2} + (C_{12}^{(k)} + \mu_{12}^{(k)}) \frac{\partial^2 U_y^{(k)}}{\partial X \partial Y} = \rho^{(k)} \frac{\partial^2 U_x^{(k)}}{\partial t^2} \tag{1}$$

$$\mu_{12}^{(k)} \frac{\partial^2 U_y^{(k)}}{\partial X^2} + C_{22}^{(k)} \frac{\partial^2 U_y^{(k)}}{\partial Y^2} + (C_{12}^{(k)} + \mu_{12}^{(k)}) \frac{\partial^2 U_x^{(k)}}{\partial X \partial Y} = \rho^{(k)} \frac{\partial^2 U_y^{(k)}}{\partial t^2} \tag{2}$$

where $C_{11}^{(k)}, C_{22}^{(k)}, C_{12}^{(k)}$ are material constants, $\mu_{12}^{(k)}$ the shear modulus. Here the superscripts $k = 1, 2$ represent upper material and lower material respectively. Now we introduce the Galilean transformation $x = X - ct, y = Y, t = t$ so that the Eqs. (1) and (2) become

$$(C_{11}^{(k)} - \rho^{(k)} c^2) \frac{\partial^2 u_x^{(k)}}{\partial x^2} + \mu_{12}^{(k)} \frac{\partial^2 u_x^{(k)}}{\partial y^2} + (C_{12}^{(k)} + \mu_{12}^{(k)}) \frac{\partial^2 u_y^{(k)}}{\partial x \partial y} = \rho^{(k)} \frac{\partial^2 u_x^{(k)}}{\partial t^2} \tag{3}$$

$$(\mu_{12}^{(k)} - \rho^{(k)} c^2) \frac{\partial^2 u_y^{(k)}}{\partial x^2} + C_{22}^{(k)} \frac{\partial^2 u_y^{(k)}}{\partial y^2} + (C_{12}^{(k)} + \mu_{12}^{(k)}) \frac{\partial^2 u_x^{(k)}}{\partial x \partial y} = \rho^{(k)} \frac{\partial^2 u_y^{(k)}}{\partial t^2} \tag{4}$$

where $u_x^{(k)}(x, y) = U_x^{(k)}(X, Y, t)$ and $u_y^{(k)}(x, y) = U_y^{(k)}(X, Y, t)$. The stresses and displacements are related by the following equations

$$\sigma_{yy}^{(k)} / \mu_{12}^{(k)} = C_{12}^{(k)} \frac{\partial u_x^{(k)}}{\partial x} + C_{22}^{(k)} \frac{\partial u_y^{(k)}}{\partial y} \tag{5}$$

$$\sigma_{xy}^{(k)} / \mu_{12}^{(k)} = \frac{\partial u_x^{(k)}}{\partial y} + \frac{\partial u_y^{(k)}}{\partial x} \tag{6}$$

$$\sigma_{xx}^{(k)} / \mu_{12}^{(k)} = C_{11}^{(k)} \frac{\partial u_x^{(k)}}{\partial x} + C_{12}^{(k)} \frac{\partial u_y^{(k)}}{\partial y} \tag{7}$$

At the plane $y = 0$ along which the crack lies, the boundary conditions to be specified as (Fig. 1)

$$\sigma_{yy}^{(1)}(x, 0^+) = \sigma_{yy}^{(2)}(x, 0^-) = 0, |x| < a, b < |x| < 1 \tag{8}$$

$$u_y^{(1)}(x, 0^+) = u_y^{(2)}(x, 0^-), a \leq |x| \leq b, |x| \geq 1 \quad (9)$$

$$u_x^{(1)}(x, 0^+) = u_x^{(2)}(x, 0^-), a \leq |x| \leq b, |x| \geq 1 \quad (10)$$

$$\sigma_{yy}^{(1)}(x, 0^+) = \sigma_{yy}^{(2)}(x, 0^-), |x| \geq 1, a \leq |x| \leq b \quad (11)$$

$$\sigma_{xy}^{(1)}(x, 0^+) = \sigma_{xy}^{(2)}(x, 0^-), -\infty < x < \infty \quad (12)$$

The displacement component u_x in x -direction is negligible in comparison to the displacement component u_y in y -direction in the case of mode-I crack. So we consider that the displacement on the crack faces $u_x^{(1)}(x, 0^+)$ and $u_x^{(2)}(x, 0^-)$ are identical. So in place of the Eq.(10) we consider

$$u_x^{(1)}(x, 0^+) = u_x^{(2)}(x, 0^-), -\infty < x < \infty.$$

Now the condition at the boundary $y = h$

$$u_y^{(1)}(x, h) = v_0^{(1)}, -\infty < x < \infty \quad (13)$$

$$u_x^{(1)}(x, h) = 0, -\infty < x < \infty \quad (14)$$

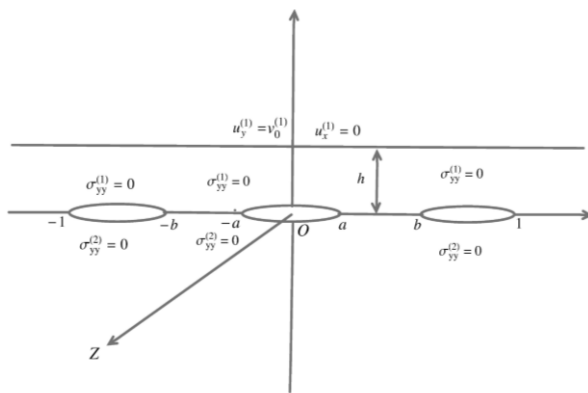


Fig.1
Geometry of the original problem.

where $v_0^{(1)}$ is constant normal displacement applied at the boundary. The above system of boundary conditions is not suitable to solve the problem. We consider a constant homogeneous normal load σ_0 superimposed on the crack surface so that the displacement on the boundary becomes zero. Then, by a trivial superposition, we may arrive at the original problem.

The new boundary conditions that the crack is subjected to normal stresses of magnitude σ_0 are (Fig. 2)

$$\sigma_{yy}^{(1)}(x, 0^+) = \sigma_{yy}^{(2)}(x, 0^-) = \sigma_0, |x| < a, b < |x| < 1 \quad (15)$$

$$u_y^{(1)}(x, 0^+) = u_y^{(2)}(x, 0^-), a \leq |x| \leq b, |x| \geq 1 \quad (16)$$

$$u_x^{(1)}(x, 0^+) = u_x^{(2)}(x, 0^-), -\infty < x < \infty \tag{17}$$

$$\sigma_{yy}^{(1)}(x, 0^+) = \sigma_{yy}^{(2)}(x, 0^-), |x| \geq 1, a \leq |x| \leq b \tag{18}$$

$$\sigma_{xy}^{(1)}(x, 0^+) = \sigma_{xy}^{(2)}(x, 0^-), -\infty < x < \infty \tag{19}$$

$$u_y^{(1)}(x, h) = 0, -\infty < x < \infty \tag{20}$$

$$u_x^{(1)}(x, h) = 0, -\infty < x < \infty \tag{21}$$

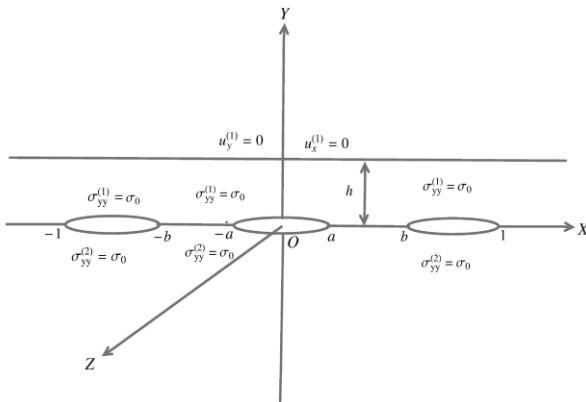


Fig.2
Geometry of the transformed problem.

