Study of the Resonant Frequency of Unimorph Triangular V-shaped Piezoelectric Cantilever Energy Harvester

R. Hosseini*
Young Researchers and Elite Club,
South Tehran Branch, Islamic Azad University, Tehran, Iran
E-mail: R.Hosseini.mech@gmail.com
*Corresponding author

M. Hamedi
Department of Mechanical Engineering,
University of Tehran, Iran
E-mail: mhamedi@ut.ac.ir

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Abstract: The main aim of the vibration energy harvesters is to locally power autonomous devices such as wireless sensors. Generally, power levels are low and the environmental benefit of the technology is to replace batteries rather than saving energy per se. Piezoelectric vibrational energy harvesters are usually inertial mass based devices, where a cantilever beam with a piezoelectric outer layer is excited into resonance by a mechanical vibration source at the root of the cantilever beam. However, the geometry of a piezoelectric cantilever beam will greatly affect its vibration energy harvesting ability. This paper deduces a remarkably precise analytical formula for calculating the fundamental resonant frequency of unimorph V-shaped cantilevers using Rayleigh-Ritz method. This analytical formula, which is convenient for mechanical energy harvester design based on piezoelectric effect, is then validated by ABAQUS simulation. This formula raises a new perspective that, among all the unimorph V-shaped cantilever beams and in comparison with rectangular one, the simplest tapered cantilever can lead to highest resonant frequency and maximum sensitivity.

Keywords: Finite Element, Mechanical Energy Harvester, Piezoelectric, Unimorph V-shaped Cantilever, Resonant Frequency

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Biographical notes: R. Hosseini received his PhD in Mechanical Engineering from University of Tehran. He is currently premier member of Young Researchers and Elite Club at Islamic Azad University South Tehran branch. His current research interest includes Energy Harvesting. **M. Hamedi** is Professor of Mechanical engineering at the University of Tehran.

1 INTRODUCTION

Energy harvesting is used to describe the scavenging of ambient energy in the environment that would otherwise be wasted. To feed the world's needs for energy, macro scale energy harvesting technologies have successfully established. On the other hand, the lack of cables induces a constraint on power supply for each low powered wireless electronic sensors. Batteries wear out with time, thus regular replacement is an integral and inevitable part of maintenance. As the dense network is employed in structures, replacing batteries becomes a major time-consuming task that is uneconomical and unmanageable, ironically contradictory to the original objective of structural health monitoring.

Scavenging energy from ambient vibrations, wind, heat or light could enable smart sensors to be functional indefinitely. Three mechanisms are available for vibration energy harvesting; using electrostatic devices, electromagnetic field and utilizing piezoelectric based materials. The flexibility associated with piezoelectric materials makes them very attractive for power scavenging. The performance of piezoelectric vibration energy harvesters is more often than other methods. Piezoelectric materials possess a large amount of mechanical energy that can be converted into electrical energy, and they can withstand large strain magnitude. Vibration from engines can stimulate piezoelectric materials, as can the heel of a shoe, or the pushing of a button.

