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Research Article

Classical Center Location Problem Under Uncertain Environment

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Abstract

This paper investigates the p-center location problem on a network in which vertex weights and distances between vertices are uncertain. The concepts of the α-p-center and the expected p-center are introduced. It is shown that the α-p-center and the expected p-center models can be transformed into corresponding deterministic models. Finally, linear time algorithms for finding the 1-center and 2-center of uncertain unweighted trees are proposed.

Keywords : Location Problem; p-center; Uncertainty theory; Uncertain Programming.

1 Introduction

Location problems are fundamental optimization models in operations research and play a considerable role in practice and theory. In a classical location problem, we want to find the optimal locations of one or more new facilities on a system (graph or space) in order to serve a given set of customers. One problem which was intensively considered in the literature is the p-center problem which can be stated as follows: Let $N = (V, E)$ be an undirected connected network with vertex set V and edge set E. The distance between two points on N is equal to the length of the shortest path connecting these two points. Each vertex is associated with a nonnegative weight that is the demand of the client at this vertex. In the p-center problem, p centers are to be located in a network, such that the maximum weighted distance from any demand point to its center is minimized.

Kariv and Hakimi [15] proved that the p-center problem on general networks is $\mathcal{NP}$-hard. For general networks, they proposed an $O(|E|^p|V|^{2p-1}\log|V|/(p-1)!)$ time algorithm to solve the weighted absolute p-center problem. Later, Tamir [28] improved the complexity bounds by using dynamic data structure, and showed that the weighted and unweighted p-center on networks can be found in $O(|E|^p|V|^p\log^2|V|)$ and $O(|E|^p|V|^p\log^3|V|)$ times. The p-center problems in tree networks are well studied [5, 8, 9, 15, 20, 21]. Bespamyatnikh, in [2], proposed efficient algorithms for centers in interval and circular-arc graphs. Frederickson and Johnson [9] solved the unweighted vertex p-center problem in cactus networks in $O(|V|.\log|V|)$ time. Also exact algorithms to solve the vertex p-center and capacitated vertex p-center problems are found in [26]. In the context of the p-center location problems, the interested reader is referred to [3, 6, 22, 23, 24]. Kariv and Hakimi [15] designed an $O(|E||V|.\log|V|)$ time algorithm for finding an absolute 1-center of a weighted network and an $O(|E||V|+|V|^2.\log|V|)$ time algorithm for find-

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ing an absolute 1-center of an unweighted network, provided that the distance matrix of the network is available. Moreover, they proposed an $O(|V|.\log|V|)$ time algorithm for the absolute (or vertex) 1-center location problem on weighted trees. Later, in [20], Megiddo showed that the weighted 1-center of a tree can be obtained in $O(|V|)$ time, since the objective function of the problem is convex on every simple path of the tree. For the unweighted case, an efficient $O(|V|)$ time algorithm was developed by Handler [14].

Location models involving uncertainty have attracted significant research efforts. This area is of great practical importance because of pervasive effects of uncertainty in real systems. There are two types of numerical parameters characterizing a network that may be subject to uncertainty, weights of vertices and distances between vertices. In 2007, Liu [18] proposed uncertainty theory to describe nondeterministic phenomena, especially expert data and subjective estimation. From then on, uncertainty theory has provided a new approach to deal with non-deterministic factors in programming problems. Some researchers employed uncertainty theory to deal with the location problems, such as uncertain models for single facility location problems by Yuan Gao [10], hierarchical facility location for the reverse logistics network design under uncertainty by Wang and Yang [29], the capacitated facility location-allocation problem under uncertain environment by Wen et al. [30], the inverse 1-median problem on a tree under uncertain cost coefficients by Nguyen and Chi [25]. For a survey on uncertain optimization problems, the reader is referred to [11, 13, 31, 32].

In this paper, the aim is to study the classical p-center location problem on a network in which vertex weights and distances between vertices are uncertain variables. We design linear time algorithms for finding the uncertain 1-center and 2-center locations on unweighted trees.