It can be easily shown that the above two problems are equivalent to some particular value σ_0 . For the problem of plane stress having this set of boundary conditions, the suitable value of σ_0 has been obtained (Georgiadis and

Papadopoulos, [10]) $-\frac{(C_{22} + \nu_{21}C_{12})v_0^{(1)}}{h}$. The solutions of Eqs. (3) and (4) are

$$u_x^{(1)}(x, y) = \frac{2}{\pi} \int_0^\infty \left[A_1(\zeta) e^{\delta_1^{(1)}y} + A_2(\zeta) e^{-\delta_1^{(1)}y} + A_3(\zeta) e^{\delta_2^{(1)}y} + A_4(\zeta) e^{-\delta_2^{(1)}y} \right] \sin(\zeta x) d\zeta, 0 \leq y \leq h \tag{22}$$

$$u_x^{(2)}(x, y) = \frac{2}{\pi} \int_0^\infty \left[A_1'(\zeta) e^{\delta_1^{(2)}y} + A_2'(\zeta) e^{\delta_2^{(2)}y} \right] \sin(\zeta x) d\zeta, -\infty < y \leq 0 \tag{23}$$

$$u_y^{(1)}(x, y) = \frac{2}{\pi} \int_0^\infty \frac{1}{\zeta} \left[\omega_1 A_1(\zeta) e^{\delta_1^{(1)}y} + \omega_2 A_2(\zeta) e^{-\delta_1^{(1)}y} + \omega_3 A_3(\zeta) e^{-\delta_2^{(1)}y} + \omega_4 A_4(\zeta) e^{-\delta_2^{(1)}y} \right] \cos(\zeta x) d\zeta, 0 \leq y \leq h \tag{24}$$

$$u_y^{(2)}(x, y) = \frac{2}{\pi} \int_0^\infty \frac{1}{\zeta} \left[\omega_1' A_1'(\zeta) e^{\delta_1^{(2)}y} + \omega_3' A_2'(\zeta) e^{\delta_2^{(2)}y} \right] \cos(\zeta x) d\zeta, -\infty < y \leq 0 \tag{25}$$

Eqs. (22-25) contain a total of six unknown functions $A_1, A_2, A_3, A_4, A_1', A_2'$ of new transformed variable ζ alone and

$$\omega_1 = \frac{\delta_1^{(1)2} \mu_{12}^{(1)} - \zeta^2 (C_{11}^{(1)} - \rho^{(1)} c^2)}{\delta_1^{(1)} (\mu_{12}^{(1)} + C_{12}^{(1)})}, \quad \omega_2 = \frac{\zeta^2 (C_{11}^{(1)} - \rho^{(1)} c^2) - (\delta_1^{(1)})^2 \mu_{12}^{(1)}}{\delta_1^{(1)} (\mu_{12}^{(1)} + C_{12}^{(1)})}$$

$$\omega_3 = \frac{(\delta_2^{(1)})^2 \mu_{12}^{(1)} - \zeta^2 (C_{11}^{(1)} - \rho^{(1)} c^2)}{\delta_2^{(1)} (\mu_{12}^{(1)} + C_{12}^{(1)})}, \quad \omega_4 = \frac{\zeta^2 (C_{11}^{(1)} - \rho^{(1)} c^2) - (\delta_2^{(1)})^2 \mu_{12}^{(1)}}{\delta_2^{(1)} (\mu_{12}^{(1)} + C_{12}^{(k)})}$$

$$\omega_1 = \frac{(\delta_1^{(2)})^2 \mu_{12}^{(2)} - \zeta^2 (C_{11}^{(2)} - \rho^{(2)} c^2)}{\delta_1^{(2)} (\mu_{12}^{(2)} + C_{12}^{(2)})}, \quad \omega_3 = \frac{(\delta_2^{(2)})^2 \mu_{12}^{(2)} - \zeta^2 (C_{11}^{(2)} - \rho^{(2)} c^2)}{\delta_2^{(2)} (\mu_{12}^{(2)} + C_{12}^{(2)})}$$

$(\delta_1^{(k)})^2, (\delta_2^{(k)})^2$ are the roots of the equation

$$C_{22}^{(k)} \delta^4 + \{ (C_{12}^{(k)})^2 + 2C_{12}^{(k)} - C_{11}^{(k)} C_{22}^{(k)} \} \zeta^2 + (1 + C_{22}^{(k)}) \rho^{(k)} c^2 \delta^2 + (C_{11}^{(k)} \zeta^2 - \rho^{(k)} c^2) (\zeta^2 - \rho^{(k)} c^2) = 0, k = 1, 2. \quad (26)$$

The expressions for stresses now become

$$\sigma_{xy}^{(1)}(x, y) = \frac{2\mu_{12}^{(1)}}{\pi} \left[\int_0^\infty \left(A_1 \delta_1^{(1)} e^{\delta_1^{(1)} y} - A_2 \delta_1^{(1)} e^{-\delta_1^{(1)} y} + A_3 \delta_2^{(1)} e^{\delta_2^{(1)} y} - A_4 \delta_2^{(1)} e^{-\delta_2^{(1)} y} \right) \sin(\zeta x) d\zeta \right. \\ \left. - \int_0^\infty \left(\omega_1 A_1 e^{\delta_1^{(1)} y} + \omega_2 A_2 e^{-\delta_1^{(1)} y} + \omega_3 A_3 e^{\delta_2^{(1)} y} + \omega_4 A_4 e^{-\delta_2^{(1)} y} \right) \sin(\zeta x) d\zeta \right] \quad (27)$$

$$\sigma_{xy}^{(2)}(x, y) = \frac{2\mu_{12}^{(2)}}{\pi} \left[\int_0^\infty \left(A_1 \delta_1^{(2)} e^{\delta_1^{(2)} y} + A_2 \delta_2^{(2)} e^{\delta_2^{(2)} y} \right) \sin(\zeta x) d\zeta \right. \\ \left. - \int_0^\infty \zeta \left(\omega_1 A_1 e^{\delta_1^{(2)} y} + \omega_3 A_2 e^{\delta_2^{(2)} y} \right) \sin(\zeta x) d\zeta \right] \quad (28)$$

$$\sigma_{yy}^{(1)}(x, y) = \frac{2\mu_{12}^{(1)}}{\pi} \left[C_{12}^{(1)} \int_0^\infty \zeta \left(A_1 e^{\delta_1^{(1)} y} + A_2 e^{-\delta_1^{(1)} y} + A_3 e^{\delta_2^{(1)} y} + A_4 e^{-\delta_2^{(1)} y} \right) \cos(\zeta x) d\zeta \right. \\ \left. + C_{22}^{(1)} \int_0^\infty \frac{1}{\zeta} \left(\omega_1 A_1 \delta_1^{(1)} e^{\delta_1^{(1)} y} - \omega_2 A_2 \delta_1^{(1)} e^{-\delta_1^{(1)} y} + \omega_3 A_3 \delta_2^{(1)} e^{\delta_2^{(1)} y} - \omega_4 A_4 e^{-\delta_2^{(1)} y} \right) \cos(\zeta x) d\zeta \right] \quad (29)$$

$$\sigma_{yy}^{(2)}(x, y) = \frac{2\mu_{12}^{(2)}}{\pi} \left[C_{12}^{(2)} \int_0^\infty \zeta \left(A_1 e^{\delta_1^{(2)} y} + A_2 e^{\delta_2^{(2)} y} \right) \cos(\zeta x) d\zeta \right. \\ \left. + C_{22}^{(2)} \int_0^\infty \left(\omega_1 A_1 \delta_1^{(2)} e^{\delta_1^{(2)} y} + \omega_3 A_2 \delta_2^{(2)} e^{\delta_2^{(2)} y} \right) \cos(\zeta x) d\zeta \right] \quad (30)$$

