Buckling of Functionally Graded Beams with Rectangular and Annular Sections Subjected to Axial Compression

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Abstract: The current study presents a new analytical method for buckling analysis of rectangular and annular beams made up of functionally graded materials with constant thickness and Poisson's ratio. The boundary conditions of the beam are assumed to be simply supported and clamped. The stability equations were obtained by using conservation of energy. The critical buckling load and first mode shape were obtained using Variational Calculus method. Increasing in buckling capacity and improvement in the behavior of functionally graded beams in comparison to homogenous beams have been investigated. After simplifying results, Duffing differential equation for homogeneous beam without oscillations was obtained and validity of this new work was proved.

Keywords: Buckling, Functionally Graded Beam, Rectangular and Annular Sections, Variational Calculus.

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1 INTRODUCTION

The current study presents a new analytical method based on Variational Calculus for buckling analyses of rectangular and annular functionally graded (FG) beams with various boundary conditions. The thickness and Poisson's ratio across the thickness are assumed to be constant. The volume fraction and power laws for expressing smooth and continues variations of mechanical properties of FG beam are used for rectangular and annular sections respectively. The stability equations were obtained by using conservation of energy. The critical buckling load and first dimensionless mode shape have been obtained. Buckling capacity increase and improvement in the behavior of functionally graded beams in comparison to homogenous beams have been investigated. After simplifying results, Duffing differential equation for homogeneous beam without oscillations was obtained and validity of this new work was proved.

B.V. Sankar in 2001 [1] made an elasticity solution for a FG beam subjected to transverse loads. He assumed that Young's modulus of the beam varies exponentially through the thickness for obtaining an exact solution for the elasticity equations. Sankar shows stress concentrations are less than homogeneous beams when the softer side of the functionally graded beam is loaded.

An asymptotic solution for buckling and initial postbuckling behavior of sandwich beams including transverse shear was shown by H. Huang and George A. Kardomateas in 2002 [2]. The asymptotic procedure by H. Huang and George A. Kardomateas is based on the nonlinear beam equation (with transverse shear included), and closed-form solutions are derived.

Analysis of postbuckling behavior of beam-column structures by stochastic finite elements has been performed by L. Graham-Brady and Benjamin W. Schafer in 2003 [3]. First-order stochastic perturbation expansions and Monte Carlo simulation techniques are used to study the effects of random elastic moduli on the postbuckling response of simple frame structures.

Thermal buckling and postbuckling of Euler–Bernoulli beams supported on nonlinear elastic foundations was investigated numerically by S. R. Li and R. C. Batra in 2007 [4]. The nonlinear boundary-value problems for postbuckling of beams are transformed into initial-value problems and analyzed by the shooting method.

Free vibration of simply supported FG beam was investigated by M. Aydogdu and V. Taskin in 2007 [5]. Applying Hamilton's principle Aydogdu and his colleague found out the governing equations and frequencies were obtained based on Navier type solution method.

An analysis of the stability of circular cylindrical columns/beams composed of FG materials was made where shear deformation is taken into account by Y. Huang and X.-F. Li in 2010 [6]. In this work the effects of radial gradient on buckling loads of elastic columns with circular cross-section made of FG materials are elucidated. The results obtained by Y. Huang and X. F. Li, show a very good agreement with the results of the Timoshenko beam theory or Reddy–Bickford beam theory.

2 BASIC EQUATIONS

The Poisson's ratio, v, across the beam thickness is assumed to be constant. When a beam is subjected to pure bending, there exists a fibre without any stresses which is referred to as the neutral axis (N.A.). In the functionally graded beam with rectangular section, the neutral axis does not coincide with the mid axis. In the current paper, stress, strain and mechanical properties of the functionally graded beam across the thickness of the beam are measured from the neutral axis; because this coordinate transformation will simplify the equations. Fig. 1 shows the neutral axis of a functionally graded beam with rectangular section which is located in position e from the mid axis.