Compared to other structural forms of beams, a cantilever beam can obtain the maximum deformation and strain under the same conditions. The larger deflection leads to more stress, strain, and consequently a higher output voltage and power. Therefore the vast majority of piezoelectric vibration energy harvesting devices use a cantilever beam structure. [1-4]. A cantilever-type energy harvester has been intensively studied. The cantilever geometrical structure plays an important role in improving the harvester's efficiency and a triangular tapered cantilever has been found to be the optimum design [5], [6], because it ensures a large constant strain in the piezoelectric layer resulting in higher power output compared with the rectangular beam with the width and length equal to the base and height of the corresponding triangular tapered cantilever beam. Most of the previous research works focused on designing a linear vibration resonator, which has maximum output power when reaching resonance frequency. Therefore the practical applications of these devices are limited due to narrow bandwidth as well as small power density. If the excitation frequency slightly shifts, the performance of the harvester will dramatically decrease. Since in the majority of practical cases, the vibration in the environment is frequency-varying or totally random with the energy distributed in a wide spectrum, how to broaden the bandwidth of harvesters becomes one of the most challenging issues before their practical deployment. When a harvester operates in an environment with multi-frequency spectra, it is desirable to design the harvester with a tailorable operating frequency band [7]. In practice, the energy harvester is a multi-degree-of-freedom system or a distributed parameter system. Certain vibration mode can be excited when the driving frequency approaches one natural frequency of the harvester. To date, one of the most important strategies to widen the bandwidth includes using a generator array consisting of small generators with different resonant frequencies. Multiple cantilever energy harvesters with different resonant frequencies can be connected in series or parallel to widen the operating frequency bandwidth of a harvesting structure. If multiple vibration modes of the harvester structure are utilized, useful power can be harvested over multiple frequency spectra, that is, wider bandwidth can be covered for efficient energy harvesting. Rather than discrete bandwidth due to the multiple modes of a single beam, multiple cantilevers or cantilever array integrated in one energy harvesting device can provide continuous wide bandwidth, if the geometric parameters of the harvester are appropriately selected. Power spectrum of a generator array is a combination of the power spectra of each small generator [7-9]. Accordingly, by division of a triangular unimorph piezoelectric beam into some V-shaped unimorph beams with different dimensions and mass and hence different resonant frequencies, can be found in an array of beams that can cover a wider range of frequencies (Fig. 1) [10]. Also in a new design, pizza model can be used to make the array of energy harvesters. The main advantage of this scheme is optimum use of space and to create higher power density (Fig. 2).

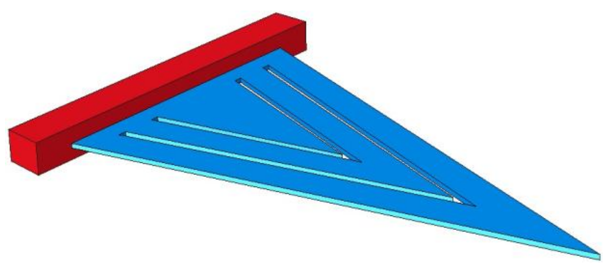

Fig. 1 Division of a triangular beam into some V-shaped beams [10]

A systematic procedure for designing mechanical band-pass filters to meet a desired frequency bandwidth is given in [11]. Such a research about rectangular shapes shows that by using some cantilevers in series connection, not only the output power increases with the use of more cantilevers, but also the frequency band is widened [12].

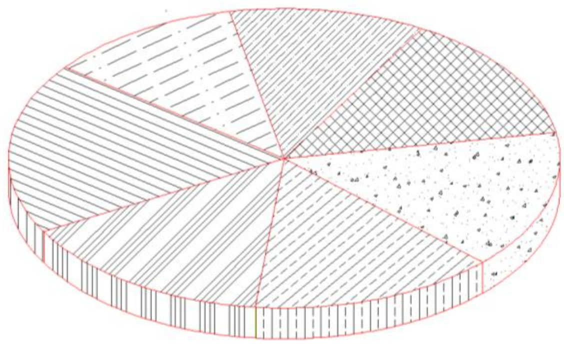

Fig. 2 The Pizza model of semi-triangular cantilever energy harvesters

The geometry of a piezoelectric cantilever beam will greatly affect its vibration energy harvesting ability. The sensitivity of resonant cantilever piezoelectric energy harvesters is directly proportional to the resonant frequency. So far, the calculation of resonant frequency of unimorph V-shaped cantilevers has not been reported in the literature and the calculations are only for a simple V-shaped cantilever beam [13]. In order to calculate the resonant frequency of V-shaped cantilevers, this paper deduces a highly precise analytical formula using Rayleigh-Ritz method, and then introduces the optimization method for enhancing the resonant frequency with this formula. This useful analytical formula is confirmed by simulation results in ABAQUS 14.1 software, and presents a strong potential to be used in the design and optimization of triangular V-shaped cantilever unimorph piezoelectric energy harvesters. It is noteworthy that a cantilever beam can have many different modes of vibration with a different resonant frequency. The first mode of vibration has the lowest resonant frequency, and typically provides the most deflection and therefore electrical energy. Accordingly, energy harvesters are generally designed to operate in the first resonant mode. This research proposes a new design for a cantilever-type unimorph piezoelectric energy harvester called V-shaped cantilever and the main focus of this paper is to study the resonant frequency of the new design in piezoelectric mechanical energy harvester.

