Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 8, No. 4, 2016 Article ID IJIM-00711, 10 pages
Research Article

On Optimal Quadrature Rule for Solving Fuzzy Fredholm Integral Equations

R. Ezzati *†, S. M. Sadatrasoul ‡

Abstract

In this paper, we present an efficient iterative procedure based on optimal fuzzy quadrature formula to solve fuzzy integral equations. Error estimation and the numerical stability analysis with respect to the choice of the first iteration are given. Some illustrative and comparative numerical experiments confirm the optimization of the successive method.

Keywords : Fuzzy integral equations; Optimal quadrature formula; Iterative methods; Numerical stability analysis.

1 Introduction

Fuzzy differential and integral equations are extremely important for studying and solving the wide variaty of problems in many topics in applied mathematics, particulary in relation to physics, medicine, biology, geography, economic, social sciences etc. Many practical problems in science and engineering can be transformed into fuzzy integral equations especially, the second kind of Fredholm type. The study of fuzzy integral equations have been started by Kaleva [16] and Siekkala [20] for the fuzzy Volterra integral equations. The Banach's fixed point principle is the powerful tool to investigate of the existence and uniqueness of the solution of the fuzzy integral equations. In [14, 18] sufficient conditions are given, under those conditions, solution of fuzzy integral equations are bounded.

So far, fuzzy integral equations are solved using numerical, analytical and numeric-analytic methods. Many researchers such as [3, 15, 17, 21, 22] have proposed various methods including iterative and direct methods for the numerical solution of these equations. As we know, iterative methods are an advantageous tool to solve these equations, in particular, methods based on quadrature formulas. There are many papers concerning the numerical computation of fuzzy integrals using various fuzzy quadrature formulas. In [3] author presented some quadrature rules for integrals of fuzzy-number-valued functions, in [10, 19] two numerical scheme for solution of nonlinear fuzzy Fredholm inegral equations using iterative method in one and two-dimensional case are proposed. Bica [4] proved the convergence of the method of successive approximations used to approximate the solution of nonlinear Hammerstein fuzzy integral equations, also in [6] he investigated the error estimation in the approximation of the solution of nonlinear fuzzy Fredholm integral equations. The authors of [7] introduced a successive iterative approach for two-dimensional

*Corresponding author. ezati@ kiau.ac.ir, Tel:+98(912)3618518.

†Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.

‡Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.

nonlinear Volterra-Fredholm integral equations. In this paper, we apply an optimal fuzzy quadrature formula to solve fuzzy integral equations. The rest of the paper is organized as follows: In Section 2, we review some elementary concepts of the fuzzy set theory such as continuity, integrability and boundedness of fuzzy-number-valued functions. In Section 3, we first introduce the optimal fuzzy quadrature formula and the error bound of this quadrature rule, then our numerical iterative method for approximating the solution of fuzzy integral equations based on optimal fuzzy quadrature formula for classes of Lipschitzian functions is presented. In addition, the convergence of this numerical methods and its numerical stability analysis with respect to the choice of the first iteration are proved in this section. Section 4 includes some numerical experiments that confirm the theoretical results and illustrate the accuracy of the numerical method.

2 Preliminaries

In this section, we briefly state some definition and results related to fuzzy numbers and fuzzy-number-valued functions from the literature, which will be referred to throughout this paper.

Definition 2.1 ([15]) *A fuzzy number is a function $u : R \to [0,1]$ satisfying the following properties:*
u is upper semicontinuous on R, $u(x) = 0$ outside of someinterval $[c,d]$, there are the real numbers a and b with $c \le a \le b \le d$, such that u is increasing on $[c,a]$, decreasing on $[b,d]$ and $u(x) = 1$ for each $x \in [a,b]$ and u is fuzzy convex set (that is $u(\lambda x+(1-\lambda)y) \ge \min\{u(x),u(y)\}, \forall x,y \in R, \lambda \in [0,1]$). The set of all fuzzy numbers denoted by R_F. Any real number $\alpha \in R$ can be interpreted as a fuzzy number $\tilde{\alpha} = \chi_{\{\alpha\}}$ and therefore $R \subset R_F$. Also, the neutral element respect to $\oplus$ in R_F denoted by $\tilde{0} = \chi_{\{0\}}$.

According to [11, 22] for any $0 < r \le 1$ an arbitrary fuzzy number is represented, in parametric form, by an ordered pair of functions (u_-^r, u_+^r), which satisfies the following properties:
u_-^r is bounded left continuous non-decreasing function over $[0,1]$, u_+^r is bounded left continuous non-increasing function over $[0,1]$ and $u_-^r \le u_+^r$.

Definition 2.2 ([12]) *For any $u \in R_F$ the r-level set of u is denoted by $[u]^r$ and defined by $[u]^r = \{x \in R \mid u(x) \ge r\}$, where $0 < r \le 1$. Also, $[u]^0 = \overline{\{x \in R : u(x) > 0\}}$ is the closure of the support of u u and is a compact set, where $\overline{A}$ denotes the closure of A. It follows that the level sets of u are closed and bounded intervals in R.*

It is well-known that the addition and multiplication operations of real numbers can be extended to R_F. In other words, for any $u, v \in R_F$ and $\lambda \in R$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by $[u \oplus v]^r = [u]^r + [v]^r$, $[\lambda \odot u]^r = \lambda[u]^r$, for all $r \in [0,1]$, where $[u]^r \oplus [v]^r$ means the usual addition of two intervals (as subset of R) and $\lambda[u]^r$ means the usual product between a scalar and a subset of R. We use the same symbol $\sum$ both for the sum of real numbers and for the sum $\oplus$ (when the terms are fuzzy numbers). Also, according to [1, 22], the following algebraic properties for any $u, v, w \in R_F$ hold:

(1) $u \oplus (v \oplus w) = (u \oplus v) \oplus w$,

(2) $u \oplus \tilde{0} = \tilde{0} \oplus u = u$,

(3) with respect to $\tilde{0}$, none $u \in (R_F - R), u \ne \tilde{0}$ has opposite in $(R_F, \oplus)$,

(4) $(a \oplus b) \odot u = a \odot u \oplus b \odot u, \forall a, b \in R$ with $ab \ge 0$,

(5) $a \odot (u \oplus v) = a \odot u \oplus a \odot v, \forall a \in R$,

(6) $a \odot (b \odot u) = (ab) \odot u, \forall a, b \in R$ and $1 \odot u = u$.

Definition 2.3 ([13, 22]) *For arbitrary fuzzy numbers $u = (u_-^r, u_+^r)$, $v = (v_-^r, v_+^r)$ the quantity $D(u,v) = \sup_{r\in[0,1]} \max\{|u_-^r - v_-^r|, |u_+^r - v_+^r|\}$ is the distance between u and v. It is demonstrated that (R_F, D) is a complete metric space and following properties hold:*

(1) $D(u \oplus w, v \oplus w) = D(u,v) \quad \forall\, u, v, w \in R_F$,

(2) $D(k \odot u, k \odot v) = |k|\, D(u,v) \quad \forall\, u, v \in R_F, \quad \forall\, k \in R$,

(3) $D(u \oplus v, w \oplus e) \le D(u,w) + D(v,e) \quad \forall\, u, v, w, e \in R_F$.

(4) $D(k_1 \odot u, k_2 \odot u) = |k_1 - k_2| D(u, \tilde{0}) \quad \forall k_1, k_2 \in R$ with $k_1 k_2 \ge 0$ and $\forall u \in R_F$,

Throughout this paper $D(.,\tilde{0})$ is denoted by $\|.\|_{\mathcal{F}}$. Also, according to [2, 3], the pair $(R_{\mathcal{F}}, D)$ is a commutative semigroup with $\tilde{0} = \chi_0$ zero element, but cannot be a group for pure fuzzy numbers. $\|.\|_{\mathcal{F}}$ has the properties of a usual norm on $R_{\mathcal{F}}$, i.e. $\|.\|_{\mathcal{F}}= 0$ iff $u = \tilde{0}$, $\|\lambda \odot u\|_{\mathcal{F}}= |\lambda|\|u\|_{\mathcal{F}}$ and $\|u \oplus v\|_{\mathcal{F}}\leq \|u\|_{\mathcal{F}}+\|v\|_{\mathcal{F}}$. Moreover, $|\|u\|_{\mathcal{F}}-\|v\|_{\mathcal{F}}|\leq D(u,v)$ and $D(u,v) \leq \|u\|_{\mathcal{F}}+\|v\|_{\mathcal{F}}$ for any $u, v \in R_{\mathcal{F}}$.

According to [5] and with our notation, if $u = (u_-^r, u_+^r)$, $v = (v_-^r, v_+^r)$ are fuzzy numbers, then $u \oplus v = ((u_-^r, u_+^r) \oplus (v_-^r, v_+^r)) = (u_-^r + v_-^r, u_+^r + v_+^r) = ((u+v)_-^r, (u+v)_+^r)$. Also, $u \odot v = ((u_-^r, u_+^r) \odot (v_-^r, v_+^r)) = (u_-^r.v_-^r, u_+^r.v_+^r) = ((u.v)_-^r, (u.v)_+^r)$, for all u, v with positive supports. In addition, for all strictly increasing and positive real function ψ, we have $\psi(u) = \psi(u_-^r, u_+^r) = (\psi(u_-^r), \psi(u_+^r))$.

Definition 2.4 *A fuzzy-number-valued function $f : A = [a,b] \to R_{\mathcal{F}}$ is said to be continuous at $s_0 \in A$ if for each $\varepsilon > 0$ there is $\delta > 0$ such that $D(f(s), f(s_0)) < \varepsilon$ whenever $|s - s_0|< \delta$. If f continuous for each $s \in A$ then we say that f is continuous on A.*

A fuzzy number $v \in R_{\mathcal{F}}$ is an upper bound for a fuzzy-number-valued function $f : A \to R_{\mathcal{F}}$ if $f(s)_-^r \leq v_-^r$ and $f(s)_+^r \leq v_+^r$ for all $s \in A$, $r \in [0,1]$. Similarly, A fuzzy number $u \in R_{\mathcal{F}}$ is a lower bound for a fuzzy-number-valued function $f : A \to R_{\mathcal{F}}$ if $u_-^r \leq f(s)_-^r$ and $u_+^r \leq f(s)_+^r$ for all $s \in A$, $r \in [0,1]$. A fuzzy-number-valued function $f : A = [a,b] \to R_{\mathcal{F}}$ is said to be bounded if it has a lower bound and upper bound.