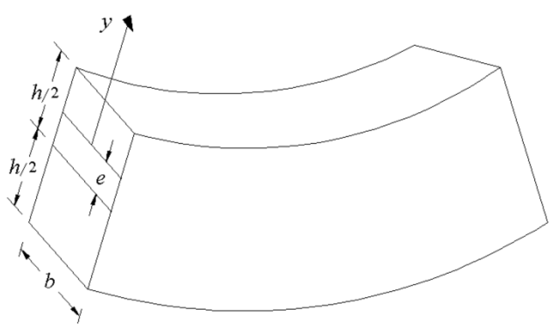

Fig. 1 Origin of y-coordinate for rectangular section

It is assumed that Young's modulus (i.e. E) varies along the thickness of the beam with rectangular section similar to the functionally graded plate (FGP), and follows the volume fraction definition as Eq. (1), where the subscripts m and c denote the metallic and ceramic constituents, respectively, and n is a material constant (which can take values greater than or equal to zero).

$$E(y) = (E_m - E_c)((h - 2y - 2e)/(2h))^n + E_c \qquad (1)$$

The FG beam is subjected to pure bending, thus the summation of all infinitesimal forces caused by bending stress (i.e. σ) must be set equal to zero.

$$\int_A \sigma(y)dA = -\int_A y\kappa E(y)dA = 0 \qquad (2)$$

In Eq. (2), A denotes the cross sectional area and κ is curvature of FG beam. Because of the same mechanical properties of prismatic FG beam in each section, the curvature of FG beam is constant. Therefore Eq. (2) reduces to Eq. (3).

$$\int_{-h/2-e}^{h/2-e} yE(y)dy = 0 \qquad (3)$$

After substituting Eq. (1) by Eq. (3), and after some mathematical manipulations, Eq. (3) gives us the amount of e for FG beam with rectangular section as bellow:

$$e = nh(E_c - E_m)/((4 + 2n)(E_m + nE_c)) \qquad (4)$$

Equation (5) shows the curvature of FG beam with rectangular section.

$$\kappa = M\Big/\int_A y^2 E(y)dA \qquad (5)$$

In the above equation, M denotes bending moment in cross section and is constant. After substituting Eq. (4) in denominator of Eq. (5), the curvature of FG beam with rectangular section will be obtained as bellow:

$$\kappa = M/E_R^* I \qquad (6)$$

In Eq. (6) the parameter I is moment of inertia for rectangular homogeneous beam and is equal to $bh^3/12$. The parameter E_R^* is presented as bellow:

$$E_R^* = \left(A_1E_c^2 + A_2E_cE_m + 12E_m^2\right)/\left(A_3E_c + A_4E_m\right) \qquad (7)$$

In the above equation, the constants A_1 through A_4 are defined as follows:

$$\begin{aligned} A_1 &= n^4 + 4n^3 + 7n^2 \\ A_2 &= 4n^3 + 16n^2 + 28n \\ A_3 &= (n^2 + 5n + 6)(n^2 + 2n) \\ A_4 &= (n^2 + 5n + 6)(n + 2) \end{aligned} \qquad (8)$$

It is assumed that Young's modulus varies along the thickness of FG beam with annular section similar to the functionally graded cylindrical reservoir, and follows the power law as in Eq. (9), where the E_0 and p are constants and both are dependant to the fabrication method of FG material (p can takes negative real values). Also the origin of polar coordinate is the center of the ring, and parameter r is radius of the cross section in polar coordinate.

$$E(r) = E_0 r^p \qquad (9)$$

In FG beam with annular section, r_1 and r_2 are inner and outer radiuses respectively. Equation (10) shows that in FG beam with annular section similar to the FG beam with rectangular section the summation of all infinitesimal forces vanish.

$$\int_0^{2\pi}\int_{r_1}^{r_2} (r\sin\theta)E(r)r\,dr\,d\theta = 0 \qquad (10)$$

Equation (11) shows the curvature of FG beam with annular section.