2 THEORETICAL ANALYSIS

2.1. Deflection Function of Rectangular Unimorph Cantilevers

Fig. 3 shows the structure of unimorph piezoelectric rectangular cantilever with length L, width W, density ρ_1 and ρ_2, thickness H_1 and H_2, and Young's modulus E_1 and E_2 for substrate and piezoelectric layers, respectively. Also the total cross-sectional area moment of inertia is $\bar{I}_z$.

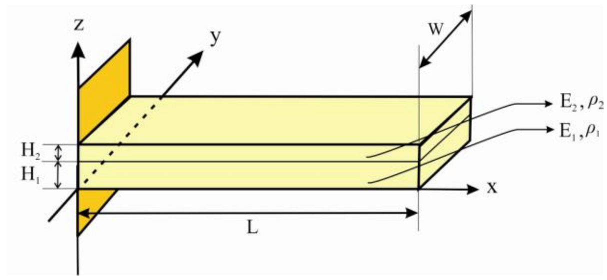

Fig. 3 The schematic drawing of a cantilever beam

For beam cross-sections that are not symmetric about the z-axis with regard to either geometry or the variation of elasticity modulus (E), a convenient method for treating bending problems is provided by the concept of the transformed section. If we choose a certain value of E as a reference value and call it E_{ref}, then we can define a transformed section and transformed width nW, where $n = E_1 / E_2$. In the case we assume that $E_{ref}=E_2$. The line of action of an axial force produces purely axial deformation, therefore passes through the centroid of the transformed section. In the case of bending with no axial force, the neutral axis also passes through this point. In this case we assume that the location of the effective centroid is determined by h (Fig. 4 and Fig. 5) [14].

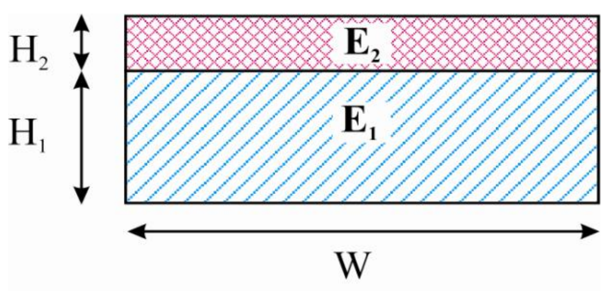

Fig. 4 The cross section of unimorph cantilever beam

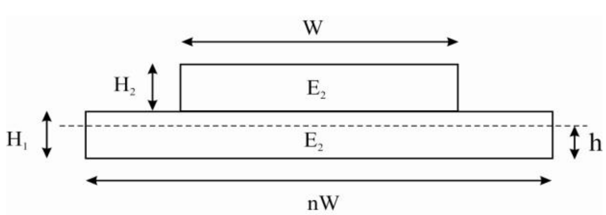

Fig. 5 The transformed section of unomorph cantilever bram

h determines the neutral axis location and can be expressed as;

$$nWH_1\frac{H_1}{2}+WH_2(H_1+\frac{H_2}{2})=(nWH_1+WH_2)h$$
$$\Rightarrow h=\frac{nH_1^{\,2}+2H_1H_2+H_2^{\,2}}{2nH_1+2H_2} \tag{1}$$

Also the total cross-sectional area moment of inertia relation is;

$$\bar{I}_z=\frac{nWH_1^{\,3}}{12}+nWH_1(h-\frac{H_1}{2})^2 +\frac{WH_2^{\,3}}{12}+WH_2(\frac{H_2}{2}+H_1-h)^2 \tag{2}$$

When applying a normal force F at the free end of the cantilever, the differential equation of the cantilever can be expressed as [15];

$$\frac{d^2z(x)}{dx^2}=\frac{F(L-x)}{\overline{EI}}=\frac{F(L-x)}{E_2\bar{I}} \tag{3}$$

Where x is the distance from the fixed end.
As one end of the cantilever is fixed, the corresponding boundary conditions are;

$$z(0)=0 \tag{4}$$

and

$$\left.\frac{dz(x)}{dx}\right|_{x=0}=0 \tag{5}$$

The solution of (3)- (5) can be expressed as;

$$z(x)=\frac{2Fx^2(3L-x)}{6E_2\bar{I}}=Ax^2(3L-x) \tag{6}$$

This is the deflection function along the length direction where A is a constant.