3 DERIVATION OF THE INTEGRAL EQUATION

Applying the boundary condition $\sigma_{xy}^{(1)}(x, 0^+) = \sigma_{xy}^{(2)}(x, 0^-)$ we obtain the following relation

$$A_1 = a_{11} A_2 - a_{22} A_3 + a_{33} A_4 + a_{44} A_1 + a_{55} A_2, \quad (31)$$

where $a_{11} = \frac{\delta_1 + \omega_2}{\delta_1 - \omega_1}, a_{22} = \frac{\delta_2 - \omega_3}{\delta_1 - \omega_1}, a_{33} = \frac{\delta_2 + \omega_4}{\delta_1 - \omega_1}$.

Using boundary condition $\sigma_{yy}^{(1)}(x, 0^+) = \sigma_{yy}^{(2)}(x, 0^-)$ we get the following relation:

$$A_2 = b_{11}A_3 + b_{22}A_4 + b_{33}A_1 + b_{44}A_2, \tag{32}$$

$$\begin{aligned} \text{where } b_{11} &= \frac{R_{11} - P_{11}a_{22}}{P_{11}a_{11} + Q_{11}}, \quad b_{22} = \frac{S_{11} + P_{11}a_{33}}{P_{11}a_{11} + Q_{11}}, \quad b_{33} = \frac{P_{11}a_{44} - T_{11}}{P_{11}a_{11} + Q_{11}}, \quad b_{44} = \frac{P_{11}a_{55} - T_{22}}{P_{11}a_{11} + Q_{11}}, \quad P_{11} = \frac{\mu_{12}^{(1)}(C_{12}^{(1)}\zeta + C_{22}^{(1)}\omega_1\delta_1^{(1)})}{\zeta}, \\ Q_{11} &= \frac{\mu_{12}^{(1)}(C_{12}^{(1)}\zeta - C_{22}^{(1)}\omega_2\delta_1^{(1)})}{\zeta}, \quad R_{11} = \frac{\mu_{12}^{(1)}(C_{12}^{(1)}\zeta + C_{22}^{(1)}\omega_3\delta_2^{(1)})}{\zeta}, \quad S_{11} = \frac{\mu_{12}^{(1)}(C_{12}^{(1)}\zeta - C_{22}^{(1)}\omega_4\delta_2^{(1)})}{\zeta}, \\ T_{11} &= \frac{\mu_{12}^{(2)}(C_{12}^{(2)}\zeta + C_{22}^{(2)}\omega_1\delta_1^{(2)})}{\zeta}, \quad T_{22} = \frac{\mu_{12}^{(2)}(C_{12}^{(2)}\zeta + C_{22}^{(2)}\omega_3\delta_2^{(2)})}{\zeta} \end{aligned}$$

Next to the boundary condition (21) yields

$$A_3 = c_{11}A_4 + c_{22}A_1 + c_{33}A_2, \tag{33}$$

where

$$\begin{aligned} c_{11} &= \frac{\left[-b_{22}a_{11}e^{(\delta_1^{(1)} - \delta_2^{(1)})h} - b_{22}e^{-2\delta_1^{(1)}h} - e^{-(\delta_1^{(1)} + \delta_2^{(1)})h} - a_{33}e^{(\delta_1^{(1)} - \delta_2^{(1)})h} \right]}{\left[1 + b_{11}a_{11}e^{(\delta_1^{(1)} - \delta_2^{(1)})h} + b_{11}e^{-2\delta_1^{(1)}h} - a_{22}e^{(\delta_1^{(1)} - \delta_2^{(1)})h} \right]} \\ c_{22} &= \frac{\left[-a_{11}b_{33}e^{(\delta_1^{(2)} - \delta_2^{(2)})h} - b_{33}e^{-2\delta_2^{(1)}h} - a_{44}e^{(\delta_1^{(2)} - \delta_2^{(2)})h} \right]}{\left[1 + b_{11}a_{11}e^{(\delta_1^{(1)} - \delta_2^{(1)})h} + b_{11}e^{-2\delta_1^{(1)}h} - a_{22}e^{(\delta_1^{(1)} - \delta_2^{(1)})h} \right]} \\ c_{33} &= \frac{\left[-b_{44}a_{11}e^{(\delta_1^{(2)} - \delta_2^{(2)})h} - b_{44}e^{-2\delta_1^{(1)}h} - a_{55}e^{(\delta_1^{(1)} - \delta_2^{(2)})h} \right]}{\left[1 + b_{11}a_{11}e^{(\delta_1^{(1)} - \delta_2^{(1)})h} + b_{11}e^{-2\delta_1^{(1)}h} - a_{22}e^{(\delta_1^{(1)} - \delta_2^{(1)})h} \right]}. \end{aligned}$$

On utilizing the boundary condition (17) we obtain

$$A_4 = d_{11}A_1 + d_{22}A_2, \tag{34}$$

where

$$\begin{aligned} d_{11} &= \frac{\left[c_{22}b_{11}a_{11} - a_{11}b_{33} + c_{22}m_{11} + c_{22}b_{11} + a_{44} - b_{33} - c_{22} - 1 \right]}{\left[b_{11}c_{11}a_{11} - a_{11}b_{22} + c_{11}m_{11} + c_{11}b_{11} + a_{33} - b_{22} - c_{11} - 1 \right]} \\ d_{22} &= \frac{\left[c_{33}b_{11}a_{11} - b_{44}a_{11} + a_{55} - b_{44} - c_{33} - 1 \right]}{\left[b_{11}c_{11}a_{11} - a_{11}b_{22} + c_{11}m_{11} + c_{11}b_{11} + a_{33} - b_{22} - c_{11} - 1 \right]}, \quad m_{11} = \frac{\delta_2^{(2)} - \omega_3'}{\delta_1^{(2)} - \omega_1'} \end{aligned}$$

Finally using $u_y^{(1)}(x, h) = 0$, we get the following relation

$$A_1' = \Delta A_2' \quad (35)$$

where $\Delta = \frac{\Delta_2}{\Delta_1}$

The mixed boundary conditions in Eqs.(15) and (16) lead to a pair of dual integral equations

$$\int_0^\infty \zeta [1 + H(\zeta)] G(\zeta) \cos(\zeta x) d\zeta = -\frac{\sigma_0 \pi}{2\mu_1^{(1)}} \quad (36)$$

$$\int_0^\infty G(\zeta) \cos(\zeta x) d\zeta = 0 \quad (37)$$

where

$$G(\zeta) = \frac{[\omega_1 e_{44} + \omega_2 e_{33} + \omega_4 e_{11} - \omega_1' \Delta - \omega_3']}{\zeta} A_2'(\zeta)$$

$$H(\zeta) = \frac{[(\zeta^2 + \omega_1 \delta_1^{(1)})e_{44} + (\zeta^2 - \omega_2 \delta_1^{(1)})e_{33} + (\zeta^2 + \omega_3 \delta_2^{(1)})e_{22} + (\zeta^2 - \omega_4 \delta_2^{(1)})e_{11}]}{\zeta [\omega_1 e_{44} + \omega_2 e_{33} + \omega_3 e_{22} + \omega_4 e_{11} - \omega_1 \Delta - \omega_3]}$$

$$e_{11} = \frac{[a_{11} b_{11} c_{11} (d_{22} + d_{11} \Delta) + a_{11} b_{11} e_{22} \Delta + a_{11} b_{11} e_{33} + a_{11} b_{22} (d_{22} + d_{11} \Delta) + a_{11} b_{33} \Delta + a_{11} c_{33} - a_{22} c_{11} (d_{22} + d_{11} \Delta) + a_{22} c_{22} \Delta + a_{22} c_{33} + a_{33} (d_{22} + d_{11} \Delta) + a_{44} \Delta + a_{55}]}{}$$

$$e_{22} = b_{11} c_{11} (d_{22} + d_{11} \Delta) + b_{11} c_{22} \Delta + b_{11} c_{33} + b_{22} (d_{22} + d_{11} \Delta) + b_{33} \Delta + c_{33}$$

$$e_{33} = c_{11} (d_{22} + d_{11} \Delta) + c_{22} \Delta + c_{33}, \quad e_{44} = (d_{22} + d_{11} \Delta)$$