The structure of this paper is organized as follows: In the next section some basic concepts and properties of the uncertainty theory are introduced. In Section 3, the p-center location problem on general networks, the 1-center and the 2-center location problems on unweighted trees are described. Section 4 gives the uncertainty distribution of the optimal objective value of the p-center problem. Then methods for finding the α-p-center and the expected p-center in an uncertain network are proposed. Finally, algorithms for the 1-center and 2-center problems on uncertain unweighted trees are presented. Section 5 gives a brief conclusion to this paper.

2 Uncertainty theory

In this section, we review some concepts of uncertainty theory [18, 19].

Let Γ be a nonempty set, $\mathcal{L}$ a σ-algebra over Γ. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms (Liu [18]):
Axiom 1: (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
Axiom 2: (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event $\Lambda \in \mathcal{L}$.
Axiom 3: (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$,
we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

The triple $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space, and the product uncertain measure on the product σ-algebra was defined by Liu [18] via the following Product axiom:
Axiom 4: (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$.
Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}\{\Lambda_k\},$$

where Λ_k are arbitrary chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

Definition 2.1 (*Liu*[*18*]) *An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the following set is an event.*

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}.$$

In order to describe an uncertain variable in practice, the uncertainty distribution was defined by Liu [18] as the following function,

$$\phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number x.

Peng and Iwamura [27] showed that a function $\mathcal{M} : \Re \to [0,1]$ is an uncertainty distribution if and only if it is a monotone increasing function except $\phi(x) \equiv 0$ and $\phi(x) \equiv 1$.

For example, an uncertain variable ξ is called zigzag if it has a zigzag uncertainty distribution

$$\phi(x) = \begin{cases} 0 & x \leq a, \\ (x-a)/[2(b-a)] & a \leq x \leq b, \\ (x+c-2b)/[2(c-b)] & b \leq x \leq c, \\ 1 & x \geq c. \end{cases}$$

denoted by $\mathcal{Z}(a,b,c)$ where a, b, c are real numbers with $a < b < c$.

Definition 2.2 (*Liu*[17]) *The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}\{\xi_i \in B_i\}\right\} = \min_{1\leq i\leq n} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

Definition 2.3 (*Liu*[19]) *An uncertainty distribution ϕ is said to be regular if its inverse function $\phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0,1)$.*

It is clear that the zigzag uncertainty distribution is regular. The inverse uncertainty distribution of the zigzag uncertain variable $\xi \sim \mathcal{Z}(a,b,c)$ is

$$\phi^{-1}(\alpha) = \begin{cases} (1-2\alpha)a + 2\alpha b & \alpha < 0.5, \\ (2-2\alpha)b + (2\alpha - 1)c & \alpha \geq 0.5. \end{cases}$$

A real-valued function $f(x_1, x_2, \ldots, x_n)$ is said to be strictly increasing if f satisfies the following conditions:

(1) $f(x_1, x_2, \ldots, x_n) \geq f(y_1, y_2, \ldots, y_n)$ when $x_i \geq y_i$ for $i = 1, 2, \ldots, n$,

(2) $f(x_1, x_2, \ldots, x_n) > f(y_1, y_2, \ldots, y_n)$ when $x_i > y_i$ for $i = 1, 2, \ldots, n$.

If $f(x_1, x_2, \ldots, x_n)$ is a strictly increasing, then $-f(x_1, x_2, \ldots, x_n)$ is a strictly decreasing function.