2.2. Resonant Frequency of Cantilevers with Arbitrary Shapes

When considering the resonant behavior of a unimorph cantilever beam with an arbitrary shape whose width function is $W(x)$, the deflection function of (6) can be used as the mode shape, and the vibration displacement at each position can be written as [10];

$$z(x,t)=Ax^2(3L-x)\sin(\omega t+\alpha) \tag{7}$$

Where A and α are constants, t is the time, and $\omega=2\pi f$ is the angular frequency.
The kinetic energy of the system is [16];

$$T=\int_0^L\frac{1}{2}(\rho_1H_1+\rho_2H_2)W(x)dx\left(\frac{\partial z}{\partial t}\right)^2 =\frac{1}{2}(\rho_1H_1+\rho_2H_2)\omega^2A^2\cos^2(\omega t+\alpha) \int_0^L W(x)x^4(3L-x)^2dx \tag{8}$$

So the maximum kinetic energy of the system is;

$$T_{\max}=\frac{1}{2}(\rho_1H_1+\rho_2H_2)\omega^2A^2 \int_0^L W(x)x^4(3L-x)^2dx \tag{9}$$

The potential energy of the system is [16];

$$V=\frac{1}{2}\iiint_V\sigma_{xx}\varepsilon_{xx}dV =\frac{1}{2}\int_0^L\int_0^W\int_{-h}^{H_1+H_2-h}\sigma_{xx}\varepsilon_{xx}dV \tag{10}$$

Where;

$$\varepsilon_{xx}=-z\frac{\partial^2W}{\partial x^2},\sigma_{xx}\big|_2=E_2\varepsilon_{xx} =-E_2z\frac{\partial^2W}{\partial x^2},\sigma_{xx}\big|_1=-E_1z\frac{\partial^2W}{\partial x^2} \tag{11}$$

Accordingly;

$$\begin{aligned}
V&=\frac{1}{2}\int_0^L\int_0^W\left\{\begin{aligned}&\int_{-h}^{H_1-h}\left(E_1z^2(\frac{\partial^2W}{\partial x^2})^2\right)dz\\&+\int_{H_1-h}^{H_1+H_2-h}\left(E_2z^2(\frac{\partial^2W}{\partial x^2})^2\right)dz\end{aligned}\right\}\\
&=\frac{W}{2}\int_0^L\left[\begin{aligned}&\frac{E_1}{3}\left(\frac{\partial^2W}{\partial x^2}\right)^2\ (H_1-h)^3+(h)^3\\&+\frac{E_2}{3}\left(\frac{\partial^2W}{\partial x^2}\right)^2\ (H_1+H_2-h)^3-(H_1-h)^3\end{aligned}\right]dx\\
&=\frac{W}{2}\int_0^L\left(\frac{\partial^2W}{\partial x^2}\right)^2\left|\begin{aligned}&\frac{E_1}{3}(H_1-h)^3+\frac{E_1}{3}h^3\\&+\frac{E_2}{3}(H_1+H_2-h)^3\\&-\frac{E_2}{3}(H_1-h)^3\end{aligned}\right|dx\\
&=\int_0^L\frac{W(x)}{2}\left[\begin{aligned}&6A(L-x)\\&\sin(\omega t+\alpha)\end{aligned}\right]^2\left|\begin{aligned}&\frac{E_1}{3}(H_1-h)^3+\frac{E_1}{3}h^3\\&+\frac{E_2}{3}(H_1+H_2-h)^3\\&-\frac{E_2}{3}(H_1-h)^3\end{aligned}\right|dx\\
&=18A^2\sin^2(\omega t+\alpha)\left|\begin{aligned}&\frac{E_1}{3}(H_1-h)^3+\frac{E_1}{3}h^3\\&+\frac{E_2}{3}(H_1+H_2-h)^3\\&-\frac{E_2}{3}(H_1-h)^3\end{aligned}\right|\\
&\int_0^L W(x)(L-x)^2dx
\end{aligned} \tag{12}$$

