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Necessary and Sufficient Optimality Conditions for a Control Fuzzy Linear Problem

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Abstract

In the present paper, we show the some properties of the fuzzy solution of the control linear fuzzy differential equations and research the optimal time problems for it.

Keywords : Fuzzy differential equations; Control problems.

1 Introduction

In recent years, the fuzzy set theory introduced by Zadeh [32] has emerged as an interesting and fascinating branch of pure and applied sciences. The applications of fuzzy set theory can be found in many branches of regional, physical, mathematical, differential equations and engineering sciences. Recently there have been new advances in the theory of fuzzy differential equations [1, 7, 6, 7, 9, 10, 11, 12, 13, 14, 16, 17, 19, 20, 21, 27, 29, 31], inclusions [28, 22] and differential inclusions with fuzzy right-hand side [17, 3, 4, 5, 8] as well as in the theory of control fuzzy differential equations [15, 23, 24, 25] and control differential inclusions with fuzzy right-hand side [18, 30].

In this article we are going to consider the some properties of the fuzzy solution of the control linear fuzzy differential equations and research the optimal time problems for it.

2 The fundamental definitions and designations

Let $comp\,(R^n)$ $(conv\,(R^n))$ be a set of all nonempty (convex) compact subsets from the space R^n,

$$h\,(A, B) = \min_{r \geqslant 0} \{S_r(A) \supset B,\ S_r(B) \supset A\}$$

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be Hausdorff distance between sets A and B, $S_r(A)$ is r-neighborhood of set A.

Let E^n be the set of all $u : R^n \to [0,1]$ such that u satisfies the following conditions:

(i) u is normal, that is, there exists an $x_0 \in R^n$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex, that is, $u(\lambda x + (1-\lambda)y) \geqslant \min\{u(x), u(y)\}$ for any $x, y \in R^n$ and $0 \leqslant \lambda \leqslant 1$;

(iii) u is upper semicontinuous,

(iv) $[u]^0 = cl\{x \in R^n : u(x) > 0\}$ is compact.

If $u \in E^n$, then u is called a fuzzy number, and E^n is said to be a fuzzy number space. For $0 < \alpha \leqslant 1$, denote:

$$[u]^\alpha = \{x \in R^n : u(x) \geqslant \alpha\}.$$

Then from (i)-(iv), it follows that the α-level set $[u]^\alpha \in conv(R^n)$ for all $0 \leqslant \alpha \leqslant 1$.

If $g : R^n \times R^n \to R^n$ is a function, then using Zadeh's extension principle we can extend $\tilde{g}$ to $E^n \times E^n \to E^n$ by the equation

$$\tilde{g}(u,v)(z) = \sup_{z=g(x,y)} \min\{u(x), v(y)\}.$$

It is well known that

$$[\tilde{g}(u,v)]^\alpha = g([u]^\alpha, [v]^\alpha)$$

for all $u, v \in E^n$, $0 \leqslant \alpha \leqslant 1$ and continuous function g. Further, we have

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [ku]^\alpha = k[u]^\alpha,$$

where $k \in R$.

Define $D : E^n \times E^n \to [0, \infty)$ by the relation

$$D(u,v) = \sup_{0 \leqslant \alpha \leqslant 1} h([u]^\alpha, [v]^\alpha),$$

where h is the Hausdorff metric defined in $comp(R^n)$. Then D is a metric in E^n.

Further we know that [26]:

(i) (E^n, D) is a complete metric space,

(ii) $D(u+w, v+w) = D(u,v)$ for all $u, v, w \in E^n$,

(iii) $D(\lambda u, \lambda v) = |\lambda| D(u,v)$ for all $u, v \in E^n$ and $\lambda \in R$.

It can be proved that

$$D(u+v, w+z) \leqslant D(u,w) + D(v,z)$$

for $u, v, w, z \in E^n$.

