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Research Article

Detecting the location of the boundary layers in singular perturbation problems with general linear non-local boundary conditions

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Abstract

Singular perturbation problems have been studied by many mathematicians. Since the approximate solutions of these problems are as the sum of internal solution (boundary layer area) and external ones, the formation or non-formation of boundary layers should be specified. This paper, investigates this issue for a singular perturbation problem including a first order differential equation with general non-local boundary condition. It needs to say that it is simple for local boundary conditions and there is no difficulty. However, the formation of boundary layers for non-local case is not as stright forward as local case. To tackle this problem generalized solution of differential equation and some necessary conditions are used.

Keywords : Generalized solution; Necessary conditions; Non-local boundary conditions; Singular perturbation problems; Fundamental solution; Uniform limit.

1 Introduction

One of the important subjects in applied mathematics is the theory of singular perturbation problems.The mathematical models for this kind of problems usually are in the form of either ordinary differential equation (O.D.E) or partial differential equation(P.D.E) in which the highest derivative is multiplied by some powers ε as a positive small parameter. The objective theory of singular perturbation problems is to solve differential equation with some initial or boundary conditions with small parameter ε. These problems are divided into 2 types: i.e regular and singular cases. Commonly in regular perturbation problems, order of resulted differential equation, after the ellipsis of zero instead of perturbation term, is the same as the order of given differential equation, see [13], [15], [16]. As an example, if we consider the motion of planets solely under the effect of the sun, an equation without small parameter will be obtained but if we consider the effect of planets‘ motion on each other, a differential equation with small parameter will be achieved. That is , the small parameter is planet's mass to the sun equals to 0.001. In singular perturbation problems, typically the order of differential equation, which results in substituting zero value instead of small parameter, is not the same as the order of main differential equation, and the solution of resulted differential equation could not hold in boundary conditions in main problem. This position is called boundary layer phenomenon. In other words, in case of singular, the solutions do not hold in boundary conditions, and the boundary layer is formed,

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see [13], [17]. In some classic textbooks, the formation and non-formation of boundary layer in singular perturbation problems includes ordinary differential equations as

$$\varepsilon y^{(n)} + f(y^{(n-1)}, y^{(n-2)}, ..., y', y, x) = 0, \quad x \in (0,1).$$

is related to hold the following uniform limit,

$$\lim_{\varepsilon\to 0^+} \lim_{x\to 0} y_\varepsilon(x) = \lim_{x\to 0} \lim_{\varepsilon\to 0^+} y_\varepsilon(x),$$
$$\lim_{\varepsilon\to 0^+} \lim_{x\to 1} y_\varepsilon(x) = \lim_{x\to 1} \lim_{\varepsilon\to 0^+} y_\varepsilon(x).$$

If these repeated uniform limits are hold at the points $x = 0$, $x = 1$, then we do not have boundary layer, otherwise we have boundary layer in the related point. Doollan and Miller and Shilders in [5] invented a uniform numerical method in order to solve the boundary value problems including linear O.D.E with local boundary conditions as singular perturbation problems. But it would be difficult for the non-local of boundary conditions and also for the identification of formation of boundary layers. As upholding of limit position of solution in above boundary conditions informs the formation of boundary layer, therefore, this boundary layer is formed on boundary points which are not known, because in theory of singular perturbation problems, the approximate solution is as the sum of internal solution (boundary layer area) and external ones. Thus, basically it should be determined on which boundary point internal solution should be written.
Therefore, the main purpose of this paper is to provide the solution for a given question.This is done by localization of non-local boundary condition to local boundary conditions. For this, an algebraic system is constructed by the given non-local boundary conditions and the resulted necessary conditions.It is remarkable the boundary and initial value problems with non local boundary conditions are appeared in some of natural phenomena and physical problems such as Reaction Diffusion System [6], Optimal Control [4] and [2], [11].