4 SOLUTION PROCEDURE

We consider the solution of the integral Eqs.(36) and (37) in the form

$$G(\zeta) = \frac{1}{\zeta} \int_0^a g(t) \sin(\zeta t) dt + \frac{1}{\zeta} \int_b^1 f(w^2) \sin(\zeta w) dw \quad (38)$$

where $g(t)$ and $f(w^2)$ are the unknown functions to be determined. By putting the value of $G(\zeta)$ and using the result

$$\int_0^\infty \frac{\sin(\zeta t) \cos(\zeta x)}{\zeta} d\zeta = \begin{cases} \frac{\pi}{2} & , t > x \\ 0 & , x > t \end{cases}$$

In Eq. (37) it can be shown that the equation is satisfied with the condition

$$\int_b^1 f(w^2) dw = 0 \quad (39)$$

Further substituting the value of $G(\zeta)$, Eq. (36) and using the above results

$$\int_0^\infty \frac{\sin(t\zeta)\sin(x\zeta)}{\zeta} d\zeta = \frac{1}{2} \log \left| \frac{t+x}{t-x} \right| \tag{40}$$

and

$$\frac{\sin(\zeta t)\sin(\zeta x)}{\zeta^2} = \int_0^x \int_0^t \frac{yzJ_0(\zeta y)J_0(\zeta z)}{\sqrt{(t^2-y^2)(x^2-z^2)}} dydz \tag{41}$$

We obtain the following integrodifferential Equations.

$$\begin{aligned} \frac{d}{dx} \int_0^a g(t) \log \left| \frac{t+x}{t-x} \right| dt + \frac{d}{dx} \int_b^1 f(w^2) \log \left| \frac{w+x}{w-x} \right| dw = 2[\tau_1 - \\ \frac{d}{dx} \int_0^a g(t) dt \int_0^x \int_0^t \frac{yzL(y,z)}{\sqrt{(t^2-y^2)(x^2-z^2)}} dydz - \frac{d}{dx} \int_b^1 f(w^2) dw \int_0^x \int_0^w \frac{yzL(y,z)}{\sqrt{(t^2-y^2)(x^2-z^2)}} dydz] \end{aligned} \tag{42}$$

where

$$L(y,z) = \int_0^\infty \zeta H(\zeta) J_0(\zeta y) J_0(\zeta z) d\zeta \tag{43}$$

It is to be noted that $L(y,z)$ represented by the semi-infinite integral given by the Eq.(43) has a slow rate of convergence. Applying a contour integration technique the semi-infinite integral has therefore been converted to the finite integrals following Mandal *et al.* [13, 35] as:

$$\begin{aligned} L(y,z) = -i \rho^{(1)} c^2 \left[\int_0^\gamma \left[\frac{C_{22}^{(1)} \left(\bar{\omega}_1 \bar{\delta}_1^{(1)} R_{11} + \bar{\omega}_2 \bar{\delta}_2^{(1)} T_{11} - \bar{\omega}_1 \bar{\delta}_1^{(1)} T_{11} - \bar{\omega}_2 \bar{\delta}_2^{(1)} Q_{11} \right)}{\pi \left\{ \bar{\omega}_1 (S_{11}\Delta - R_{11} + 1) + \bar{\omega}_2 (Q_{11}\Delta - T_{11} - \Delta) \right\}} + \right. \right. \\ \left. \frac{C_{22}^{(1)} \left(\bar{\omega}_3 \bar{\delta}_1^{(1)} R_{11} + \bar{\omega}_4 \bar{\delta}_2^{(1)} T_{11} - \bar{\omega}_3 \bar{\delta}_1^{(1)} T_{11} - \bar{\omega}_4 \bar{\delta}_2^{(1)} Q_{11} \right)}{\pi \left\{ \bar{\omega}_3 (S_{11}\Delta - R_{11} + 1) + \bar{\omega}_4 (Q_{11}\Delta - T_{11} - \Delta) \right\}} \right] J_0(\rho^{(1)} c^2 \eta y) H_0^{(1)}(\rho^{(1)} c^2 \eta z) d\eta \tag{44} \\ - \int_\gamma^1 \left[\frac{C_{22}^{(1)} \left(\hat{\omega}_2 \hat{\delta}_2^{(1)} T_{11} - \hat{\omega}_2 \hat{\delta}_2^{(1)} Q_{11} \right)}{\pi \left\{ \hat{\omega}_1 (R_{11} - S_{11}\Delta) - \hat{\omega}_2 (1 + T_{11} + Q_{11}\Delta) \right\}} + \right. \\ \left. \frac{C_{22}^{(1)} \left(\hat{\omega}_4 \hat{\delta}_2^{(1)} T_{11} - \hat{\omega}_4 \hat{\delta}_2^{(1)} Q_{11} \right)}{\pi \left\{ \hat{\omega}_3 (R_{11} - S_{11}\Delta) - \hat{\omega}_4 (1 + T_{11} + Q_{11}\Delta) \right\}} \right] J_0(\rho^{(1)} c^2 \eta y) H_0^{(1)}(\rho^{(1)} c^2 \eta z) d\eta \end{aligned}$$

where

$$\gamma = \frac{1}{\sqrt{C_{11}^{(1)}}}, \quad \zeta = \rho^{(1)} c^2 \eta, \quad \bar{\delta}_1^{(1)} = \left[\frac{1}{2} \left\{ r_1 - \sqrt{r_1^2 - 4\bar{r}_2} \right\} \right]^{\frac{1}{2}}, \quad \bar{\delta}_2^{(1)} = \left[\frac{1}{2} \left\{ r_1 + \sqrt{r_1^2 - 4\bar{r}_2} \right\} \right]^{\frac{1}{2}},$$