Definition 2.1. *[9] A mapping $F : [0,T] \to E^n$ is strongly measurable if for all $\alpha \in [0,1]$ the set-valued map $F_\alpha : [0,T] \to conv(R^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable.*

Definition 2.2. *[9] A mapping $F : [0,T] \to E^n$ is said to be integrably bounded if there is an integrable function $h(t)$ such that $\|x(t)\| \leqslant h(t)$ for every $x(t) \in F_0(t)$.*

Definition 2.3. *[9] The integral of a fuzzy mapping $F : [0,T] \to E^n$ is defined levelwise by $\left[\int\limits_0^T F(t)dt\right]^\alpha = \int\limits_0^T F_\alpha(t)dt$. The set of all $\int\limits_0^T f(t)dt$ such that $f : [0,T] \to R^n$ is a measurable selection for F_α for all $\alpha \in [0,1]$.*

Definition 2.4. *[9] A strongly measurable and integrably bounded mapping $F : [0,T] \to E^n$ is said to be integrable over $[0,T]$ if $\int\limits_0^T F(t)dt \in E^n$.*

Note that if $F : [0,T] \to E^n$ is strongly measurable and integrably bounded, then F is integrable. Further if $F : [0,T] \to E^n$ is continuous, then it is integrable.

Definition 2.5. *[9] A mapping $u : [0,T] \to E^n$ is called differentiable at $t \in [0,T]$ if, for any $\alpha \in [0,1]$, the set-valued mapping $u_\alpha(t) = [x(t)]^\alpha$ is Hukuhara differentiable at point t with $D_H u_\alpha(t)$ and the family $\{D_H u_\alpha(t) : \alpha \in [0,1]\}$ define a fuzzy number $u'(t) \in E^n$.*

If $u : [0,T] \to E^n$ is differentiable at $t \in [0,T]$, then we say that $u'(t)$ is the fuzzy derivative of $u(\cdot)$ at the point $t \in [0,T]$.

Let $\tilde{E}^n \subset E^n$ such that u is strongly normal, that is, there exists a unique $x_0 \in R^n$ such that $u(x_0) = 1$.

Consider the following control linear fuzzy differential equations

$$u' = A(t)\, u + g(t,w), \quad u(0) = u_0, \tag{2.1}$$

and the following nonlinear fuzzy differential equations

$$u' = f(t,u,w), \quad u(0) = u_0\,, \tag{2.2}$$

where $t \in R_+$ is the time; $u \in \tilde{E}^n$ is the state; $w \in R^m$ is the control; $A(t)$ is $(n \times n)$-dimensional matrix-valued function; $g : R_+ \times R^m \to \tilde{E}^n$, $f : R_+ \times \tilde{E}^n \times R^m \to \tilde{E}^n$ are the fuzzy maps, $u_0 \in \tilde{E}^n$.

Let $W : R_+ \to conv(R^m)$ be the measurable multivalued map.

Definition 2.6. *Set LW of all single-valued branches of the multivalued map $W(t)$ is the set of the possible controls.*

Obviously, the control fuzzy differential equation (4.7) turns into the ordinary fuzzy differential equation

$$u' = \phi(t,u)\,, \ u(0) = u_0, \tag{2.3}$$

if the control $\tilde{w}(\cdot) \in LW$ is fixed and $\phi(t,u) \equiv f(t,x,\tilde{w}(t))$.

Definition 2.7. *[9] A mapping $u : [0,T] \to E^n$ is a fuzzy solution to the problem (2.3) if it is continuous and satisfies the integral equation*

$$u(t) = u_0 + \int\limits_0^t \phi(s,u(s))\, ds \tag{2.4}$$

for all $t \in [0,T]$.

The fuzzy differential equations (2.3) has the fuzzy solution, if right-hand side of the fuzzy differential equation (2.3) satisfies some conditions [9, 10, 11, 12, 13, 16, 17, 19, 20, 21, 27, 29].