Lemma 2.1 ([1]) *If $f, g : A = [a,b] \subseteq R \to R_{\mathcal{F}}$ are fuzzy continuous functions, then the function $F : A \to R_+$ defined by $F(x) = D(f(x), g(x))$ is continuous on A.*

Also, $D(f(x),\tilde{0}) \leq M$, $\forall x \in A, M > 0$, that is f is bounded. Equivalently we get $\chi_{-M} \leq f(x) \leq \chi_M$, $\forall x \in [a,b]$.
On the set $C(A, R_{\mathcal{F}}) = \{f : A \to R_{\mathcal{F}} : f \text{ is continuous}\}$, we define

$$D^*(f,g) = \sup D(f(t), g(t)),\ \forall f, g \in C(A, R_{\mathcal{F}}).$$

It is obvious that $(C(A, R_{\mathcal{F}}), D^*)$ is a complete metric space.

Definition 2.5 *In [13, 15, 21] the fuzzy Riemann integral for a fuzzy-number-valued function f on $[a,b]$ was introduced. For $\Delta_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$, a partition of the interval $[a,b]$, we define the modulus of the partition Δ_n by $\|\Delta_n\|= \sup_{1\leq i\leq n}|x_i - x_{i-1}|$. The function called fuzzy Riemann integrable on $[a,b]$ to $I \in R_{\mathcal{F}}$ if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that for any Δ_n satisfying $\|\Delta_n\|< \delta$ and any intermediate points $\xi_i \in [x_{i-1}, x_i]$, $i = 1, ..., n$, we have*

$$D\left(\sum_{i=1}^{n}(x_i - x_{i-1}) \odot f(\xi_i), I\right) < \varepsilon.$$

Then I is called the fuzzy Riemann integral of f on $[a,b]$ and denoted by $(FR)\int_a^b f(t)dt$. If f is fuzzy Riemann integrable to $I \in R_{\mathcal{F}}$, then for each $r \in [0,1]$, the crisp functions $f_-^r(x), f_+^r(x)$ are Riemann integrable on $[a,b]$ to I_-^r, I_+^r, respectively. If f is continuous on $[a,b]$, then for each $r \in [0,1]$, $f_-^r(x), f_+^r(x)$ are continuous on $[a,b]$, and hence, f is fuzzy Riemann integrable on $[a,b]$.

Lemma 2.2 ([22, 13]) *If $f \in C([a,b], R_{\mathcal{F}})$, its definite integral exists, and also, $((FR)\int_a^b f(t)dt)_-^r = \int_a^b f_-^r(t)dt$ and $((FR)\int_a^b f(t)dt)_+^r = \int_a^b f_+^r(t)dt$, for all $r \in [0,1]$.*

Lemma 2.3 ([3]) *If $f : [a,b] \to R_{\mathcal{F}}$ is an integrable bounded function, then for any fixed $u \in [a,b]$, the function $\eta_u : [a,b] \to R_+$, defined by $\eta_u(x) = D(f(u), f(x))$ is Lebesgue integrable on $[a,b]$.*

Lemma 2.4 ([1]) *If f and g are (FR)-integrable functions on $[a,b]$ and $D(f(x), g(x))$ is Lebesgue integrable, then*

$$D((FR)\int_a^b f(x)dx, (FR)\int_a^b g(x)dx) \leq (L)\int_a^b D(f(x), g(x))dx. \tag{2.1}$$

Theorem 2.1 ([4]) *If $f \in C([a,b] \times [a,b], R_{\mathcal{F}})$, $g \in C([a,b], R_{\mathcal{F}})$, and $h \in C([a,b], R_+)$ then the functions $h.g : [a,b] \to R_{\mathcal{F}}$ and $U : [a,b] \to R_{\mathcal{F}}$ given by $(h.g)(t) = h(t) \odot g(t)$, $\forall t \in [a,b]$ and $U(t) = (FR)\int_a^b f(t,s)ds$ are continuous.*

Definition 2.6 *For $0 < \alpha \leq 1$, a function $f : [a, b] \to R_F$ is said to be L-Lipschitz of order α if $D(f(x), f(y)) \leq L|x-y|^\alpha$ for any $x, y \in [a, b]$.*

Similarly, a function $g : [a, b] \times [c, d] \to R$ is said to be L-Lipschitz of order α if $|g(x, y) - g(s, t)| \leq L((x-s)^2 + (y-t)^2)^{\frac{\alpha}{2}}$ for any $(x, y), (s, t) \in [a, b] \times [c, d]$. According to [3, 9], it is clear that any Lipschitz function with constant L and order α is continuous.

Remark 2.1 *It is easy to see that if $F \in C([a, b], R_F)$ and $K \in C([a, b] \times [a, b], R_+)$, be Lipschitz functions with constants L_1, L_2 and orders α_1, α_2, respectively, then the function $V : [a, b] \times [a, b] \to R_F$ given by $V(s, t) = K(s, t) \odot F(s)$, is L-Lipschitz of order α, i.e. there exist $L > 0$ and $0 < \alpha \leq 1$, such that*

$$D(V(x, y), V(s, t)) \leq L((x-s)^2 + (y-t)^2)^{\frac{\alpha}{2}},$$

for all $(x, y), (s, t) \in [a, b] \times [a, b]$.