Theorem 2.1 (*Liu*[19]) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\phi_1, \phi_2, \ldots, \phi_n$, respectively. If the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_n$, then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable with inverse uncertainty distribution*

$$\psi^{-1}(\alpha) = f(\phi_1^{-1}(\alpha), \phi_2^{-1}(\alpha), \ldots, \phi_n^{-1}(\alpha)).$$

3 The p-center problem on networks

Let $N = (V, E)$ be a connected network with vertex set V, $\mid V \mid= n$, edge set E, $\mid E \mid= m$, and let p be a constant with $p \leq n$. Every edge $e \in E$ has a positive length and each vertex $v_i \in V$ is associated with nonnegative weight w_i that is the demand of vertex v_i. Let $d(x, y)$ denote the distance between x, $y \in N$ which is equal to the length of the shortest path connecting these two points. We say that point x lies in N, if x coincides with a vertex or lies on an edge of N. In the absolute p-center problem, the aim is to find a set of p pairwise different point $x_1, \ldots, x_p$ on the network N which minimize the maximum of weighted distances from each vertex to its closest facility:

$$\min_{X_p \subset N, |X_p|=p} \max_{v_i \in V} w_i d(v_i, X_p) \tag{3.1}$$

where

$$d(v_i, X_p) = \min_{j=1,2,\ldots,p} d(v_i, x_j), X_p = \{x_1, \ldots, x_p\}.$$

An optimal solution X_p^* of this problem is called absolute p-center.

If in problem (3.1) we restrict the facilities to be located only on the vertices of the network, then we say that the optimal solution X_p^* is a vertex p-center location of N. We can formulate the vertex p-center problem as follows [7]:

Let $d_{ij} = d(v_i, v_j)$ be the distance from demand vertex v_i to candidate facility at vertex v_j. Also let $w = \{w_i | \, v_i \in V\}$ and $d = \{d_{ij} | \, v_i, v_j \in V\}$ denote the set of vertex weights and the set of distances between vertices, respectively. Then in network $N = (V, E)$, the optimal objective value of the p-center problem is a function of w and d, which is denoted as f_{pc} in this paper. Using this notation, the vertex p-center problem is as:

$$
\begin{aligned}
\min \quad & \max_{1\le i\le n} w_i \sum_{j=1}^{n} d_{ij} y_{ij} \\
s.t. \quad & \textstyle\sum_{j=1}^{n} y_{ij} = 1 \qquad i = 1,\dots,n, \\
& \textstyle\sum_{j=1}^{n} x_j = p, \\
& y_{ij} \le x_j \qquad i,j = 1,\dots,n, \\
& y_{ij}, x_j \in \{0,1\} \qquad i,j = 1,\dots,n,
\end{aligned}
\tag{3.2}
$$

in which, y_{ij} is the variable that is equal to 1 if demands of the vertex v_i are served by a facility at the vertex v_j, and 0 otherwise. Also the variable x_j is equal to 1 if there is an open facility at the vertex v_j, and 0 otherwise.

The objective function of model (3.2) minimizes the maximum distance between a demand vertex and the closest facility to the vertex. The first set of constraints states that all of the demand at vertex v_i must be assigned to a facility at some vertex for all vertex v_j. The second set of constraints stipulates that p facilities be located. The third set of constraints states that demands at vertex v_i cannot be assigned to a facility at vertex v_j unless a facility is located at vertex v_j. Finally, the last constraints are the integrality constraints.

For a general network and for variable values of p, the p-center problem is $\mathcal{NP}$-hard [15]. This is true for both the vertex and absolute p-center problems. In the following, we investigate the vertex (absolute) 1-center and 2-center problems on an unweighted tree.

3.1 The 1-center and 2-center problems on unweighted trees

Let $T = (V, E)$ be an unweighted tree network with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set E, $|E| = n-1$. The distances between vertices are positive. Obviously, a vertex v^* is a 1-center of the unweighted tree T if

$$\max_{i=1,\dots,n} d(v^*, v_i) \le \max_{i=1,\dots,n} d(v, v_i), \quad \forall v \in V.$$

As a consequence of Handler [14], we have the following theorem:

Theorem 3.1 *In an unweighted tree network the midpoint of the longest path is an absolute 1-center. The closest vertex to the absolute 1-center is a vertex 1-center of the given network.*

Moreover, Handler [14] proved the following lemma:

Lemma 3.1 *The absolute 1-center of an unweighted tree network is unique.*

Based on Theorem 3.1, to solve the vertex (absolute) 1-center problem on an unweighted tree, we can use the following algorithm:

Algorithm 1

1. Pick any point on the tree and find the vertex that is farthest away from the point that was picked. Call this vertex v_1.
2. Find the vertex that is farthest from v_1 and call this vertex v_2.
3. The absolute 1-center of the unweighted tree is at the midpoint of the (unique) path from v_1 to v_2. The vertex 1-center of the unweighted tree is at the vertex of the tree that is closest to the absolute 1-center. (Note that there may be two vertices that satisfy this condition if the absolute 1-center is at the midpoint of a edge).