Let $u(t)$ denotes the fuzzy solution of the differential equation (2.3), then $u(t,w)$ denotes the fuzzy solution of the control differential equation (4.7) for the fixed $w(\cdot) \in LW$.

Definition 2.8. *The set $Y(T) = \{u(T,w) : w(\cdot) \in LW\}$ be called the attainable set of the fuzzy system (4.7).*

3 The some properties of the fuzzy solution

In this section, we consider the some properties of the fuzzy solution of the control fuzzy differential equation (2.1).

Let us the following condition is true.

Condition A:

A1. $A(t)$ is measurable on $[0,T]$;

A2. There exist $a > 0$ such that $\|A(t)\| \leqslant a$ almost everywhere $t \in [0,T]$;

A3. The multivalued map $W : [0,T] \to conv(R^m)$ is measurable on $[0,T]$;

A4. The fuzzy map $g : R_+ \times R^m \to \tilde{E}^n$ is strongly measurable on $[0,T]$ and is continuous on R^m;

A5. There exist $v(\cdot) \in L_2[0,T]$ and $l(\cdot) \in L_2[0,T]$ such that

$$|W(t)| \leqslant v(t), \quad |g(t,w)| \leqslant l(t)$$

almost everywhere on $[0,T]$ and every $w \in R^m$;

A6. If $\{w_k(t)\}_{k=1}^{\infty}$, $w_k(t) \in LW$, $k = \overline{1,\infty}$ week converges to $w^*(t) \in LW$ then $\{g(t,w_k(t))\}_{k=1}^{\infty}$ week converges to $g(t,w^*(t))$.

Theorem 3.1. *Let the condition A is true.*

Then there exists a unique solution $u(t,w)$ of (2.1) for every $w(\cdot) \in LW$.

Proof. The proof is easy consequence of the [9, 10, 13, 19, 21, 27]. □

Theorem 3.2. *Let the condition A is true.*

Then the attainable set $Y(T)$ is compact i.e. the set $[Y(T)]^\alpha$ is compact for all $\alpha \in [0,1]$.

Proof. We show that $[Y(T)]^\alpha$ is closed for any $\alpha \in [0,1]$. Let $u_k^\alpha \in [Y(T)]^\alpha$, $k = 1,2,\ldots$, and $\lim u_k^\alpha = u_*^\alpha$. By Definition 2.8 we have the sequences $\{u(t,w_k)\}_{k=1}^{\infty}$ and $\{w_k(t)\}_{k=1}^{\infty}$ such that

1) $u(t,w_k)$ is solution of the problem (2.1) for all $k = 1,2,\ldots$;

2) $[u(T,w_k)]^\alpha = u_k^\alpha$ for all $k = 1,2,\ldots$;

3) $w_k(t) \in LW$ for all $k = 1,2,\ldots$.

Then there exists subsequence $\{w_{k_n}(t)\}_{n=1}^{\infty}$ such that the subsequence $\{w_{k_n}(t)\}_{n=1}^{\infty}$ week converges to $w_*(t) \in LW$.

By Definition 2.7 we have

$$h\left([u(T,w_{k_n})]^\alpha,[u(T,w_*)]^\alpha\right)=$$

$$=h\left(\left[u_0+\int_o^T\left(A(t)u(t,w_{k_n})+g(t,w_{k_n})\right)dt\right]^\alpha,\left[u_0+\int_0^T\left(A(t)u(t,w_*)+g(t,w_*)\,dt\right)\right]^\alpha\right)\leqslant$$

$$\leqslant a\int_0^T h\left([u(t,w_{k_n})]^\alpha,[u(t,w_*)]^\alpha\right)dt+h\left(\left[\int_0^T g(t,w_{k_n})\,dt\right]^\alpha,\left[\int_0^T g(t,w_*)\,dt\right]^\alpha\right)\leqslant$$

$$\leqslant h\left(\left[\int_0^T g(t,w_{k_n})\,dt\right]^\alpha,\left[\int_0^T g(t,w_*)\,dt\right]^\alpha\right)e^{aT}.$$

Hence we have $\lim_{n\to\infty} h\left([u(T,w_{k_n})]^\alpha,[u(T,w_*)]^\alpha\right)=0$. Observe that $[u(t,w_*)]^\alpha=u_*^\alpha$, i.e the set $[Y(T)]^\alpha$ is closed.