Therefore, the maximum potential energy of the system is;

$$V_{\max} = 18A^2 \begin{bmatrix} \frac{E_1}{3}(H_1-h)^3 + \frac{E_1}{3}h^3 \\ +\frac{E_2}{3}(H_1+H_2-h)^3 \\ -\frac{E_2}{3}(H_1-h)^3 \end{bmatrix} \int_0^L W(x)(L-x)^2 dx \tag{13}$$

According to conservation law of mechanical energy;

$$T_{\max} = V_{\max} \tag{14}$$

Hence, the resonant frequency can be obtained as;

$$f(W(x)) = \frac{\omega}{2\pi} = \frac{\sqrt{3}}{\pi}\sqrt{\frac{E_1(H_1-h)^3 + E_1h^3 + E_2(H_1+H_2-h)^3 - E_2(H_1-h)^3}{\rho_1H_1+\rho_2H_2}} \sqrt{\frac{\int_0^L W(x)(L-x)^2dx}{\int_0^L W(x)x^4(3L-x)^2dx}} \tag{15}$$

In particular, for the case of a rectangular cantilever with length L_1 and width W_1, the resonant frequency can be deduced from (15);

$$f_{rect} = \sqrt{\frac{E_1(H_1-h)^3+E_1h^3 +E_2(H_1+H_2-h)^3-E_2(H_1-h)^3}{\rho_1H_1+\rho_2H_2}} \frac{\sqrt{3}}{\pi}\sqrt{\frac{\int_0^L W_1(L_1-x)^2dx}{\int_0^L W_1x^4(3L_1-x)^2dx}} = \sqrt{\frac{E_1(H_1-h)^3+E_1h^3 +E_2(H_1+H_2-h)^3-E_2(H_1-h)^3}{\rho_1H_1+\rho_2H_2}} \frac{1}{\pi L_1^2}\sqrt{\frac{35}{33}}$$
$$\Rightarrow f_{rect} = \frac{\sqrt{1155}}{33\pi L_1^2}\sqrt{\frac{E_1(H_1-h)^3+E_1h^3 +E_2(H_1+H_2-h)^3-E_2(H_1-h)^3}{\rho_1H_1+\rho_2H_2}} \tag{16}$$

2.3. Resonant Frequency of unimorph triangular V-shaped Cantilevers

Fig. 6(a) shows that a typical unimorph triangular V-shaped cantilever can be treated as the difference between two unimorph triangular cantilevers, with lengths L_0 and L_1, and with widths W_0 and W_1 respectively. It can be easily confirmed by (15), that due to the mirror symmetry of unimorph triangular V-shaped cantilever, we need only analyze half of it, which is a quadrilateral cantilever as shown in Fig. 6(b).

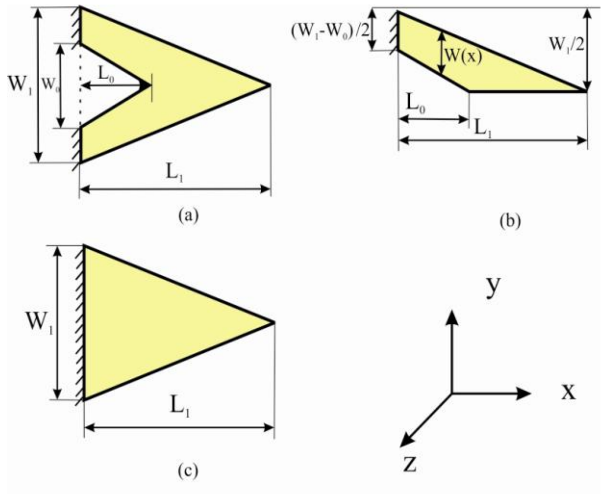

Fig. 6 Shape and dimension of (a) unimorph V-shaped cantilever (b) half of the unimorph V-shaped cantilever (c) triangular tapered cantilever [10]

Obviously, the width function of the quadrilateral cantilever is a piecewise-continuous function of *x*, that is;

$$W(x) = \begin{cases} \frac{W_1}{2}\left(1-\frac{x}{L_1}\right) - \frac{W_0}{2}\left(1-\frac{x}{L_0}\right), x \in [0, L_0] \\ \frac{W_1}{2}\left(1-\frac{x}{L_1}\right), x \in [L_0, L_1] \end{cases} \tag{17}$$