Corollary 2.1 **([3, 6])** *Let $f : [a, b] \to R_F$ be an L-Lipschitz function. Then:*

$$D\Big((FR) \int_a^b f(t)dt, [(x-a) \odot f(u) \oplus (b-x) \odot f(v)]\Big) \leq L\Big[\frac{(b-a)^2}{4} + (x - \frac{a+b}{2})^2\Big],$$

for any $x \in [a, b], u \in [a, x]$ and $v \in [x, b]$. Choosing $u = a$, $v = b$ and $x = (a+b)/2$, we obtain the fuzzy trapezoidal quadrature formula

$$D\Big((FR) \int_a^b f(t)dt, [\frac{b-a}{2} \odot f(a) \oplus \frac{b-a}{2} \odot f(b)]\Big) \leq L\frac{(b-a)^2}{2},$$

which can be extended for uniform partitions, $\Delta : a = t_0 < t_1 < ... < t_n = b$ with $t_i = a + ih$, $h = (b-a)/n$, $\forall i = 0, 1, ..., n$, it can be seen that for uniform partitions, the following trapezoidal inequality holds true:

$$D\Big((FR) \int_a^b f(t)dt, \sum_{i=1}^{n-1} \frac{(t_{i+1} - t_i)}{2} \odot [f(t_i) \oplus f(t_{i+1})]\Big) \tag{2.2}$$

$$\leq L\frac{(b-a)^2}{4n}. \tag{2.3}$$

3 Optimal fuzzy quadrature formula

In [23], author found out the optimal fuzzy quadrature formula with given knots among all fuzzy quadrature formulas for classes of fuzzy number valued functions of Lipschitz type as follows:

Theorem 3.1 **([23])** *Let $f : [a, b] \to R_F$ be a fuzzy Riemann integrable and L-Lipschitz of order α function, then the following fuzzy quadrature formula*

$$\widetilde{I}_n(f) = \Big((\frac{\xi_1 + \xi_2}{2} - a) \odot f(\xi_1)\Big) \oplus \Big(\sum_{i=2}^{n-1} \frac{\xi_{i+1} - \xi_{i-1}}{2} \odot f(\xi_i)\Big) \oplus \Big((b - \frac{\xi_{n-1} + \xi_n}{2}) \odot f(\xi_n)\Big), \tag{3.4}$$

to approximate $(FR) \int_a^b f(x)dx$ has the minimal error among all quadrature formulas that use given knots $a \leq \xi_1 < ... < \xi_n \leq b$.

Remark 3.1 *Note that $\widetilde{I}_n(f)$ is the fuzzy variant of classical trapezoidal rule, which is based on the fuzzy data $f(\xi_i)$, the assumptions that $f(a) = f(\xi_1)$, $f(b) = f(\xi_n)$, and the formula*

$$\widetilde{I}_n(f) = ((\xi_1 - a) \odot f(\xi_1)) \oplus \sum_{i=2}^{n-1} \frac{\xi_{i+1} - \xi_i}{2} \odot (f(\xi_i) \oplus f(\xi_{i+1})) \oplus (b - \xi_n) \odot f(\xi_n)).$$

Theorem 3.2 (**[23]**) *Let $f : [a,b] \to R_{\mathcal{F}}$ be a fuzzy Riemann integrable function on $[a,b]$ of Lipschitz type with constant L and order α, then the following fuzzy variant of classical trapezoidal rule*

$$\widetilde{S}_n(f) = \frac{b-a}{n} \odot \sum_{i=1}^{n} f(a + \frac{(2i-1)(b-a)}{2n}), \tag{3.5}$$

is the optimal fuzzy quadrature formula for $(FR)\int_a^b f(x)dx$ among all formulas (3.4). Also, we have

$$\begin{aligned} &E_n(\widetilde{S}_n(f), (FR)\int_a^b f(x)dx) \\ &= \sup_{x\in[a,b]} D(\widetilde{S}_n(f), (FR)\int_a^b f(x)dx) \\ &\leq \frac{L(b-a)^{\alpha+1}}{(\alpha+1)2^\alpha n^\alpha}. \end{aligned} \tag{3.6}$$

3.1 Main Results

Now, we apply the fuzzy optimal quadrature formula to obtain numerical solution of the fuzzy integral equations of Fredholm type

$$F(t) = f(t) \oplus \lambda \odot (FR)\int_a^b K(s,t) \odot F(s)ds, \tag{3.7}$$

where $\lambda > 0, K(s,t)$ is a crisp kernel function on square $[a,b] \times [a,b]$, and $f : [a,b] \to R_{\mathcal{F}}$.

Theorem 3.3 (**[8]**) *Let $K(s,t)$ be continuous for $a \leq s,t \leq b$, $\lambda > 0$ and $f(t)$ be a continuous fuzzy function of t. If $\Gamma = \lambda M(b-a) < 1$, where $M = \max_{a\leq s,t\leq b}|K(s,t)|$, then the integral equation (3.7) has unique solution in $C([a,b], R_{\mathcal{F}})$, F^*, and the following sequence of successive approximations*

$$\begin{aligned} F_m(t) = f(t) \oplus \lambda \odot (FR)\int_a^b K(s,t) \odot F_{m-1}(s)dt, \\ \forall s \in [a,b], \end{aligned} \tag{3.8}$$

where $F_m \in C([a,b], R_{\mathcal{F}}), m \in N^*$, converges to F^* in $C([a,b], R_{\mathcal{F}})$ for any choice of $F_0 \in C([a,b], R_{\mathcal{F}})$. Also, the following error estimates hold true:

$$D(F^*(t), F_m(t)) \leq \frac{\Gamma^{m+1}}{1-\Gamma} M_0, \tag{3.9}$$

where $M_0 = \|f\|_{\mathcal{F}}$. In addition, the sequence of successive approximation (3.8) is uniformly bounded and solution F^* is bounded too.