$$\kappa = M\Big/\int_A (r\sin\theta)^2 E(r)dA \qquad (11)$$

After substituting Eq. (9) in denominator of Eq. (11), the curvature of FG beam with annular section will be obtained as bellow:

$$\kappa = M/E_A^* I \qquad (12)$$

In Eq. (12) the parameter I is moment of inertia for annular homogeneous beam with inner and outer radiuses equal to r1 and r2 respectively and is equal to $\pi(r_2^4 - r_1^4)/4$. The parameter E_A^* is as bellow:

$$E_A^* = 4E_0\left(r_2^{p+4} - r_1^{p+4}\right)/\left((p + 4)(r_2^4 - r_1^4)\right) \qquad (13)$$

The external work done by compressive axial load can be determined by Eq. (14). In the Eq. (14) P, L and Y are the axial load, length and deflection of the FG beam respectively. Y' denotes the first derivative of deflection with respect to x. The x axis locates at the neutral axis and is parallel to the neutral axis.

$$W_E = P\Delta L = P\left(\int_0^L \sqrt{1+\left(Y'\right)^2}\,dx - L\right) \quad (14)$$

After approximating above integration by two terms of Taylor series, Eq. (14) was changed as bellow:

$$W_E = \int_0^L P\left(Y'\right)^2 / 2\,dx \quad (15)$$

The internal work can be shown as following equation. Y'' denotes the second derivative of deflection with respect to *x*.

$$W_I = \int_0^L MY'' / 2\,dx \quad (16)$$

In above equation *M* can be replaced by the equivalent expressions in Eqs. (6) and (12) for the FG beams with rectangular and annular cross sections respectively. The curvature of the FG beam i.e. κ is equal to Y'' approximately. By using conservation of energy the axial load i.e. *P*, can be obtained as bellow:

$$P = \left(E^*_{FG} I\right)\int_0^L \left(Y''\right)^2 dx \Big/ \int_0^L \left(Y'\right)^2 dx \quad (17)$$

The parameter E^*_{FG} for FG beams with rectangular and annular sections is equal to Eqs. (7) and (13) respectively. Also the parameter *I* is moment of inertia for homogeneous beams with similar shape and dimensions.

3 FORMULATION

For transforming axial load in Eq. (17) to the critical buckling load, Eq. (18) must be satisfied.

$$\partial\left(\int_0^L \left(Y''(x)\right)^2 dx \Big/ \int_0^L \left(Y'(x)\right)^2 dx\right) \Big/ \partial Y(x) = 0 \quad (18)$$

In the bellow equation, $\eta(x)$ is an arbitrary function that at least has one derivative and $\eta(0) = \eta(L) = 0$.

$$\ddot{y}(x) = y(x) + \xi\eta(x) \quad (19)$$

The value of $\int_0^L f dx$ in the bellow integration is minimum if ξ approaches to zero, in which *f* is assumed to be a function of x, $y(x)$, and $y''(x)$ in general.

$$\lim_{\xi \to 0} \partial\int_0^L f\left(x, \ddot{y}, \ddot{y}', \ddot{y}''\right)dx \Big/ \partial\xi \quad (20)$$

After some manipulations, equation (20) was changed to Euler–Lagrange equation.

$$g(f) = f_Y - \partial f_{Y'} / \partial x + \partial^2 f_{Y''} / \partial x^2 = 0 \quad (21)$$

In Eq. (21) f_y, $f_{y'}$ and $f_{y''}$ are partial derivatives of function f with respect to y, y' and y'' respectively. For numerator and denominator in Eq. (17), the function f in Eq. (21) was assumed equal to $f_n = (y'')^2$ and $f_d = (y')^2$ respectively. Therefore $g(f_d)P = g(f_n)E^*_{FG}I$ and the ordinary differential equation can be obtained as follows:

$$d^4Y(x)/dx^4 + \left(P/E^*_{FG}I\right)d^2Y(x)/dx^2 = 0 \quad (22)$$

After replacing E instead of E^*_{FG} (i.e. *n*=*p*=0), the above ODE is changed into an especial case of Duffing equation for homogeneous beam without oscillation. Therefore the validity of this new work was proved. Solution for the above ODE is as follows:

$$Y(x) = c_0 + c_1 x + c_2 \sin kx + c_3 \cos kx \quad (23)$$

In which *k* is equal to $(P/E^*_{FG})^{1/2}$.