The running time of this algorithm is $O(n)$.

To solve the vertex (absolute) 2-center problem on the unweighted tree T, we can modify Algorithm 1 outlined as follows:

Algorithm 2

1. Use Algorithm 1 to find the absolute 1-center.
2. Delete from the tree the edge containing the absolute
1-center (if the absolute 1-center is on a vertex delete one of the edges incident on the absolute 1-center which is on the path from v_1 to v_2). This divides the tree into two disconnected subtrees.
3. Use Algorithm 1 to find the vertex (absolute) 1-center of each of the subtrees. These locations constitute a solution to the vertex (absolute) 2-center problem.

The running time of this algorithm is $O(n)$.

4 The p-center problem on uncertain networks

In this section, we first investigate the p-center problem on a network in which vertex weights and distances between vertices are uncertain. Then we present algorithms for finding the 1-center and 2-center locations on an uncertain unweighted tree.

Consider network $N = (V, E)$ with the uncertain vertex weights and the uncertain vertex distances. Some assumptions are listed as follows:

1. The undirected uncertain network is connected.
2. The weight of each vertex v_i is a positive uncertain variable η_i.
3. The distance (shortest path length) between two vertices v_i and v_j is a positive uncertain variable ξ_{ij}.
4. All the uncertain variables η_i and ξ_{ij} are independent.

Define $\eta = \{\eta_i | \; v_i \in V\}$ and $\xi = \{\xi_{ij} | \; v_i, v_j \in V\}$. We denote the network with uncertain vertex weights and uncertain vertex distances as $N = (V, E, \eta, \xi)$. The optimal objective value of the p-center problem may be comprehended as an uncertain variable $f_{pc}(\eta, \xi)$, where f_{pc} is a strictly increasing function. For the uncertain network N, assume that the uncertainty distribution of $f_{pc}(\eta, \xi)$ is $\psi(x)$.

Definition 4.1 *Let $V_p \subseteq V$, $|V_p| = p$ and let y_{ij} be the variable that is equal to 1 if the location of the nearest facility to the vertex v_i in V_p is the vertex v_j, and 0 otherwise. Also let the variable x_j be equal to 1 if $v_j \in V_p$, and 0 otherwise.*

Using the above notations, V_p is a p-facility location (i.e. the location of p facilities to satisfy the demands at the vertices) if and only if

$$\begin{cases} \sum_{j=1}^{n} y_{ij} = 1 & i = 1, \ldots, n, \\ y_{ij} \le x_j & i, j = 1, \ldots, n, \\ \sum_{j=1}^{n} x_j = p, & \\ y_{ij}, x_j \in \{0, 1\} & i, j = 1, \ldots, n. \end{cases}$$

Now, let V_p be a p-facility location. Define

$$M(V_p) = \max_{1 \le i \le n} \sum_{j=1}^{n} \eta_i \xi_{ij} y_{ij}.$$

It is clear that $M(V_p)$ is also an uncertain variable.

According to Theorem 2.1, we can easily obtain the inverse uncertainty distribution of $f_{pc}(\eta, \xi)$.

Theorem 4.1 *In network $N = (V, E, \eta, \xi)$, let η_i and ξ_{ij} be independent uncertain variables with regular uncertainty distributions φ_i and ϕ_{ij}, respectively. Then, the inverse uncertainty distribution of $f_{pc}(\eta, \xi)$ is determined by*

$$\psi^{-1}(\alpha) = f_{pc}(\varphi^{-1}(\alpha), \phi^{-1}(\alpha)),$$

where $\varphi^{-1}(\alpha) = \{\varphi_i^{-1}(\alpha) | v_i \in V\}$ and $\phi^{-1}(\alpha) = \{\phi_{ij}^{-1}(\alpha) | v_i, v_j \in V\}$.