Here, we introduce a numerical method to solve (3.7) using optimal fuzzy quadrature formula for classes of fuzzy-number-valued functions of Lipschitz type. For any partition $a \leq \xi_1 < \xi_2 < ... < \xi_n \leq b$ of interval $[a,b]$, let $\sigma = (b-a)/n$. We apply fuzzy quadrature formula (3.5) in the computation of the terms of the sequence of successive approximations (3.8).

Theorem 3.4 *Let $K(s,t)$ be the kernel having constant sign on $[a,b]\times[a,b]$, $\lambda > 0$, f is Lipschitz function with constant L and order α and $M =$* $\max_{a\leq s,t\leq b}|K(s,t)|$.

Let us consider $\tilde{F}_0(t) = f(t)$ and

$$\begin{aligned} \tilde{F}_m(t) &= f(t) \\ &\oplus \lambda\sigma \odot \sum_{i=0}^{n} K(a + \frac{\sigma}{2}(2i-1), t) \\ &\odot \tilde{F}_{m-1}(a + \frac{\sigma}{2}(2i-1)), \end{aligned} \tag{3.10}$$

then the above iterative procedure converges to the unique solution of (3.7), F^*, as $m, n \to \infty$, and the following error estimates hold true:

$$\begin{aligned} D^*(F^*, \tilde{F}_m) &\leq \frac{\Gamma^{m+1}}{1-\Gamma} M_0 \\ &+ \frac{L\sigma^\alpha}{(1-\Gamma)(\alpha+1)2^\alpha}(b-a), \end{aligned} \tag{3.11}$$

where $M_0 = \|f\|_{\mathcal{F}}$.
Proof. Since

$$\begin{aligned} &D(F^*(t), \tilde{F}_m(t)) \\ &\leq D(F^*(t), F_m(t)) + D(F_m(t), \tilde{F}_m(t)), \end{aligned} \tag{3.12}$$

and in (3.9) the a priori estimate is

$$D(F^*(t), F_m(t)) \leq \frac{\Gamma^{m+1}}{1-\Gamma} M_0, \tag{3.13}$$

it is remain to obtain the estimates for $D(F_m(t), \tilde{F}_m(t))$. From (3.8) for $m = 1$ and by (3.10), (3.18) we obtain

$$\begin{aligned} &D(F_1(t), \tilde{F}_1(t)) \leq D(R_{1,n}, \tilde{0}) \\ &\leq \frac{L\sigma^\alpha}{(\alpha+1)2^\alpha}(b-a). \end{aligned} \tag{3.14}$$

Table 1: Numerical errors on the level sets for Example 4.1 at $t_0 = 1.5$.

	n=10, m=5			
r	$E_-^r(opti)$	$E_-^r(class)$	$E_+^r(opti)$	$E_+^r(class)$
0.00	0.9e-8	0.7e-7	4.4e-5	2.0e-4
0.25	4.9e-6	2.5e-5	3.9e-5	1.8e-4
0.50	5.3e-6	5.1e-5	2.0e-5	1.5e-4
0.75	7.7e-6	7.6e-5	1.6e-5	1.3e-4
1.00	8.7e-6	1.0e-4	8.7e-6	1.0e-4

Table 2: Numerical errors on the level sets for Example 4.1 at $t_0 = 1.5$

	n=100, m=7			
r	$E_-^r(opti)$	$E_-^r(class)$	$E_+^r(opti)$	$E_+^r(class)$
0.00	0.8e-9	0.5e-8	8.0e-6	6.8e-5
0.25	5.1e-7	0.9e-5	7.1e-6	6.0e-5
0.50	8.8e-7	1.7e-5	5.9e-6	5.1e-5
0.75	1.3e-6	2.6e-5	5.6e-6	4.3e-5
1.00	3.7e-6	3.4e-5	3.7e-6	3.4e-5

Table 3: Numerical errors on the level sets for Example 4.2 at $t_0 = 0.5$.

	n=20, m=10			
r	$E_-^r(opti)$	$E_-^r(class)$	$E_+^r(opti)$	$E_+^r(class)$
0.00	5.21e-5	3.43e-4	7.21e-5	5.62e-4
0.25	5.43e-5	3.76e-4	6.97e-5	5.59e-4
0.50	5.82e-5	4.28e-4	6.89e-5	5.33e-4
0.75	6.01e-5	4.70e-4	6.56e-5	5.11e-4
1.00	6.46e-5	4.95e-4	6.46e-6	4.95e-4

Table 4: Numerical errors on the level sets for Example 4.2 at $t_0 = 0.5$.

	n=200, m=30			
r	$E_-^r(opti)$	$E_-^r(class)$	$E_+^r(opti)$	$E_+^r(class)$
0.00	6.66e-10	8.02e-7	6.54e-9	1.19e-6
0.25	6.87e-10	8.23e-7	4.07e-9	9.92e-7
0.50	7.13e-10	8.39e-7	3.41e-9	9.72e-7
0.75	8.11e-10	8.68e-7	1.52e-9	9.10e-7
1.00	1.31e-9	8.80e-7	1.31e-9	8.80e-7