4 RESULTS AND DISCUSSION

In Fig. 2(a), buckling of simply supported FGB was considered. By satisfying boundary conditions for first mode shape of buckling in Fig. 2(a), the constants c_0, c_1 and c_3 were omitted. In this figure at $x = L/2$, dimensionless deflection (i.e. y/c_2) is equal to one. Therefore critical buckling load for this case is equal to $E^*_{FG}I(\pi/L)^2$. In Fig. 2(b), buckling of clamped FGB was considered. By satisfying boundary conditions, dimensionless deflection for this case is $\pm(\cos(2\pi x/L) - 1)$. Therefore critical buckling load for clamped FGB is $4E^*_{FG}I(\pi/L)^2$. In Fig. 2(c), buckling of simply supported FGB with freedom for drift at one end was considered. Dimensionless deflection for this case is $\pm(\cos(\pi x/(2L))-1)$ and critical buckling load is $E^*_{FG} I(\pi/L)^2/4$. In Fig. 2(d), buckling of clamped FGB with freedom for drift at one of the ends was considered. Dimensionless deflection for this case is $\pm(\cos(\pi x/L)-1)$ and critical buckling load is $E^*_{FG} I(\pi/L)^2$. Critical load for FG beam with pinned and clamped ends after satisfying boundary conditions can be obtained. For this case a nontrivial solution exists when $\tan\sqrt{P/E^*_{FG}} = \sqrt{P/E^*_{FG}}$. The lowest amount of P is $2.04575\ E^*_{FG} I(\pi/L)^2$.

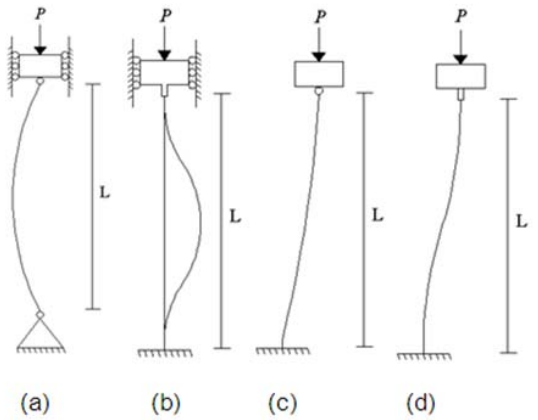

Fig. 2 Buckling of FGB with various boundary conditions

Buckling critical load for FG beam under the same condition is E^*_{FG}/E times the homogeneous beam. Therefore the ratio of the mentioned expression for comparison between FG and homogeneous beams with similar boundary condition, shape and dimensions must be elaborated. For FG beam with rectangular cross section the ratio of $\overline{P}=E^*_{FG}/E$ is dependant to the ratio of $\overline{E}=E_m/E_c$. If $\overline{E}$ is greater than one then FG beam has a greater buckling critical load with respect to the homogeneous beam made of ceramic or vice versa. If the ratio of $\overline{E}$ is less than one then FG beam has greater buckling critical load with respect to the homogeneous metallic beam or vice versa. For the case that $\overline{E}=1$, the ratio of $\overline{P}$ is equal to one The dimensionless plot in Fig. 3 shows three sample diagrams corresponding to ratios of $\overline{E}$ less than, equal to and greater than one. Fig. 3 shows the comparison between critical buckling loads of FG beam and homogeneous beam made of ceramic.