For calculation convenience, it is reasonable to define the width ratio u and the length ratio v of the two unimorph tapered cantilevers;

$$u = \frac{W_0}{W_1}, v = \frac{L_0}{L_1} \tag{18}$$

Substituting (17) and (18) into (15), the resonant frequency formula of the quadrilateral cantilever (just the resonant frequency of unimorph triangular V-shaped cantilever) is obtained.

$$f(W(x))=\frac{\sqrt{3}}{\pi}\sqrt{\frac{E_1(H_1-h)^3+E_1h^3+E_2(H_1+H_2-h)^3-E_2(H_1-h)^3}{\rho_1H_1+\rho_2H_2}}\sqrt{\frac{\int_0^{L_1}W(x)(L_1-x)^2dx}{\int_0^{L_1}W(x)x^4(3L_1-x)^2dx}}$$
$$=\frac{\sqrt{3}}{\pi}\sqrt{\frac{E_1(H_1-h)^3+E_1h^3+E_2(H_1+H_2-h)^3-E_2(H_1-h)^3}{\rho_1H_1+\rho_2H_2}}\sqrt{\frac{\frac{W_1L_1^3}{8}-\frac{W_0L_1^2L_0}{4}+\frac{W_0L_1L_0^2}{6}-\frac{W_0L_0^3}{24}}{\frac{7W_1L_1^7}{80}-\frac{3W_0L_1^2L_0^5}{20}+\frac{W_1L_1L_0^6}{14}-\frac{W_0L_0^7}{112}}}$$
$$=\frac{\sqrt{3}}{\pi}\sqrt{\frac{E_1(H_1-h)^3+E_1h^3+E_2(H_1+H_2-h)^3-E_2(H_1-h)^3}{\rho_1H_1+\rho_2H_2}}\sqrt{\frac{70(3W_1L_1^3-6W_0L_1^2L_0+4W_0L_1L_0^2-W_0L_0^3)}{3(49W_1L_1^7-84W_0L_1^2L_0^5+40W_0L_1L_0^6-5W_0L_0^7)}}$$
$$=\frac{\sqrt{70}}{\pi L_1^2}\sqrt{\frac{E_1(H_1-h)^3+E_1h^3+E_2(H_1+H_2-h)^3-E_2(H_1-h)^3}{\rho_1H_1+\rho_2H_2}}\sqrt{\frac{3-6uv+4uv^2-uv^3}{49-84uv^5+40uv^6-5uv^7}} \quad (19)$$

In order to represent the relationship between the resonant frequency and the two ratios *u* and *v,* we can define a characteristic function;

$$g(u,v)=\sqrt{\frac{3-6uv+4uv^2-uv^3}{49-84uv^5+40uv^6-5uv^7}} \quad (20)$$
$$u\in[0,1], v\in[0,1]$$

Thus, the resonant frequency of V-shaped cantilever is;

$$f(W(x))=\frac{\sqrt{70}}{\pi L_1^2}g(u,v)\sqrt{\frac{E_1(H_1-h)^3+E_1h^3+E_2(H_1+H_2-h)^3-E_2(H_1-h)^3}{\rho_1H_1+\rho_2H_2}} \quad (21)$$

As shown in Fig. 7, $g(u,v)$ reaches the maximum value $\frac{\sqrt{3}}{7}\approx 0.2474$, when $v=0$ or $v=1$ or $u=0$. That means unimorph V-shaped cantilever achieves maximum resonant frequency only when $L_0=0$ or $L_0=L_1$ or $W_0=0$. Apparently, when $L_0=0$ or $W_0=0$, the V-shaped cantilever turns into a tapered cantilever as shown in Fig. 6(c). When $L_0=L_1$, the unimorph V-shaped cantilever turns into two side by side unimorph triangular tapered cantilevers, however, this peculiar shape is difficult to carry out in practice.

Anyway, triangular tapered cantilever, a special kind of V-shaped cantilever and easy for micro-fabrication, can reach the maximum resonant frequency and thus the highest sensitivity.