Considering (3.10) and (3.17) we have

$$D(F_m(t), \tilde{F}_m(t)) = D\Big(f(t)$$

$$\oplus \lambda \odot (FR) \int_a^b K(s,t) \odot F_{m-1}(s)ds,$$

$$f(t) \oplus \lambda \odot (FR) \int_a^b K(s,t) \odot \tilde{F}_{m-1}(s)ds \Big)$$

$$\leq D(R_{m,n}, \tilde{0}) + \lambda\sigma D\Big($$

$$\sum_{i=0}^{n} K(a + \frac{\sigma}{2}(2i-1), t) \odot F_{m-1}(a + \frac{\sigma}{2}(2i-1)),$$

$$\sum_{i=0}^{n} K(a + \frac{\sigma}{2}(2i-1), t) \odot \tilde{F}_{m-1}(a + \frac{\sigma}{2}(2i-1))\Big)$$

$$\leq \frac{L\sigma^{\alpha}}{(\alpha+1)2^{\alpha}}(b-a)$$

$$+ \lambda\sigma M \sum_{i=0}^{n} D\Big(F_{m-1}(a + \frac{\sigma}{2}(2i-1)),$$

$$\tilde{F}_{m-1}(a+\frac{\sigma}{2}(2i-1))\Big).$$

From above inequality and (3.14) for $m = 2$, it follows that

$$D(F_2(t),\tilde{F}_2(t)) \leq \frac{L\sigma^\alpha}{(\alpha+1)2^\alpha}(b-a)+\lambda\sigma M$$
$$\sum_{i=0}^{n} D\Big(F_1(a+\frac{\sigma}{2}(2i-1)),\tilde{F}_1(a+\frac{\sigma}{2}(2i-1))\Big)$$
$$\leq \frac{L\sigma^\alpha}{(\alpha+1)2^\alpha}(b-a)$$
$$+\lambda\sigma M(\frac{L\sigma^\alpha}{(\alpha+1)2^\alpha}(b-a)).n$$
$$\leq \frac{L\sigma^\alpha}{(\alpha+1)2^\alpha}(b-a)(1+\lambda M(b-a)),$$

therefore

$$D(F_2(t),\tilde{F}_2(t)) \leq \frac{L\sigma^\alpha}{(\alpha+1)2^\alpha}(b-a)(1+\Gamma). \quad (3.15)$$

By induction, for $m \geq 3$, we derive that

$$D(F_m(t),\tilde{F}_m(t)) \leq \frac{L\sigma^\alpha}{(\alpha+1)2^\alpha}(b-a)(1+\Gamma+\Gamma^2+...+\Gamma^{m-1}) = \frac{L\sigma^\alpha}{(\alpha+1)2^\alpha}(b-a)\frac{1-\Gamma^m}{1-\Gamma}.$$

Since $\Gamma < 1$, we obtain

$$D(F_m(t),\tilde{F}_m(t)) \leq \frac{L\sigma^\alpha}{(1-\Gamma)(\alpha+1)2^\alpha}(b-a), \quad (3.16)$$

hence, from (3.12), (3.13) and (3.16) the required inequality (3.11) follows. That is

$$D^*(F^*,\tilde{F}_m) \leq \frac{\Gamma^{m+1}}{1-\Gamma}M_0 + \frac{L\sigma^\alpha}{(1-\Gamma)(\alpha+1)2^\alpha}(b-a).$$

Regarding to $\Gamma < 1$ and $\sigma = (b-a)/n$, it is easy to see that

$$\lim_{m,n\to\infty} D^*(F^*,\tilde{F}_m) = 0,$$

that shows the convergence of the method.

Remark 3.2 *Throughout the proof of this theorem we suppose that the functions $f, F, F_m, \tilde{F}_m$ and $R_{m,n}$ are Lipschitz functions with constant L and order α such that L, α are the maximum of all constants and orders of these functions, respectively.*

Also, using (3.5), (3.6) we infer that

$$F_m(t) = f(t)\oplus\lambda\sigma\odot\sum_{i=0}^{n} K(a+\frac{\sigma}{2}(2i-1),t) \odot \tilde{F}_{m-1}(a+\frac{\sigma}{2}(2i-1))+R_{m,n}, \quad (3.17)$$

and

$$D(R_{m,n},\tilde{0}) \leq \frac{L\sigma^\alpha}{(\alpha+1)2^\alpha}(b-a). \quad (3.18)$$

3.2 The numerical stability analysis

In order to investigate the numerical stability of the computed values with respect to small perturbations in the first iteration we consider an another first iteration term $G_0 \in C([a,b],R_F)$ such that there exists $\varepsilon > 0$ for which $D(F_0(t),G_0(t)) < \varepsilon$, for all $t \in [a,b]$. It should be noted that all the symbols and variables that we use in this subsection is quite similar before with respect to new first iteration term, and just for convenience we put a mark bar over the priori symbols. The new sequence of successive approximations is:

$$G_m(t) = f(t)\oplus\lambda\odot(FR)\int_a^b K(s,t)\odot G_{m-1}(s)dt,$$
$$\forall s \in [a,b]. \quad (3.19)$$

Using the same numerical method (3.5) to solve (3.7) we have $\tilde{G}_0(t) = G_0(t)$ and

$$\tilde{G}_m(t) = f(t)\oplus\lambda\sigma\odot\sum_{i=0}^{n} K(a+\frac{\sigma}{2}(2i-1),t) \odot \tilde{G}_{m-1}(a+\frac{\sigma}{2}(2i-1)). \quad (3.20)$$