Through Theorem 4.1, we can get the uncertainty distribution of $f_{pc}(\eta, \xi)$ in a numerical sense easily.

4.1 The α-p-center

Here we give the concept of the α-p-center in an uncertain network $N = (V, E, \eta, \xi)$.

Definition 4.2 *In network $N = (V, E, \eta, \xi)$, V_p^* is a p-facility location. Then V_p^* is called the α-p-center if*

$$\begin{aligned} &min\{W \mid \mathcal{M}\{M(V_p^*) \le W\} \ge \alpha\} \\ &\le min\{W \mid \mathcal{M}\{M(V_p) \le W\} \ge \alpha\} \end{aligned}$$

for all p-facility location $V_p \subseteq V$, where $\alpha \in (0, 1)$ is a predetermined confidence level.

Uncertain constrained programming offers a powerful tool for modeling uncertain decision systems. The essential idea of uncertain constrained programming of α-p-center model is to minimize W value in network subject to uncertain constraint holding with confidence level α. In order to find α-p-center, we propose the following α-p-center model:

$$\begin{aligned} \min \quad & W \\ s.t. \quad & \mathcal{M}\{\max_{1 \le i \le n} \textstyle\sum_{j=1}^{n} \eta_i \xi_{ij} y_{ij} \le W\} \ge \alpha, \\ & \textstyle\sum_{j=1}^{n} y_{ij} = 1 \qquad i = 1, \ldots, n, \\ & y_{ij} \le x_j \qquad i, j = 1, \ldots, n, \\ & \textstyle\sum_{j=1}^{n} x_j = p, \\ & y_{ij}, x_j \in \{0, 1\} \qquad i, j = 1, \ldots, n, \end{aligned} \tag{4.3}$$

where α is a predetermined confidence level.

Since each η_i and ξ_{ij} have the regular uncertainty distributions and $\max_{1\leq i\leq n}\sum_{j=1}^{n}\eta_i\xi_{ij}y_{ij}$ is a strictly increasing function with respect to η_i and ξ_{ij}, according to Theorem 2.1, the model (4.3) can be equivalently transformed to the following deterministic model:

$$\begin{aligned} \min \quad & W \\ s.t. \quad & \max_{1\leq i\leq n}\sum_{j=1}^{n}\varphi_i^{-1}(\alpha)\phi_{ij}^{-1}(\alpha)y_{ij} \leq W, \\ & \sum_{j=1}^{n} y_{ij} = 1 \qquad i = 1,\ldots,n, \\ & y_{ij} \leq x_j \qquad i,j = 1,\ldots,n, \\ & \sum_{j=1}^{n} x_j = p, \\ & y_{ij}, x_j \in \{0,1\} \qquad i,j = 1,\ldots,n. \end{aligned} \tag{4.4}$$

Or

$$\begin{aligned} \min \quad & \max_{1\leq i\leq n}\sum_{j=1}^{n}\varphi_i^{-1}(\alpha)\phi_{ij}^{-1}(\alpha)y_{ij} \\ s.t. \quad & \sum_{j=1}^{n} y_{ij} = 1 \qquad i = 1,\ldots,n, \\ & y_{ij} \leq x_j \qquad i,j = 1,\ldots,n, \\ & \sum_{j=1}^{n} x_j = p, \\ & y_{ij}, x_j \in \{0,1\} \qquad i,j = 1,\ldots,n. \end{aligned} \tag{4.5}$$

In fact, the solution to model (4.5) is just the p-center of the deterministic network $\overline{N} = (\overline{V}, \overline{E})$, $\overline{V} = V$, $\overline{E} = E$, and the distance between vertices v_i, v_j is $\phi_{ij}^{-1}(\alpha)$, and the weight of vertex $v_i \in \overline{V}$ is $\varphi_i^{-1}(\alpha)$. According to this explanations, we have the following theorem.