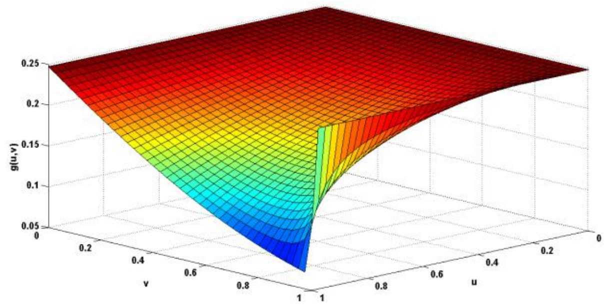

Fig. 7 The function image of $g(u,v)$

3 VERIFICATION BY SIMULATION RESULTS

In order to assess the accuracy of (21), relative error δ is introduced to compare the calculation results using this formula with the corresponding simulation results.

$$\delta=\frac{f-f'}{f} \quad (22)$$

Where *f* refers to the calculation results with (21), and *f'* refers to simulation results with ABAQUS modal analysis.

Consider a unimorph rectangular cantilever beam, assuming ρ_1=8740 kg/m^3, ρ_2=7800 kg/m^3, E_1=9.7×10^{10} Pa, E_2=6.6×10^{10}Pa, H_1=1mm, H_2=1mm, W_1=80mm and L_1=100mm. The frequency calculation according to (16) is 6.34 Hz and the corresponding simulation result with ABAQUS is 6.30 Hz. Hence the relative error is only 0.66% and an excellent agreement is obtained between the calculation results and the simulation results, yielding little relative error. The simulated shape is shown in Fig. 8.

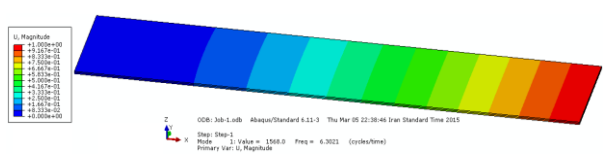

Fig. 8 Deformed shaped for the first vibration mode of unimorph piezoelectric cantilever

Also consider a series of V-shaped cantilevers with different shapes, assuming, ρ_1=8740 kg/m^3, ρ_2=7800 kg/m^3, E_1=9.7×10^{10} Pa, E_2=6.6×10^{10} Pa, H_1=0.6 mm, H_2=0.4 mm, W_1=80 mm , W_0=40mm, L_1=100mm and changing L_0, the calculation according to (21) and the corresponding simulation results with ABAQUS are listed in Table 1.

Table 1 The comparison between the calculation results and the simulation results of the resonant frequencies of unimorph triangular V-shaped cantilevers

L_0(mm)	f (Hz)	f' (Hz)	δ
0	101.68	93.49	8.05
10	96.81	89.8	7.24
20	92.39	85.83	7.10
30	88.46	83.12	6.04
40	85.14	79.85	6.21
50	82.58	78.86	4.50
60	81.04	77.714	4.10
70	80.9	76.09	5.95
80	82.85	78.19	5.62
90	88.38	83.28	5.77
100	101.68	94.03	7.52

It can be seen from Table 1 that, a very good agreement is obtained between the calculation results and the simulation results, yielding little relative error (less than 8.1%). When L_0=70mm, the simulated shape is shown in Fig. 9.

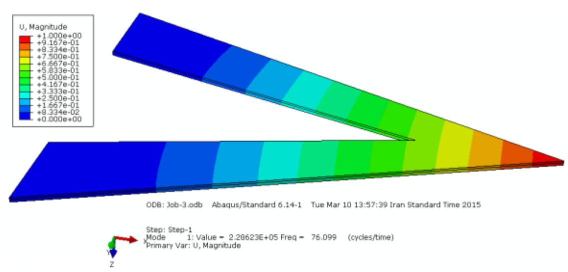

Fig. 9 Deformed shaped for the first vibration mode of unimorph piezoelectric cantilever

4 APPLICATION

The resonant frequency formula presented in this paper is useful for many applications. First, this simple formula can be effectively used to determine the resonant frequency of unimorph triangular V-shaped cantilevers of any dimensions and material properties. Another significant application is the optimization of unimorph V-shaped cantilever vibration energy harvesters. The sensitivity of resonant cantilever vibration energy harvesters is directly proportional to the resonant frequency, and the resonant frequency is a key parameter to design a mechanical energy harvester. As mentioned above, with given length L_1, given width W_1, given thickness H_1 *and* H_2 and given material properties E_1, E_2, ρ_1 and ρ_2, triangular tapered cantilever-a special kind of V-shaped cantilevers can reach the maximum resonant frequency and highest sensitivity.