2 Mathematical statement of problem

We consider the following perturbation problem,

$$lx \equiv \varepsilon\dot{x}(t) + x(t) = t, t \in (0,1) \tag{2.1}$$

$$\alpha_0 x(0) + \alpha_1 x(1) = \alpha. \tag{2.2}$$

Where ε is the small parameter, and α_0, α_1 and α are real constants. $x(t)$ is real unknown function on [0,1].
First, we obtain the adjoint equation of problem (2.1)-(2.2).
For this, the arbitrary function $y(t)$ is multiplied on both sides of equation (2.1), then integration on [0,1] yields:

$$< lx, y >= \varepsilon \int_0^1 \dot{x}(t)y(t)dt + \int_0^1 x(t)y(t)dt$$
$$= \varepsilon x(t)y(t)\Big|_0^1 + \int_0^1 x(t)[-\varepsilon\dot{y}(t) + y(t)]dt.$$

In conclusion, the adjoint equation is as follows,

$$l^* y \equiv -\varepsilon\dot{y}(t) + y(t). \tag{2.3}$$

3 Generalized solution and necessary condition

Now we obtain the Generalized solution (or fundamental solution) of equation (2.3). For doing this, we solve the following non-homogeneous equation.

$$-\varepsilon\dot{y}(t) + y(t) = f(t), \quad t \in (0,1). \tag{3.4}$$

the particular solution of above equation is as follows

$$y(t) = \frac{1}{\varepsilon}\int_t^1 e^{\frac{t-\tau}{\varepsilon}} f(\tau)d\tau.$$

Therefore, we conclude that the following function is a fundamental solution of equation (3.4),

$$Y(t-\tau) = \frac{\theta(t-\tau)}{\varepsilon} e^{-\frac{\tau-t}{\varepsilon}}. \tag{3.5}$$

That is,

$$\varepsilon\dot{Y}(t-\tau) + \dot{Y}(t-\tau) = \delta(t-\tau).$$

Where the function $\delta(t-\tau)$ is Delta-Dirac-function, and $\theta(t-\tau)$ is the Heaviside function, see [12], [19]. Now we obtain the necessary condition on equation (2.1). For doing this, the equation (2.1) is multiplied by (3.5) and integrated on interval [0,1].

$$\varepsilon\int_0^1 \dot{x}(t)Y(t-\tau)dt + \int_0^1 x(t)Y(t-\tau)dt$$
$$= \int_0^1 tY(t-\tau)dt.$$

Applying the integration parts rule for the first term yields,

$$\varepsilon x(t)Y(t-\tau)\Big|_0^1 + \int_0^1 x(t)[-\varepsilon\dot{Y}(t-\tau) + Y(t-\tau)]dt = \int_0^1 tY(t-\tau)dt.$$

According to the property of Delta-Dirac function we have,

$$\varepsilon x(0)Y(-\tau) - \varepsilon x(1)Y(1-\tau) + \int_0^1 tY(t-\tau)dt$$

$$= \int_0^1 [\varepsilon\dot{Y}(t-\tau) + Y(t-\tau)]x(\tau)dt$$

$$= \int_0^1 x(\tau)\delta(t-\tau)dt$$

$$= \begin{cases} x(\tau), & \tau \in (0,1) \\ \frac{1}{2}x(\tau), & \tau = 0, \quad \tau = 1 \end{cases} \tag{3.6}$$

If $\tau = 0$, a trivial relation is reached and If $\tau = 1$, then the following necessary condition is achieved.Hence, for any arbitrary solution of equation (2.1), we have:

$$x(1) = x(0)e^{-\frac{1}{\varepsilon}} + \frac{1}{\varepsilon}\int_0^1 te^{-\frac{1-t}{\varepsilon}}\theta(1-t)dt \tag{3.7}$$

In other words, any solution of equation (2.1) satisfies in (3.7). This relation is called a necessary condition for arbitrary solution of equation (2.1), see [1], [7], [12].

4 Localization of boundary conditions

Now we consider the boundary condition (2.2) and the necessary condition (3.7) together, as an algebraic system with unknowns $x(0)$ and $x(1)$,

$$\begin{cases} \alpha_0 x(0) + \alpha x(1) = \alpha, \\ x(1) - x(0)e^{-\frac{1}{\varepsilon}} = \frac{1}{\varepsilon}\int_0^1 te^{-\frac{1-t}{\varepsilon}}\theta(1-t)dt \end{cases} \tag{4.8}$$

If we assume

$$\Delta = \begin{vmatrix} \alpha_0 & \alpha_1 \\ -e^{-\frac{1}{\varepsilon}} & 1 \end{vmatrix} \neq 0 \tag{4.9}$$