Theorem 4.2 *In network $N = (V, E, \eta, \xi)$, let η_i and ξ_{ij} be independent uncertain variables with regular uncertainty distributions φ_i and ϕ_{ij}, respectively. Then, the α-p-center of $N = (V, E, \eta, \xi)$ is just the p-center of the network $\overline{N} = (\overline{V}, \overline{E})$, $\overline{V} = V$, $\overline{E} = E$ and the weight of the vertex v_i is $\varphi_i^{-1}(\alpha)$ and the shortest distance between the vertices v_i and v_j is $\phi_{ij}^{-1}(\alpha)$.*

Kariv and Hakimi proved that the absolute (vertex) p-center problem for a vertex-weighted or a vertex-unweighted network is $\mathcal{NP}$-hard even when the network is a planar graph of maximum vertex degree 3 [15]. Thus according to Theorem 4.2, we have the following theorem:

Theorem 4.3 *The uncertain p-center problem is $\mathcal{NP}$-hard on general networks.*

The α-1-center and α-2-center on uncertain unweighted trees

According to Theorem 4.2, we can obtain the α-1-center and α-2-center on an uncertain unweighted tree using the following algorithm:

Algorithm 3

1. Give a predetermined confidence level α and calculate $\phi_{ij}^{-1}(\alpha)$ for each ξ_{ij}.
2. Construct the corresponding deterministic network.
3. Use Algorithm 1 and Algorithm 2 to find the 1-center and 2-center.
4. Return 1-center and 2-center as the α-1-center and α-2-center.

Now we give an example to illustrate the conclusions presented above.

4.1.1 Example

Consider the unweighted tree $T = (V, E)$ given in Figure 1 with uncertain edge lengths in Table 1. Given a predetermined confidence level $\alpha = 0.9$.

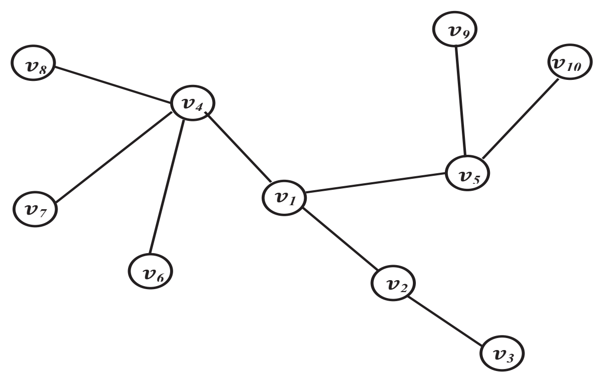

Figure 1: Tree T for Example *4.1.1*

Using the data in Table 1, we compute $\phi_{ij}^{-1}(\alpha)$ for $(v_i, v_j) \in E$ and construct the deterministic tree network (see Table 2). By applying Algorithm 1, we can find the vertex 0.9-1-center as follows:

Table 1: List of the edge lengths

Edges	ξ_{ij}	Edges	ξ_{ij}
(v_1, v_2)	$\mathcal{Z}(14, 16, 18)$	(v_3, v_{10})	$\mathcal{Z}(15, 16, 18)$
(v_1, v_4)	$\mathcal{Z}(15, 18, 20)$	(v_4, v_6)	$\mathcal{Z}(16, 17, 18)$
(v_2, v_3)	$\mathcal{Z}(10, 11, 12)$	(v_4, v_7)	$\mathcal{Z}(17, 19, 20)$
(v_2, v_5)	$\mathcal{Z}(20, 21, 22)$	(v_4, v_8)	$\mathcal{Z}(8, 9, 10)$
(v_3, v_9)	$\mathcal{Z}(12, 13, 15)$		