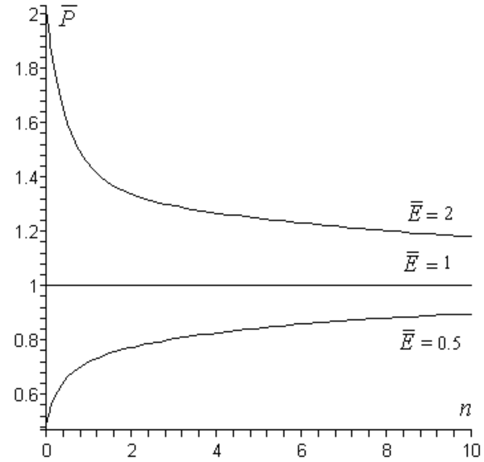

Fig. 3 $\overline{P}$ for rectangular ceramic beam

Fig. 4 shows comparison between critical buckling loads of FG beam and homogeneous metallic beam. In Fig. 3 by approaching n to infinity the dimensionless parameter $\overline{P}$ approaches to 1 and for $n=0$ the parameter $\overline{P}$ is equal to $\overline{E}$. In Fig. 4 the variation of $\overline{P}$ with respect to n has occurred in a vice versa manner. By approaching n to infinity $\overline{P}$ approaches to $1/\overline{E}$ and for $n=0$ the parameter $\overline{P}$ is equal to 1.

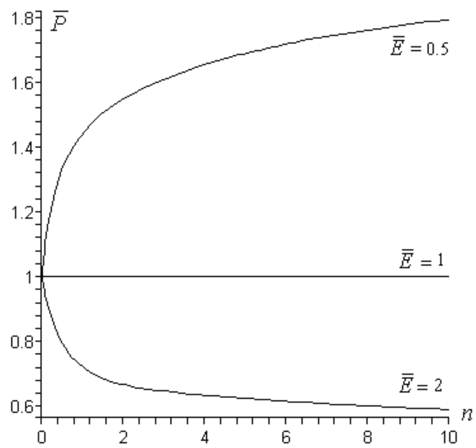

Fig. 4 $\overline{P}$ for rectangular metallic beam

Fig. 5 shows three sample diagrams corresponding to ratios of $\overline{E}$ less than, equal to and greater than one for dimensionless depth of neutral axis (i.e. e/h) vs. n. By approaching n to infinity, dimensionless depth of neutral axis in rectangular FG beam approaches to zero. The extreme amount of e/h occurs for $n=(2\overline{E})^{1/2}$.