Definition 3.1 *we say that the method of successive approximations applied to solve (3.7) is numerically stable with respect to the choice of the first iteration iff there exist positive numbers $p, q > 0$ and constants $k_1, k_2, k_3 > 0$ which are independent from $\sigma = (b-a)/n$ such that*

$$D(\tilde{F}_m(t),\tilde{G}_m(t)) < k_1\varepsilon + k_2\sigma^p + k_3\sigma^q. \quad (3.21)$$

To obtain the numerical stability using Theorem 3.4 we observe that

$$D(G_m(t),G_0(t)) \leq \frac{\Gamma}{1-\Gamma}\bar{M}_0,$$

$$D(G_m(t),\tilde{0}) \leq D(G_m(t),G_0(t)) + D(G_0(t),\tilde{0})$$
$$\leq \frac{\Gamma}{1-\Gamma}\bar{M}_0 + \bar{M}_0. \quad (3.22)$$

Also

$$D(G_m(t),\tilde{G}_m(t)) \leq \frac{\overline{L}\sigma^{\overline{\alpha}}}{(1-\Gamma)(\overline{\alpha}+1)2^{\overline{\alpha}}}(b-a). \quad (3.23)$$

From (3.20), (3.22) and (3.23) we obtain

$$D(\tilde{F}_m(t),\tilde{G}_m(t)) \leq D(\tilde{F}_m(t),F_m(t))$$
$$+ D(F_m(t),G_m(t)) + D(G_m(t),\tilde{G}_m(t))$$
$$\leq D(F_m(t),G_m(t))$$
$$+ \frac{L\sigma^{\alpha}}{(1-\Gamma)(\alpha+1)2^{\alpha}}(b-a)$$
$$+ \frac{\overline{L}\sigma^{\overline{\alpha}}}{(1-\Gamma)(\overline{\alpha}+1)2^{\overline{\alpha}}}(b-a),$$

and

$$D(F_0(t),G_0(t)) < \varepsilon,$$
$$D(F_1(t),G_1(t)) \leq D(f(t),f(t))$$
$$+ \lambda\int_a^b D(K(s,t)\odot F_0(s),K(s,t)\odot G_0(s))ds$$
$$\leq \lambda M\int_a^b D(F_0(s),G_0(s)) \leq \lambda M(b-a)\varepsilon.$$

By induction for $m \geq 2$, since $\Gamma = \lambda M(b-a) < 1$ we get

$$D(F_m(t),G_m(t))$$
$$\leq \lambda M\int_a^b D(F_{m-1}(s),G_{m-1}(s))ds$$
$$< (\lambda M(b-a))^m\varepsilon = \Gamma^m\varepsilon < \varepsilon.$$

Thus, for all $t \in [a,b]$ and $m \in N^*$ we conclude that

$$D(\tilde{F}_m(t),\tilde{G}_m(t)) \leq \varepsilon$$
$$+ \left[\frac{L(b-a)}{(1-\Gamma)(\alpha+1)2^{\alpha}}\right]\sigma^{\alpha}$$
$$+ \left[\frac{\overline{L}(b-a)}{(1-\Gamma)(\overline{\alpha}+1)2^{\overline{\alpha}}}\right]\sigma^{\overline{\alpha}}. \quad (3.24)$$

Comparing (3.24) and (3.21) follows

$$k_1 = 1,\ k_2 = \frac{L(b-a)}{(1-\Gamma)(\alpha+1)2^{\alpha}},$$
$$k_3 = \frac{\overline{L}(b-a)}{(1-\Gamma)(\overline{\alpha}+1)2^{\overline{\alpha}}},\ p = \alpha \text{ and } q = \overline{\alpha}.$$

So, stability of the numerical method is proved.

4 Numerical experiments

Now, we present two illustrative example to show the accuracy and the convergence of the method. Comparing the error between calculated numerical results in introduced optimal method and classical trapezoidal quadrature rule for Lipschitzian functions confirms the correctness of our claim. The methods were implemented using Mathematica.

Example 4.1 *[[10]] Consider the following linear fuzzy Fredholm integral equation:*

$$F(t) = f(t) \oplus (FR)\int_1^2 K(s,t)\odot F(s)ds,$$
$$f_-^r(t) = rt - \frac{3}{26}rt^2 - \frac{3}{52}r,$$
$$f_+^r(t) = 2t - rt - \frac{3}{13}t^2 - \frac{3}{26} + \frac{3}{26}rt^2 + \frac{3}{52}r,$$

and kernel $K(s,t) = \frac{s^2+t^2-2}{13}$ where $s,t \in [1,2]$ and $r \in [0,1]$. The exact solution in this case is given by $(F_-^r(t), F_+^r(t)) = (rt,(2-r)t)$. To compare the errors in optimal and classical rules with $n = 10, m = 5$ and $n = 100, m = 7$, see Tables 1-2.

Example 4.2 *The integral equation*

$$F(t) = (e^t - \frac{0.2(e^{t+1}-1)}{t+1})\odot\rho$$
$$\oplus\, 0.2\odot\int_0^1 e^{st}\odot F(s)ds,$$

where $\rho = (r,2-r)$, $s,t \in [0,1]$ and $0 \leq r \leq 1$ has the exact solution $F(t) = e^t\odot\rho$. Applying optimal and classic trapezoidal fuzzy quadrature rules we have calculated the error estimate between exact and approximate solutions at $t_0 = 0.5$. These results are shown in Tables 3-4.