Table 2: List of $\phi_{ij}^{-1}(0.9)$

Edges	$\phi_{ij}^{-1}(0.9)$	Edges	$\phi_{ij}^{-1}(0.9)$
(v_1, v_2)	17.6	(v_3, v_{10})	17.6
(v_1, v_4)	19.6	(v_4, v_6)	17.8
(v_2, v_3)	11.8	(v_4, v_7)	19.8
(v_2, v_5)	21.8	(v_4, v_8)	9.8
(v_3, v_9)	14.6		

Table 3: List of the expected values

Edges	$E(\xi_{ij})$	Edges	$E(\xi_{ij})$
(v_1, v_2)	16	(v_3, v_{10})	16.25
(v_1, v_4)	17.75	(v_4, v_6)	17
(v_2, v_3)	11	(v_4, v_7)	18.75
(v_2, v_5)	21	(v_4, v_8)	9
(v_3, v_9)	13.25		

We begin by picking any point on tree T. Suppose that we pick vertex v_9. Vertex v_7 is the farthest vertex from vertex v_9. The farthest vertex from vertex v_7 is the vertex v_{10}. The absolute 0.9-1-center is the midpoint of the path from v_7 to v_{10} on tree T, or 13.8 units from vertex v_2 on the edge connecting vertices v_1 and v_2. The vertex v_1 is closest to the absolute 0.9-1-center. Thus v_1 is vertex 0.9-1-center. The absolute 1-center lies on edge (v_1, v_2). Removing this edge, the tree T divides to two subtrees. After applying the Algorithm 1 to each of these subtrees, we obtain the absolute 0.9-1-center locations on edges (v_4, v_7) and (v_3, v_{10}). Thus the vertex 0.9-2-center is $\{v_3, v_4\}$.

4.2 The expected p-center

Expected value is the average value of an uncertain variable in the sense of uncertain measure and represents the size of uncertain variable.

In [18], Liu introduced the concept of an expected value. The expected value of an uncertain variable is defined as follows: Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{\infty} \mathcal{M}\{\xi \geq r\}dr - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\}dr$$

provided that at least one of the two integrals is finite.

Theorem 4.4 *(Liu [18]) Let ξ be an uncertain variable with uncertainty distribution ϕ. If the expected value exists, then*

$$E[\xi] = \int_0^{\infty} (1 - \phi(x))dx - \int_{-\infty}^{0} \phi(x)dx.$$

Theorem 4.5 *(Liu [19]) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\phi_1, \phi_2, \ldots, \phi_n$, respectively. If the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_n$ then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an expected value*

$$E[\xi] = \int_0^1 f(\phi_1^{-1}(\alpha), \phi_2^{-1}(\alpha), \ldots, \phi_n^{-1}(\alpha))d\alpha.$$

As a useful representation of expected value, it has been proved by Liu [18] that

$$E[\xi] = \int_0^1 \phi^{-1}(\alpha)d\alpha$$

where ϕ^{-1} is the inverse uncertainty distribution of uncertain variable ξ. Note that if $\eta = \mathcal{Z}(a,b,c)$ and φ be its uncertainty distribution, then

$$E(\eta) = \int_0^1 \varphi^{-1}(\alpha)d\alpha = (a+2b+c)/4.$$

Now, we give the concept of the expected p-center in an uncertain network $N = (V, E, \eta, \xi)$.

Definition 4.3 *In network $N = (V, E, \eta, \xi)$, V_p^e is a p-facility location. Then V_p^e is called the expected p-center if*

$$E(M(V_p^e)) \leq E(M(V_p))$$

for all p-facility location $V_p \subseteq V$.