$$\begin{aligned} \hat{\delta}_1^{(1)} &= \left[\frac{1}{2} \left\{ -r_1 + \sqrt{r_1^2 + 4r_2} \right\} \right]^{\frac{1}{2}}, \quad \hat{\delta}_2^{(1)} = \left[\frac{1}{2} \left\{ r_1 + \sqrt{r_1^2 + 4r_2} \right\} \right]^{\frac{1}{2}}, \\ r_1 &= \frac{1}{C_{22}^{(1)}} \left[\left(C_{12}^2 + 2C_{12}^{(1)} - C_{11}^{(1)} C_{22}^{(1)} \right) \rho^{(1)} c^2 \eta^2 + \left(1 + C_{22}^{(1)} \right) \right], \quad \bar{r}_2 = \frac{C_{11}^{(1)}}{C_{22}^{(1)}} \left[\left(1 - \rho^{(1)} c^2 \eta^2 \right) \left(\frac{1}{C_{11}^{(1)}} - \rho^{(1)} c^2 \eta^2 \right) \right] \\ r_2 &= \frac{C_{11}^{(1)}}{C_{22}^{(1)}} \left[\left(1 - \rho^{(1)} c^2 \eta^2 \right) \left(\rho^{(1)} c^2 \eta^2 - \frac{1}{C_{11}^{(1)}} \right) \right], \quad \bar{\omega}_1 = \frac{\bar{\delta}_1^{(1)2} \mu_{12}^{(1)} - \zeta^2 \left(C_{11}^{(1)} - \rho^{(1)} c^2 \right)}{\bar{\delta}_1^{(1)} \left(\mu_{12}^{(1)} + C_{12}^{(1)} \right)}, \\ \bar{\omega}_2 &= \frac{\zeta^2 \left(C_{11}^{(1)} - \rho^{(1)} c^2 \right) - \bar{\delta}_1^{(1)2} \mu_{12}^{(1)}}{\bar{\delta}_1^{(1)} \left(\mu_{12}^{(1)} + C_{12}^{(1)} \right)}, \quad \bar{\omega}_3 = \frac{\bar{\delta}_2^{(1)2} \mu_{12}^{(1)} - \zeta^2 \left(C_{11}^{(1)} - \rho^{(1)} c^2 \right)}{\bar{\delta}_2^{(1)} \left(\mu_{12}^{(1)} + C_{12}^{(1)} \right)}, \\ \bar{\omega}_4 &= \frac{\zeta^2 \left(C_{11}^{(1)} - \rho^{(1)} c^2 \right) - \bar{\delta}_1^{(1)2} \mu_{12}^{(1)}}{\bar{\delta}_1^{(1)} \left(\mu_{12}^{(1)} + C_{12}^{(1)} \right)}, \quad \hat{\omega}_1 = \frac{\hat{\delta}_1^{(1)2} \mu_{12}^{(1)} - \zeta^2 \left(C_{11}^{(1)} - \rho^{(1)} c^2 \right)}{\hat{\delta}_1^{(1)} \left(\mu_{12}^{(1)} + C_{12}^{(1)} \right)} \\ \hat{\omega}_2 &= \frac{\zeta^2 \left(C_{11}^{(1)} - \rho^{(1)} c^2 \right) - \hat{\delta}_1^{(1)2} \mu_{12}^{(1)}}{\hat{\delta}_1^{(1)} \left(\mu_{12}^{(1)} + C_{12}^{(1)} \right)}, \quad \hat{\omega}_3 = \frac{\hat{\delta}_2^{(1)2} \mu_{12}^{(1)} - \zeta^2 \left(C_{11}^{(1)} - \rho^{(1)} c^2 \right)}{\hat{\delta}_2^{(1)} \left(\mu_{12}^{(1)} + C_{12}^{(1)} \right)} \\ \hat{\omega}_4 &= \frac{\zeta^2 \left(C_{11}^{(1)} - \rho^{(1)} c^2 \right) - \hat{\delta}_2^{(1)2} \mu_{12}^{(1)}}{\hat{\delta}_2^{(1)} \left(\mu_{12}^{(1)} + C_{12}^{(1)} \right)}, \quad \hat{\beta} = \frac{\hat{\omega}_1 + \hat{\delta}_1^{(1)}}{\hat{\omega}_1 + \hat{\delta}_1^{(1)}} \end{aligned}$$

Employing the series expansions for the Bessel function J_0 and the Hankel function $H_0^{(1)}$ as:

$$J_0 \left(\eta \rho^{(1)} c^2 y \right) H_0^{(1)} \left(\eta \rho^{(1)} c^2 z \right) = \frac{2i}{\pi} \log \left(\rho^{(1)} c^2 \right) + \left[1 + \frac{2i}{\pi} \left(y + \log \left(\frac{\eta z}{2} \right) \right) \right]$$

In Eq.(44) we have

$$L(y, z) = M \rho^{(1)} c^2 \log \left(\rho^{(1)} c^2 \right) + O \left(c^2 \right) \tag{45}$$

where

$$\begin{aligned} M &= \frac{2}{\pi^2} \left[\int_0^\gamma \left[\frac{C_{22}^{(1)} \left(\bar{\omega}_1 \bar{\delta}_1^{(1)} R_{11} + \bar{\omega}_2 \bar{\delta}_2^{(1)} T_{11} - \bar{\omega}_1 \bar{\delta}_1^{(1)} T_{11} - \bar{\omega}_2 \bar{\delta}_2^{(1)} Q_{11} \right)}{\left\{ \bar{\omega}_1 \left(S_{11} \beta - R_{11} + 1 \right) + \bar{\omega}_2 \left(Q_{11} \beta - T_{11} - \beta \right) \right\}} + \right. \\ &\left. \frac{C_{22}^{(1)} \left(\bar{\omega}_3 \bar{\delta}_1^{(1)} R_{11} + \bar{\omega}_4 \bar{\delta}_2^{(1)} T_{11} - \bar{\omega}_3 \bar{\delta}_1^{(1)} T_{11} - \bar{\omega}_4 \bar{\delta}_2^{(1)} Q_{11} \right)}{\pi \left\{ \bar{\omega}_3 \left(S_{11} \beta - R_{11} + 1 \right) + \bar{\omega}_4 \left(Q_{11} \beta - T_{11} - \beta \right) \right\}} \right] d\eta \\ &- \int_\gamma^1 \left[\frac{C_{22}^{(1)} \left(\hat{\omega}_2 \hat{\delta}_2^{(1)} T_{11} - \hat{\omega}_2 \hat{\delta}_2^{(1)} Q_{11} \right)}{\pi \left\{ \hat{\omega}_1 \left(R_{11} - S_{11} \beta \right) - \hat{\omega}_2 \left(1 + T_{11} + Q_{11} \beta \right) \right\}} + \frac{C_{22}^{(1)} \left(\hat{\omega}_4 \hat{\delta}_2^{(1)} T_{11} - \hat{\omega}_4 \hat{\delta}_2^{(1)} Q_{11} \right)}{\pi \left\{ \hat{\omega}_3 \left(R_{11} - S_{11} \beta \right) - \hat{\omega}_4 \left(1 + T_{11} + Q_{11} \beta \right) \right\}} \right] d\eta \end{aligned} \tag{46}$$

Let us expand $g(t)$ and $f(w^2)$ in the form

$$g(t) = g_0(t) + \rho^{(1)}c^2 \log\left(\sqrt{\rho^{(1)}c}\right)g_1(t) + O(c^2) \tag{47}$$

$$f(w^2) = f_0(w^2) + \rho^{(1)}c^2 \log\left(\sqrt{\rho^{(1)}c}\right)f_1(w^2) + O(c^2) \tag{48}$$