5 Conclusion

In order to solve fuzzy integral equations using quadrature formulas hitherto many various of such formulas have been used. In this paper, we have applied the optimal fuzzy quadrature formula to approximate fuzzy integrals for classes of fuzzy-number-valued functions of Lipschitz type. The proof of convergence of optimal quadrature formula is discussed in Theorem 18 and the investigation of numerical stability of the iterative algorithm with respect to the choice of the first iteration is given. To confirm the accuracy of the optimal method two illustrative and comparative numerical examples are presented so that the numerical results indicate the validity of our claim.

References

[1] G. A. Anastassiou, Fuzzy mathematics: Approximation theory, *Springer-Verlag Berlin Heidelberg*, (2010).

[2] G. A. Anastassiou, S. G. Gal, On a fuzzy trigonometric approximation theorem of Weierstrass-type, *Journal of Fuzzy Mathematics* 9 (2001) 701-708.

[3] B. Bede, S. G. Gal, Quadrature rules for integrals of fuzzy-number-valued functions, *Fuzzy Sets and Systems* 145 (2004) 359-380.

[4] A. M. Bica, Constantin Popescu, Approximating the solution of nonlinear Hammerstein fuzzy integral equations, *Fuzzy Sets and Systems* 245 (2014) 1-17.

[5] A. M. Bica, Algebraic structure for fuzzy numbers from categorical point of view, *Soft Computing* 11 (2007) 1099-1105.

[6] A. M. Bica, Error estimation in the approximation of the solution of nonlinear fuzzy Fredholm integral equations, *Information Sciences* 178 (2008) 1279-1292.

[7] A. H. Borzabadi, M. Heidari, A successive iterative approach for two dimensional nonlinear Volterra-Fredholm integral equations, *Iranian Journal of Numerical Analysis and Optimization* 4 (2014) 95-104.

[8] W. Congxin, M. Ma, On the integrals, series and integral equations of fuzzy set-valued functions, *J. of Harbin Inst. of Technology* 21 (1990) 11-19.

[9] L.Coroianu, Lipschitz functions and fuzzy number approximations, *Fuzzy Sets and Systems* 200 (2012) 116-135.

[10] R. Ezzati, S. Ziari, Numerical solution of nonlinear fuzzy Fredholm integral equations using iterative method, *Applied Mathematics and Computation* 225 (2013) 33-42.

[11] M. Friedman, M. Ma, A. Kendal, Numerical solution of differential and integral equations, *Fuzzy Sets and Systems* 106 (1999) 35-48.

[12] M. Friedman, M. Ma, A. Kandel, Solutions to fuzzy integral equations with arbitrary kernels, *International Journal of Approximate Reasoning* 20 (1999) 249-262.

[13] S. G. Gal, Approximation theory in fuzzy setting, in: G.A. Anastassiou (Ed.), *Handbook of Analytic Computational Methods in Applied Mathematics*, Chapman & Hall, CRC Press, Boca Raton, London, New York, Washington DC, (2000) (Chapter 13).

[14] D. N. Georgiou, I. E. Kougias, Bounded solutions for fuzzy integral equations, *International Journal of Mathematics and Mathematical* 31 (2002) 109-114.

[15] R. Goetschel, W. Voxman, Elementary fuzzy calculus, *Fuzzy Sets and Systems* 18 (1986) 31-43.

[16] O. Kaleva, Fuzzy differential equations, *Fuzzy Sets and Systems* 24 (1987) 301-317.

[17] S. Nanda, On itegration of fuzzy mappings, *Fuzzy Sets and Systems* 32 (1989) 95-101.

[18] J. J. Nieto, R. Rodriguez-López, Bounded solutions for fuzzy differential and integral equations, *Chaos Solitons Fractals* 27 (2006) 1376-1386.

[19] S. M. Sadatrasoul, R. Ezzati, Quadrature rules and iterative method for numerical solution of two-dimensional fuzzy integral equations, *Abstract and Applied Analysis* (2014) Article ID 413570, 18 pages, http://dx.doi.org/10.1155/2014/413570/.

[20] S. Siekkala, On the fuzzy initial value problem, *Fuzzy Sets and Systems* 24 (1987) 319-330.

[21] H. C. Wu, The fuzzy Riemann integral and its numerical integration, *Fuzzy Sets and Systems* 110 (2000) 1-25.

[22] C. Wu, Z. Gong, On Henstock integral of fuzzy-number-valued unctions (I), *Fuzzy Sets and Systems* 120 (2001) 523-532.

[23] Xuezhi. Wu, Optimal fuzzy quadrature formula for classes of fuzzy-number-valued functions of Lipschitz type, *Fuzzy Sets and Systems* 223 (2013) 100-108.

Reza Ezzati received his PhD degree in applied mathematics from IAU Science and Research Branch, Tehran, Iran in 2006. He is an professor in the Department of Mathematics at Islamic Azad University, Karaj Branch, (Iran) from 2015. His current interests include numerical solution of differential and integral equations, fuzzy mathematics, especially, on solution of fuzzy systems, fuzzy integral equations, and fuzzy interpolation.

Seyyed Mohsen Sadat Rasoul was born in Karaj, Iran, in 1977. He received his B.S. degree in pure mathematics in 1999 from Tehran Teacher Education University and M.S. degree and Ph.D. degree from IAU- karaj branch in 2012 and 2015, respectively. He is currently the provisional member of board at the CFU- karaj campus. He has published several papers in international journals on fuzzy integral equations, also he has authored about 10 books on mathematics educations.