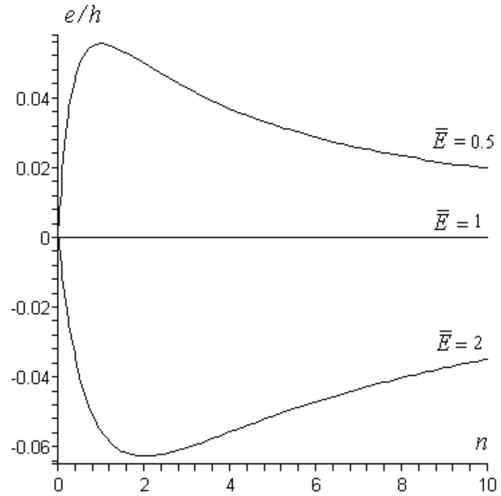

Fig. 5 Dimensionless depth of N.A. for rectangular FGB

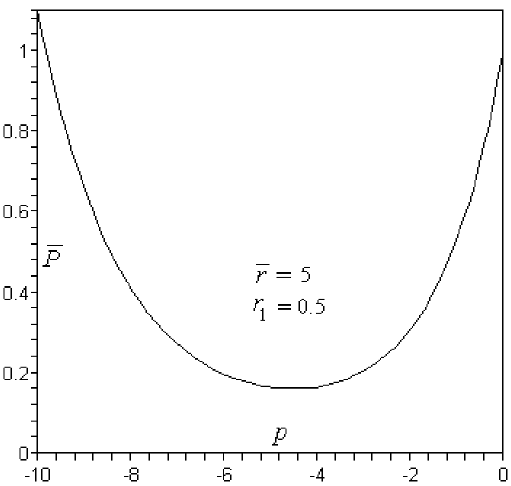

Fig. 7 Local minimum for $\overline{P}$ in annular FGB

Fig. 6 shows the comparison between critical buckling loads of FG annular beam and homogeneous beam. The parameter $\overline{P}$ in FG beam with annular section when *p* has approached to negative infinity approaches to zero for $r_1 \geq 1$ and approaches to infinity for $r_1 < 1$. For $r_1 < 1$ and little amounts of $\overline{r} = r_2/r_1$ the parameter $\overline{P}$ increases as *p* decreases. For $r_1 < 1$ and great amounts of $\overline{r}$ the parameter $\overline{P}$ has a local minimum. For *p=0* the FG beam with annular section is transformed into a homogeneous beam and parameter $\overline{P}$ for this case is equal to 1.

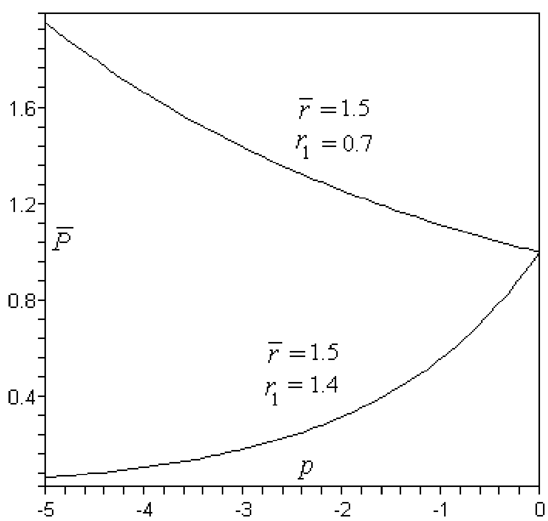

Fig. 6 $\overline{P}$ for FG and homogeneous annular beams

Fig. 7 shows the local minimum of dimensionless parameter $\overline{P}$ for $\overline{r} = 5$ and $r_1 = 0.5$.

5 CONCLUSION

This paper has presented a novel method for analyzing the buckling behavior of rectangular and annular FG beams. The main contributions of this research can be summarized as follows:

- Exact analytical solutions for buckling of functionally graded beams with rectangular and annular cross sections were presented. The critical buckling load and dimensionless first mode shape were obtained using variation calculus for arbitrary boundary conditions. Validity of solutions was proved by simplifying the results and obtaining well-known relations.

- This work proves that dimensionless first mode shape of buckling for prismatic functionally graded beams is similar to prismatic homogeneous beams. Because of symmetrical conditions for prismatic functionally graded beams, first mode shape in this case is similar to first mode shape of homogeneous beams and also is dependent upon boundary conditions of the beam.

- Increasing or decreasing in capacity of rectangular functionally graded beams in comparison to the capacity of rectangular homogeneous beams was investigated for all the possible conditions. Critical buckling load of rectangular functionally graded beam made of constituent volume fractions of metal and ceramic ($\overline{E}$=Em/Ec<1) has increased by increasing n. Critical buckling load of rectangular functionally graded beam for n >0 is between critical buckling load of metallic and ceramic beams.

- Increasing or decreasing the capacity of annular functionally graded beams in comparison to the capacity of annular homogeneous beams was precisely investigated for all the possible conditions. For $r_1 < 1$ and little amounts of r_2/r_1 critical buckling load of annular functionally graded beams with respect to critical buckling load of annular homogeneous beams has increased by decreasing p. For $r_1 < 1$ and great amounts of r_2/r_1 critical buckling load of annular functionally graded beams with respect to critical buckling load of annular homogeneous beams for little amounts of absolute of p decreases and afterwards increases after a local minima by increasing the absolute amount of p.

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