For a triangular tapered cantilever, substituting $v=0$ into (21), the maximum resonant frequency is obtained.

$$\begin{aligned} f_{tap} &= \frac{\sqrt{70}}{\pi L_1^2}\sqrt{\frac{E_1(H_1-h)^3+E_1h^3+E_2(H_1+H_2-h)^3-E_2(H_1-h)^3}{\rho_1H_1+\rho_2H_2}}\,g(u,0) \\ &= \frac{\sqrt{70}}{\pi L_1^2}\sqrt{\frac{E_1(H_1-h)^3+E_1h^3+E_2(H_1+H_2-h)^3-E_2(H_1-h)^3}{\rho_1H_1+\rho_2H_2}}\,\frac{\sqrt{3}}{7} \\ &= \frac{\sqrt{210}}{7\pi L_1^2}\sqrt{\frac{E_1(H_1-h)^3+E_1h^3+E_2(H_1+H_2-h)^3-E_2(H_1-h)^3}{\rho_1H_1+\rho_2H_2}} \\ &\approx \frac{0.659}{L_1^2}\sqrt{\frac{E_1(H_1-h)^3+E_1h^3+E_2(H_1+H_2-h)^3-E_2(H_1-h)^3}{\rho_1H_1+\rho_2H_2}} \end{aligned} \tag{23}$$

Apparently, the resonant frequency of a unimorph tapered cantilever is unrelated to its width W_1. It is necessary to point out that, for a tapered cantilever, when increasing W_1 and keeping other parameters fixed, its resonant frequency will remain constant. It is worth comparing (16) and (23), and we can get the resonant frequency ratio of unimorph tapered cantilever and unimorph rectangular cantilever.

$$\frac{f_{tap}}{f_{rect}} = \frac{\dfrac{0.659}{L_1^2}\sqrt{\dfrac{E_1(H_1-h)^3+E_1h^3+E_2(H_1+H_2-h)^3-E_2(H_1-h)^3}{\rho_1H_1+\rho_2H_2}}}{\dfrac{0.3278}{L_1^2}\sqrt{\dfrac{E_1(H_1-h)^3+E_1h^3+E_2(H_1+H_2-h)^3-E_2(H_1-h)^3}{\rho_1H_1+\rho_2H_2}}} = 2.0104 > 2 \tag{24}$$

Hence, the unimorph tapered cantilevers can lead to much higher resonant frequency and higher sensitivity than that of unimorph rectangular cantilevers.

5 CONCLUSION

Energy harvesters provide a very small amount of power for low-energy electronics. Vibration energy harvesters are generally designed to operate in the first resonant mode. The piezoelectric effect converts mechanical vibration strain into electric current or voltage. This paper deduces a highly precise explicit formula to calculate the fundamental resonant frequency of unimorph V-shaped piezoelectric cantilever beams based on Rayleigh-Ritz method. With this analytical formula, the calculation results are in perfect agreement with the simulation results, yielding little relative error (less than 8.1%). This error for a unimorph rectangular cantilever reduces to only 0.66%. Triangular tapered cantilever, a special kind of V-shaped cantilever and easy for micro-fabrication, can reach the maximum resonant frequency and thus the highest sensitivity. In the first mode of vibration, the exact shape of the cantilever is not identical to the static deflection profile. Accordingly the velocity distribution is not exactly proportional to the static deflection profile. This is why the natural frequency estimates are slightly different from the simulation values. Because of simplicity of the derived formula, it is an easily learned and easily applied procedure for approximately calculating or recalling some value, or for making some determination. Finally, an application for calculating frequency of unimorph V-shaped cantilever energy harvesters, is presented with this formula in order to achieve a Multi-Modal energy harvester. This formula can be commonly used in the design and optimization of vibration energy harvesters. Experimental analysis can validate the results and expand the research for output voltage and power density.

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