Therefore, if the decision maker wants to find a scheme in the sense of expected value, then the expected value model for problem (3.2) can be written as follows:

$$\begin{array}{lll} \min & \max\limits_{1\leq i\leq n} E(\sum\limits_{j=1}^{n} \eta_i \xi_{ij} y_{ij}) & \\ s.t. & \sum_{j=1}^{n} y_{ij} = 1 & i = 1,\dots,n, \\ & y_{ij} \leq x_j & i,j = 1,\dots,n, \\ & \sum_{j=1}^{n} x_j = p, & \\ & y_{ij}, x_j \in \{0,1\} & i,j = 1,\dots,n. \end{array} \quad (4.6)$$

Equivalently, it can be rewritten as:

$$\begin{array}{lll} \min & \max\limits_{1\leq i\leq n} \sum\limits_{j=1}^{n} E(\eta_i)E(\xi_{ij})y_{ij} & \\ s.t. & \sum_{j=1}^{n} y_{ij} = 1 & i = 1,\dots,n, \\ & y_{ij} \leq x_j & i,j = 1,\dots,n, \\ & \sum_{j=1}^{n} x_j = p, & \\ & y_{ij}, x_j \in \{0,1\} & i,j = 1,\dots,n. \end{array} \quad (4.7)$$

In other words, it can be represented as:

$$\begin{array}{lll} \min & \max\limits_{1\leq i\leq n} (\int_0^1 \varphi_i^{-1}(\alpha)d\alpha) \sum\limits_{j=1}^{n} (\int_0^1 \phi_{ij}^{-1}(\alpha)d\alpha) y_{ij} & \\ s.t. & \sum_{j=1}^{n} y_{ij} = 1 & i = 1,\dots,n, \\ & y_{ij} \leq x_j & i,j = 1,\dots,n, \\ & \sum_{j=1}^{n} x_j = p, & \\ & y_{ij}, x_j \in \{0,1\} & i,j = 1,\dots,n. \end{array} \quad (4.8)$$

According to the above statements, we have the following theorem:

Theorem 4.6 *In network $N = (V, E, \eta, \xi)$, let η_i and ξ_{ij} be independent uncertain variables with regular uncertainty distributions φ_i and ϕ_{ij}, respectively. Then, the expected p-center of $N = (V, E, \eta, \xi)$ is just the p-center of the network $\overline{N} = (\overline{V}, \overline{E})$, $\overline{V} = V$, $\overline{E} = E$ and the weight of the vertex v_i is $E(\eta_i)$ and the shortest distance between the vertices v_i and v_j is $E(\xi_{ij})$.*

The expected 1-center and expected 2-center on uncertain unweighted trees

According to Theorem 4.6, we can obtain the expected 1-center and expected 2-center on an unweighted tree with uncertain vertex distances by using the following algorithm:

Algorithm 4

1. Calculate $E(\xi_{ij})$ for $v_i, v_j \in V$.
2. Construct the corresponding deterministic network.
3. Use Algorithm 1 and Algorithm 2 to find the 1-center and 2-center.
4. Return 1-center and 2-center as the expected 1-center and expected 2-center.

4.2.1 Example

Consider Example 4.1.1. By using the data in Table 1 and according to definition of expected value of a zigzag uncertain variable, we obtain the values of Table 3. By using Algorithm 1 and Algorithm 2, we can find the vertex expected 1-center and 2-center as follows: The longest path on tree T is the path from v_7 to v_{10}. The absolute expected 1-center is the midpoint of the path from v_7 to v_{10} on edge (v_1, v_2) or 12.625 units from vertex v_2 on the edge connecting vertices v_1 and v_2. The vertex v_1 is closest to the absolute expected 1-center. Thus v_1 is the vertex expected 1-center. Removing the edge (v_1, v_2), the tree T divides to two subtrees. After applying the Algorithm 1 to each of these subtrees, we obtain the absolute expected 1-center locations on edges (v_4, v_7) and (v_3, v_{10}). Then the vertex expected 2-center is $\{v_3, v_4\}$.

5 Conclusion

In this paper, we considered the p-center location problem on a network in which vertex weights and vertex distances are uncertain variables. We first investigated the uncertainty distribution of the optimal objective value of the p-center problem and introduced the concepts of the α-p-center and the expected p-center. Then we showed that the α-p-center and the expected p-center models can be transformed into corresponding deterministic models. Finally, we proposed linear time algorithms for finding the 1-center and 2-center of uncertain unweighted trees.

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