We have from (4.8) by Cramer rule,

$$x(0) = \frac{1}{\Delta}[\alpha - \frac{\alpha_1}{\varepsilon}\int_0^1 te^{-\frac{1-t}{\varepsilon}}\theta(1-t)dt] \tag{4.10}$$

$$x(1) = \frac{1}{\Delta}[\frac{\alpha_0}{\varepsilon}\int_0^1 te^{-\frac{1-t}{\varepsilon}}\theta(1-t)dt + \alpha e^{-\frac{1}{\varepsilon}}] \tag{4.11}$$

Where $\Delta = \alpha_0 + \alpha_1 e^{-\frac{1}{\varepsilon}}$.
Therefore, we obtain the boundary values of unknown function as detached. Now we could assume equation (2.1) with conditions (4.10) and (4.11) is a boundary value problem with local boundary conditions (Dirichlet conditions).For this problem we can easily apply classic methods for recognizing formation or nonformation of boundary layers. If we let:

$$x(0) = a_0, \quad x(1) = a_1.$$

Then these conditions with arbitrary constants a_0, a_1, could not give with equation (2.1) as a boundary value problem.Because equation (2.1) is a first order equation, but the number of above conditions equals 2. But these conditions could be given as a combination like boundary condition (2.2).
Now from (3.6) for $t \in (0,1)$, the general solution of equation (2.1) is obtained which is independent of boundary condition (2.2),

$$x(\tau) = x(0)e^{-\frac{\tau}{\varepsilon}} + \int_0^1 \frac{t}{\varepsilon}e^{-\frac{t-\tau}{\varepsilon}}\theta(\tau-t)dt \tag{4.12}$$

Now, by considering the following relation

$$\lim_{\varepsilon\to 0^+} \frac{\theta(t)}{2\varepsilon}e^{-\frac{t}{\varepsilon}} = \delta(t). \tag{4.13}$$

We will obtain the limits of relations (4.10), (4.11), when $\varepsilon \to 0^+$, and it the repeated uniform limits when $\tau \to 0, \tau \to 1$ for recognizing formation or nonformation of boundary layers in boundary points.We give the main results as following theorems.

5 Main results

Theorem 5.1 *In singular perturbation problem (2.1)-(2.2) , for any arbitrary values of* α_0, α_1 *and* α, *there is no boundary layer at the point* $t=1$; *provided* α_0, α_1 *are not zero together.*

Proof. We have from (4.11):

$$\lim_{\varepsilon\to 0^+} x(1) = \lim_{\varepsilon\to 0^+} \frac{1}{\alpha_0+\alpha_1 e^{-\frac{1}{\varepsilon}}} [\frac{\alpha_0}{\varepsilon}\int_0^1 t e^{-\frac{1-t}{\varepsilon}}\theta(1-t)dt + \alpha e^{-\frac{1}{\varepsilon}}] = 1.$$

On the other hand, for the repeated uniform limits when $\tau \to 1$, we have

$$1 = \lim_{\varepsilon\to 0^+}\lim_{\tau\to 1} x_\varepsilon(\tau) = \lim_{\varepsilon\to 0^+} x(1) = \lim_{\varepsilon\to 0^+} x(\tau) = \lim_{\tau\to 1}\lim_{\varepsilon\to 0^+} x_\varepsilon(\tau) = 1 \quad (5.14)$$

Therefore for arbitrary values of α_0, α_1 and α, we have no boundary layers at the boundary point $t=1$, because this uniform limit is established independently from α, α_0 and α_1.

Theorem 5.2 *Under the conditions and hypothesis of theorem 5.1, there is no boundary layer at the point* $t=0$, *provided* $\alpha=\alpha_1$.