Substituting the above expansion and the value of $L(y, z)$ from Eq. (45) and using (47) and (48) in Eq. (42) we obtain

$$\begin{aligned} & \frac{d}{dx} \int_0^a \left[g_0(t) + \rho^{(1)}c^2 \log\left(\sqrt{\rho^{(1)}c}\right)g_1(t) \right] \log\left|\frac{t+x}{t-x}\right| dt + \\ & \frac{d}{dx} \int_b^1 \left[f_0(w^2) + \rho^{(1)}c^2 \log\left(\sqrt{\rho^{(1)}c}\right)f_1(w^2) \right] \log\left|\frac{w+x}{w-x}\right| dw \\ & = 2\left[\tau_1 - \frac{d}{dx} \int_0^a \left[g_0(t) + \rho^{(1)}c^2 \log\left(\sqrt{\rho^{(1)}c}\right)g_1(t) \right] dt \int_0^x \int_0^t \frac{yzM \rho^{(1)}c^2 \log\left(\sqrt{\rho^{(1)}c}\right)}{\sqrt{(t^2-y^2)(x^2-z^2)}} dydz - \right. \\ & \left. \frac{d}{dx} \int_b^1 \left[f_0(w^2) + \rho^{(1)}c^2 \log\left(\sqrt{\rho^{(1)}c}\right)f_1(w^2) \right] dw \int_0^x \int_0^w \frac{yzM \rho^{(1)}c^2 \log\left(\sqrt{\rho^{(1)}c}\right)}{\sqrt{(w^2-y^2)(x^2-z^2)}} dydz \right] \end{aligned} \tag{49}$$

Equating the coefficients of the like powers of $\rho^{(1)}c^2$ the following equations are derived

$$\frac{d}{dx} \int_0^a g_0(t) \log\left|\frac{t+x}{t-x}\right| dt + 2 \int_b^1 \frac{wf_0(w^2)}{w^2-x^2} dw = 2\tau_1 \tag{50}$$

$$\frac{d}{dx} \int_0^a g_1(t) \log\left|\frac{t+x}{t-x}\right| dt + 2 \int_b^1 \frac{wf_1(w^2)}{w^2-x^2} dw = -2M \left[\int_0^a t g_0(t) dt + \int_b^1 wf_0(w^2) dw \right] \tag{51}$$

From (39) we obtain

$$\int_b^1 f_i(w^2) dw = 0 \tag{52}$$

Rewriting Eq. (50) as:

$$\int_0^a g_0(t) \log\left|\frac{t+x}{t-x}\right| dt = \pi F_1(x) \tag{53}$$

where $F_1(x) = -\int_0^x \left[\frac{\tau_0}{2\mu_1^{(1)}} + \frac{2}{\pi} \int_b^1 \frac{wf_0(w^2)}{w^2-v^2} dw \right] dv$ and applying the Cook's result we got the following expression:

$$g_0(t) = -\frac{\tau_0 t}{\mu_{12}^{(1)} \sqrt{a^2 - t^2}} - \frac{2t}{\pi \sqrt{a^2 - t^2}} \int_b^1 \frac{\sqrt{w^2 - a^2} f_0(w^2)}{w^2 - t^2} dw \quad (54)$$

Substituting the value of $g_0(t)$ from (53) in Eq. (50) the following singular integral equation is formed:

$$\int_b^1 \frac{\sqrt{w^2 - a^2} f_0(w^2)}{w^2 - x^2} dw = -\frac{\pi \tau_0}{2\mu_{12}^{(1)}} \quad (55)$$

Applying Hilbert-transform we have

$$f_0(w^2) = -\frac{\tau_0}{\mu_{12}^{(1)}} \sqrt{\frac{w^2(w^2 - b^2)}{(w^2 - a^2)(1 - w^2)}} + \frac{wD_1}{\sqrt{(w^2 - a^2)(w^2 - b^2)(1 - w^2)}} \quad (56)$$

where D_1 is the unknown constant to be determined from Eq. (52). Now using Eq. (56) in Eq. (54) we obtain $g_0(t)$

$$g_0(t) = -\frac{\tau_0}{\mu_{12}^{(1)}} \sqrt{\frac{t^2(b^2 - t^2)}{(a^2 - t^2)(1 - t^2)}} + \frac{tD_1}{\sqrt{(a^2 - t^2)(b^2 - t^2)(1 - t^2)}} \quad (57)$$

From Eq. (51) we have $\int_0^a t g_0(t) dt + \int_b^1 w f_0(w^2) dw = -\frac{\tau_0}{\mu_{12}^{(1)}} [I_0^a + J_b^1] + D_1 [K_b^1 - L_0^a]$

where $I_0^a = \int_0^a \frac{t^2 \sqrt{b^2 - t^2}}{\sqrt{(a^2 - t^2)(1 - t^2)}} dt$, $J_b^1 = \int_b^1 \frac{t^2}{\sqrt{(a^2 - t^2)(b^2 - t^2)(1 - t^2)}} dt$, $L_0^a = \int_0^a \frac{w^2 \sqrt{w^2 - b^2}}{\sqrt{(w^2 - a^2)(1 - w^2)}} dw$,

$$K_b^1 = \int_b^1 \frac{w^2}{\sqrt{(w^2 - a^2)(w^2 - b^2)(1 - w^2)}} dw$$

Rewriting Eq. (51) as:

$$\int_0^a g_1(t) \log \left| \frac{t+x}{t-x} \right| dt = \pi F_2(x) \quad (58)$$

where $F_2(x) = -\frac{1}{\pi} \int_0^x \left[\frac{2MQ}{\pi} + 2 \int_b^1 \frac{w f_1(w^2)}{w^2 - v^2} dw \right] dv$, $Q = -\frac{\tau_0}{\mu_{12}^{(1)}} [I_0^a + J_b^1] + D_1 [K_b^1 - L_0^a]$ with the help of

Cook's result (1970) the solution of the integral Eq. (58) is found to be

$$g_1(t) = -\frac{2MQ}{\pi^2} \frac{t}{\sqrt{a^2 - t^2}} - \frac{2t}{\pi \sqrt{a^2 - t^2}} \int_b^1 \frac{\sqrt{w^2 - a^2}}{w^2 - t^2} f_1(w^2) dw \quad (59)$$

Substituting the value of $g_1(t)$ from Eq. (59) in (51) the singular integral equation becomes

$$\int_b^1 \frac{\sqrt{w^2 - a^2}}{w^2 - x^2} f_1(w^2) dw = -\frac{2MQ}{\pi} \tag{60}$$

Next using the finite Hilbert-transform technique the solution of the integral Eq.(60) is found to be of the form

$$f_1(w^2) = -\frac{2MQ}{\pi^2} \sqrt{\frac{w^2(w^2 - b^2)}{(w^2 - a^2)(1 - w^2)}} + \frac{wD_2}{\sqrt{(w^2 - a^2)(w^2 - b^2)(1 - w^2)}} \tag{61}$$

where D_2 is the unknown constant to be determined from the Eq. (52). Substituting the value of $f_1(w^2)$ from Eq. (61) into Eq.(59) $g_1(t)$ is obtained as:

$$g_1(t) = -\frac{2MQ}{\pi^2} \sqrt{\frac{t^2(b^2 - t^2)}{(a^2 - t^2)(1 - t^2)}} - \frac{tD_2}{\sqrt{(a^2 - t^2)(b^2 - t^2)(1 - t^2)}} \tag{62}$$

The values of the unknown constants D_1 and D_2 we utilize Eq. (52) and given by

$$D_i = A_i \left[(1 - a^2) \frac{E}{F} - (b^2 - a^2) \right], i = 1, 2 \tag{63}$$

where $A_1 = \frac{\tau_0}{\mu_{12}^{(1)}}$, $A_2 = \frac{2MQ}{\pi^2}$ and $F = F\left(\frac{\pi}{2}, p\right)$, $E = E\left(\frac{\pi}{2}, p\right)$ are the elliptic integrals of a first and second kind respectively and $p = \sqrt{\frac{1 - b^2}{1 - a^2}}$. On substituting one of the values D_i ($i = 1, 2$) given by the Eq. (63) in the Eqs. (56)-(57) and Eqs. (61)-(62) yield

$$g_{i-1}(t) = -A_i \left[(1 - a^2) \frac{E}{F} + (a^2 - t^2) \right] \frac{t}{\sqrt{(a^2 - t^2)(b^2 - t^2)(1 - t^2)}} \tag{64}$$

$$f_{i-1} = A_i \left[(1 - a^2) \frac{E}{F} - (w^2 - a^2) \right] \frac{w}{\sqrt{(w^2 - a^2)(w^2 - b^2)(1 - w^2)}}, i = 1, 2 \tag{65}$$