Proof. At first, we calculate the limits of relations (4.10) and (4.11) when $\varepsilon\to 0$ as follows:

$$\lim_{\varepsilon\to 0^+} x(0) = \lim_{\varepsilon\to 0^+} \frac{1}{\alpha_0+\alpha_1 e^{-\frac{1}{\varepsilon}}} [\alpha - \frac{\alpha_1}{\varepsilon}\int_0^1 t e^{-\frac{1-t}{\varepsilon}}\theta(1-t)dt] = \frac{1}{\alpha_0}(\alpha-\alpha_1) \quad (5.15)$$

$$\lim_{\varepsilon\to 0^+} x(\tau) = \lim_{\varepsilon\to 0^+} x(0)e^{-\frac{\tau}{\varepsilon}} + \lim_{\varepsilon\to 0^+}\int_0^1 \frac{t}{\varepsilon} e^{-\frac{\tau-t}{\varepsilon}}\theta(\tau-t)dt = \tau, \quad \tau\in(0,1). \quad (5.16)$$

Consequently the following relations are resulted:

$$\lim_{\varepsilon\to 0^+} x(0) = \frac{1}{\alpha_0}(\alpha-\alpha_1),$$
$$\lim_{\varepsilon\to 0^+} x(\tau) = \tau.$$

On the other hand, we can consider the repeated uniform limits when $\tau\to 0$, from above relations.

$$0 \neq \frac{1}{\alpha_0}(\alpha-\alpha_1) = \lim_{\varepsilon\to 0^+}\lim_{\tau\to 0} x_\varepsilon(\tau) \neq \lim_{\tau\to 0}\lim_{\varepsilon\to 0^+} x_\varepsilon(\tau) = 0. \quad (5.17)$$

This relation shows, if $\alpha=\alpha_1$, then we have no boundary layer at $t=0$. Because uniform limits are satisfied.

Theorem 5.3 *Under the conditions and hypothesis of theorem 5.1, there is a boundary layer at the point* $t=0$, *provided* $\alpha\neq\alpha_1$.

Proof. Let us consider again the uniform limits relation (3.7). We see that if $\alpha\neq\alpha_1$, then this uniform limit is not satisfied.In other words, the boundary condition (2.2) is not hold.Therefore, for the values α,α_1 which $\alpha\neq\alpha_1$ there will be a boundary layer at $t=0$.

6 Unsolved problems

Problem 1. Consider the following singular perturbation problem,

$$\varepsilon\ddot{x}(t)+p(t)\dot{x}(t)+q(t)x(t)=f(t), \quad t\in(a,b). \quad (6.18)$$

With general non-local linear boundary conditions:

$$\sum_{j=0}^{1}[\alpha_{ij}x^{(j)}(a)+\beta_{ij}x^{(j)}(b)]=0, \quad i=1,2. \quad (6.19)$$

Where $\varepsilon\geq 0$ is the small parameter and $p(t), q(t)$ and $f(t)$ are real continuous functions and $p(t)\neq 0$. Also suppose α_{ij},β_{ij} $(i,j=1,2)$ are real constants and the linear non-local boundary conditions (6.19) are independent.
Give sufficient conditions for each following cases separately,
I- In none of boundary points, the given problem does not have any boundary layer.
II- Boundary layer formed only in boundary point $t=a$.
III- Boundary layer formed only in boundary point $t=b$.
IV-Boundary layer formed at the both of boundary points $t=a$, $t=b$.
Problem 2. Consider the following singular perturbation problem,

$$\varepsilon\dot{x}(t)+[p(t)+\varepsilon q(t)]x(t)=f(t), \quad t\in(a,b), \quad (6.20)$$

$$\alpha x(a)+\beta x(b)=\gamma. \quad (6.21)$$

Where $p(t), q(t)$ are $n\times n$ real matrix functions and α,β are $n\times n$ real constant matrices, $x(t)$

and $f(t)$ are column vectors and γ is real constant vector with n components.
Give sufficient conditions for formation or nonformation of boundary layers in boundary points $t=a$ and $t=b$.
Problem 3. Consider the following singular perturbation problem,

$$\varepsilon x^{(iv)}(t)+\ddot{x}(t)+px(t)=0,\ t\in(a,b), \quad (6.22)$$

$$\sum_{j=0}^{3}[\alpha_{ij}x^{(j)}(a)+\beta_{ij}x^{(j)}(b)]=\gamma_i, \quad i=1,2,3,4. \quad (6.23)$$

Where $\varepsilon\geq 0$ is small parameter and p,α_{ij}, β_{ij}, γ_i are real constants.
Give sufficient conditions for formation or nonformation of boundary layers in boundary points $t=a$ and $t=b$ like problem 2.1.

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