5 STRESS INTENSITY FACTOR BAND CRACK OPENING DISPLACEMENT

Substituting the values of the functions $g(t)$ and $f(w^2)$ the stress component $\sigma_{yy}^{(1)}$ can be evaluated from Eqs.(29). The stress intensity factor K_a , K_b and K_1 at the tip of the cracks at $x = a$, and $x = 1$ respectively are found to be

$$K_a = \lim_{x \rightarrow a^+} \left| \frac{\sigma_{yy}^{(1)}(x, 0^+)(x - a)^{\frac{1}{2}}}{\sigma_0} \right|_{a < x < b} = \sqrt{\frac{a(1 - a^2)}{a^2 - b^2}} \frac{E}{F} \left[1 - \frac{2M}{\pi^2} N \rho c^2 \log(\sqrt{\rho c}) \right] \tag{66}$$

$$K_b = \lim_{x \rightarrow b^-} \left| \frac{\sigma_{yy}^{(1)}(x, 0^+) (b-x)^{\frac{1}{2}}}{\sigma_0} \right|_{a < x < b} \quad (67)$$

$$= \sqrt{\frac{b}{(1-b^2)(b^2-a^2)}} \left[(1-a^2) \frac{E}{F} - (b^2-a^2) \right] \left[1 - \frac{2M}{\pi^2} N \rho c^2 \log(\sqrt{\rho c}) \right]$$

$$K_1 = \lim_{x \rightarrow 1^+} \left| \frac{\sigma_{yy}^{(1)}(x, 0^+) (x-1)^{\frac{1}{2}}}{\sigma_0} \right|_{x > 1} = \sqrt{\frac{1-a^2}{1-b^2}} \left[1 - \frac{E}{F} \right] \left[1 - \frac{2M}{\pi^2} N \rho c^2 \log(\sqrt{\rho c}) \right] \quad (68)$$

where $N = I_0^a + J_b^1 + \left[(1-a^2) \frac{E}{F} - (b^2-a^2) \right] (K_0^a - L_b^1)$.

The crack opening displacement is given by

$$\Delta v(x, 0) = v(x, 0^+) - v(x, 0^-) = \begin{cases} 2 \int_x^a g(t) dt & 0 \leq x \leq a \\ 2 \int_x^1 f(w^2) dw & b \leq x \leq 1 \end{cases}$$

with the help of the Eqs.(64) and Eqs. (65) we get the expressions of crack opening displacement as follows:

$$\Delta v(x, 0) = \frac{2\tau_0}{\mu} \left[1 - \frac{4M}{\pi^2} N \rho c^2 \log \sqrt{\rho c} \right] \times \left[\sqrt{1-a^2} F(\beta, p) \left\{ \frac{E(\beta, p)}{F(\beta, p)} - \frac{E}{F} \right\} - \sqrt{\frac{(1-x^2)(a^2-x^2)}{a^2-x^2}} \right] \quad (69)$$

where $\sin \beta = \sqrt{\frac{a^2-x^2}{b^2-x^2}}$ and $\sin \lambda = \sqrt{\frac{1-x^2}{1-b^2}}$

6 NUMERICAL CALCULATIONS AND DISCUSSIONS

From the expressions of the stress intensity factors K_a , K_b and K_1 at the tip of the cracks given by the Eqs. (66),(67) and (68), it is clear that SIF depends on the material constants and width (h) of the upper medium. The values of SIF can be plotted graphically against the crack velocity (c). Keeping the central crack length fixed ($a = 0.2$) stress intensity factors at the tip of the central and outer cracks have been plotted against crack velocity for different outer crack lengths ($b = 0.4, 0.6, 0.8$) (Fig. 3, Fig. 5). It is observed from the figures, that the stress intensity factors K_a and K_1 initially decrease with an increase in the value of the crack velocity (c) and after it increases. Also with the increase in the distance between the central crack and outer crack, the graph of stress intensity factors become flattered and the graph tends to be linear. Again, on keeping the outer crack length fixed $b = 0.7$, SIFs are shown for different central crack lengths ($a = 0.4, 0.6, 0.8$) in Figs. 6, 7, 8. It is found that the nature of stress intensity factors remains the same in this case as well. The graphs are shown only for Table 1 and Table 2 materials as the nature of the graphs are a similar type for various type of orthotropic materials. The values of engineering elastic constants are given in the following table

Table 1
Engineering elastic constants.

Elastic constants	
Medium-1	E-type glass-epoxy composite
Medium-2	Steel-Mylar composite

Table 2
Engineering elastic constants

Elastic constants	
Medium-1	Boron-epoxy composite
Medium-2	Steel-Mylar composite

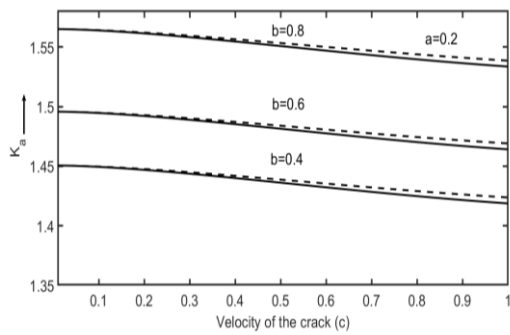


Fig.3
Stress intensity factor K_a vs crack velocity C with fixed central crack length ($a = 0.2$) and layer thickness ($h = 0.3$), (—)Table 1; (- -)Table 2.

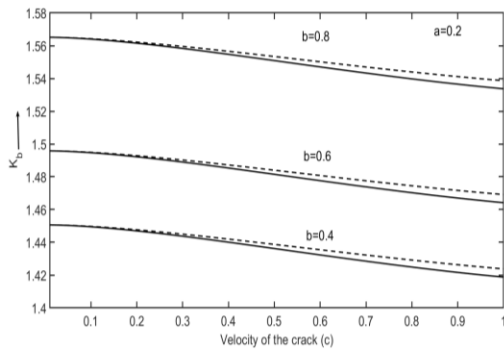


Fig.4
Stress intensity factor K_b vs crack velocity C with fixed central crack length ($a = 0.2$) and layer thickness ($h = 0.3$), (—)Table 1; (- -)Table 2.

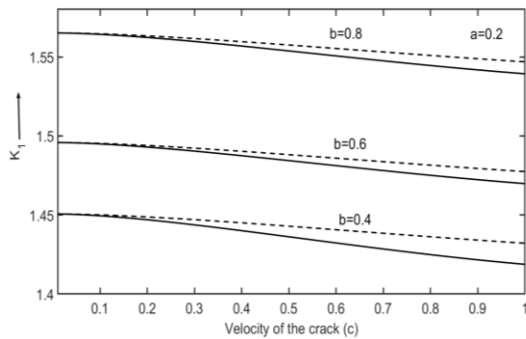


Fig.5
Stress intensity factor K_1 vs crack velocity C with fixed central crack length ($a = 0.2$) and layer thickness ($h = 0.3$), (—)Table 1; (- -)Table 2.

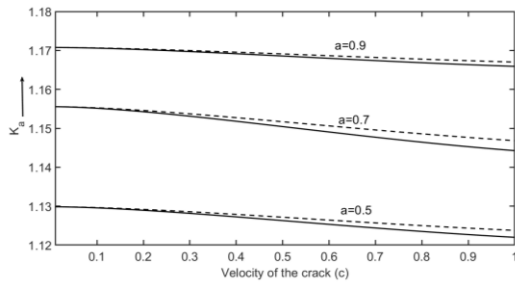


Fig.6
Stress intensity factor K_a vs crack velocity C with fixed outer crack length ($b = 0.4$) and layer thickness ($h = 0.3$), (—) Table 1; (- - -) Table 2.

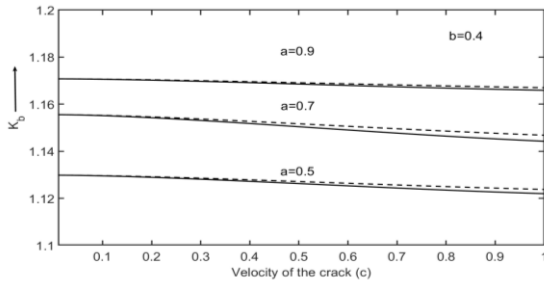


Fig.7
Stress intensity factor K_b vs crack velocity C with fixed outer crack length ($b = 0.4$) and layer thickness ($h = 0.3$), (—) Table 1; (- - -) Table 2.

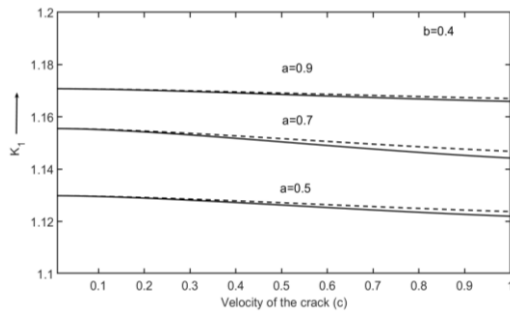


Fig.8
Stress intensity factor K_1 vs crack velocity C with fixed outer crack length ($b = 0.4$) and layer thickness ($h = 0.3$), (—) Table 1; (- - -) Table 2.

From the all figures (Fig. 3-Fig. 8) it is clear that the values of SIF depends on the geometry of the crack and the applied load distribution. If the applied load is constant then the SIF decreases with increasing value of crack velocity which means that crack will not propagate further if applied load does not exceed critical value of the load. The crack opening displacement (COD) has been plotted for different crack lengths. It is observed from Fig. 9 and Fig. 10, that the peak of crack opening displacement increases as the length of the crack increases. For the central crack, crack opening displacement gradually decreases slowly and for the outer crack, crack opening displacement gradually increases and obtaining its maximum value, decreases slowly thereafter.

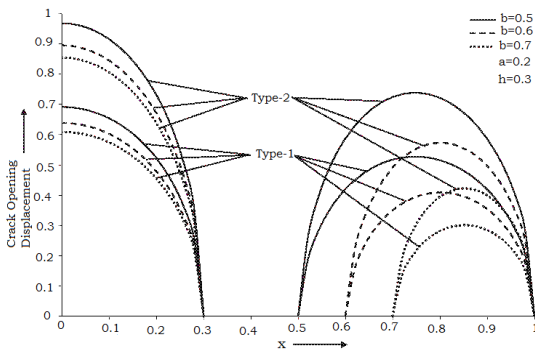


Fig.9
Crack opening displacement vs distance for generalized plane stress.

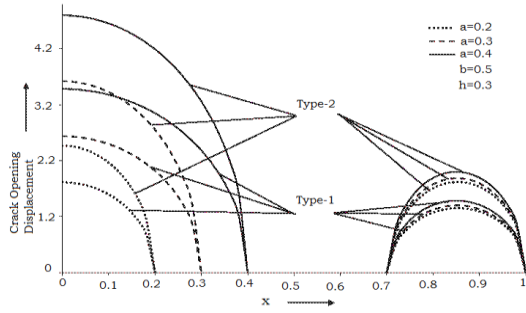


Fig.10 Crack opening displacement vs distance for generalized plane stress.

7 COMPARISION

In this part, we have compared our results with some published articles to prove the correctness and validation of our problem. Interaction of moving interface collinear Griffith cracks under anti plane shear waves is solved by S. Das [21]. In present problem, we considered the moving three collinear Griffith cracks at orthotropic interface under *P*-waves. Geometrically both problems are same but the medium and wave forms are different. Das [21] considered in isotropic medium and antiplane shear wave but in our work we are considered orthotropic medium and *P*-wave. For correctness in our problem, we are converting the medium from orthotropic to isotropic medium. For isotropic media we can write $C_{11}=C_{22}=\lambda+2\mu$, $C_{12}=\lambda$ where λ and μ are Lamé’s constant. The term *M* of kernel function (Eq. (45)) has been converted by the expression of Lamé’s constant. In both works, the expression of Stress Intensity Factor (SIF) for central crack are same. Only the term *P* of K_b in Das [21] is slightly different with the term *M* of K_a in our work because of wave forms are different.

For more validation of this work, we plotted the graphs of SIF against the layer thickness like the numerical result of S. Das [21]. The following graphs are showing the correctness of our work

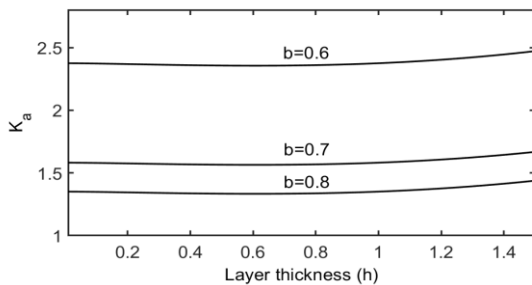


Fig.11 Stress intensity factor K_a vs layer thickness (*h*) with fixed central crack length ($a = 0.5$).

The nature of the above graph is same with comparing the graph of SIF in S. Das [21]. Similar type of problem was solved by Debnath [26]. They considered the interaction between Griffith cracks in a sandwich orthotropic layers and we have taken same type of problems with single layer. Both problems have been solved by using integral equation method and it produced like same results (SIF). The accuracy of our work is confirmed by following the graphs. These graphs are plotted against layer thickness

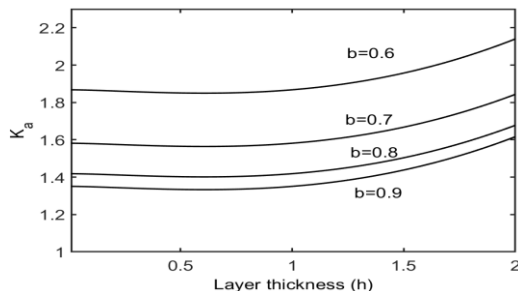


Fig.12 Stress intensity factor K_a vs layer thickness (*h*) with fixed central crack length ($a = 0.5$).

The graphs of SIF are similar in nature with Debnath [26]. In the above graph, the SIF increases with increasing the values of layer thickness.

8 CONCLUSION

An interfacial crack problem with a Griffith Crack between two dissimilar orthotropic media has been solved and numerical computation has been done with a pair of composite materials. The SIF and COD have been obtained at the tips of the crack on the orthotropic bi-material interface subject to P wave interaction. The singularities and discontinuities associated with the P waves and crack have been predicted in the solution. The graphs of Stress Intensity Factor and Crack Opening Displacement have been plotted to show the effects of various parameters on these quantities. From all the graphs it can be concluded that the value of SIF can be controlled and arrested within a certain range by varying a parameter i.e. crack velocity by monitoring applied loads. Moreover COD can also be controlled by varying same parameters. Finally the expressions for the stress intensity factor in case of isotropic medium have also been obtained and verified with the results already obtained earlier.

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