Let $UY^\alpha=\bigcup\limits_{u^\alpha\in[Y(T)]^\alpha}u^\alpha$ and let $Z^\alpha=\{u\ :\ u\in comp(R^n),\ u\in UY^\alpha\}$. Since the set UY^α is compact, the set Z^α is compact too. By the definition of Z^α, we obtain $[Y(T)]^\alpha\subset Z^\alpha$. Hence the set $[Y(T)]^\alpha$ is compact. The theorem is proved. □

Remark 3.1. *If $g:[0,T]\times R^m\to E^n$ then the theorem 3.2 is true.*

Remark 3.2. *If $g:[0,T]\times R^m\to conv(R^n)$ and $u_0\in conv(R^n)$ then we obtain the control differential equations with Hukuhara derivative*

$$D_Hu=A(t)u+g(t,w),\ \ u(0)=u_0$$

and the theorem 3.2 is true, also.

Hitherto we obtained the basic properties of the fuzzy solution of systems (2.1). Now, we are going to consider the some control fuzzy problem.

4 The optimal time problem

Consider the control linear fuzzy differential equation (2.1), when

$$g(t,w)=B(t)w+q(t),\tag{4.5}$$

where

B1. $B(\cdot)$ is measurable on $[0,T]$;

B2. The norm $\|B(t)\|$ of the matrix $B(t)$ is integrable on $[0,T]$;

B3. The fuzzy map $q:[0,T]\to\tilde{E}^n$ is measurable on $[0,T]$;

B4. There exists $f(\cdot)\in L_2[0,T]$ such that

$$|q(t)|\leqslant f(t)$$

almost everywhere on $[0,T]$.

Consider the following optimal control problem: It is necessary to find the minimal time T and the control $w^*(\cdot) \in LW$ such that the fuzzy solution of system (2.1) satisfies the following condition:

$$x(T, w^*) \cap S_k \neq \emptyset \tag{4.6}$$

where $S_k \in E^n$ is the terminal set.

Clearly, these time optimal problems are different from the ordinary time optimal problem.

Definition 4.1. *We shall say that the pair $(w^*(\cdot), u(\cdot, w^*))$ satisfies the maximum principle on $[0, T]$, if there exists the vector-function $\psi(\cdot)$, which is the solution of the system*

$$\psi' = -A^T(t)\psi, \;\; \psi(T) \in S_1(0)$$

and the following conditions are hold

1) *The maximum condition:*

$$C(B(t)w^*(t), \psi(t)) = \max_{w \in W(t)} C(B(t)w, \psi(t))$$

almost everywhere on $[0, T]$;

2) *The transversal condition:*

$$C\left([u(T, w^*)]^1, \psi(T)\right) = -C\left([S_k]^1, -\psi(T)\right),$$

where $C(Z, \psi) = \max_{z \in Z}(z, \psi)$, $Z \in comp(R^n)$.

Theorem 4.1. (Necessary optimal condition.) *Let the conditions A1-A3,A5,B1-B4 are true and the pair $(T, w^*(\cdot))$ is optimality.*

Then the pair $(w^*(\cdot), u(\cdot, w^*))$ satisfies the maximum principle on $[0, T]$.

Proof. Let $w^*(\cdot)$ is the optimal control and $u(\cdot, w^*)$ is the optimal fuzzy solution of the system (2.1), i.e.

1) $u(T, w^*) \in Y(T)$;

2) $u(T, w^*) \cap S_k \neq \emptyset$.

From 1) and 2) we have $\max_{u \in [Y(T)]^1} C(u, \psi) \geqslant -C\left([S_k]^1, -\psi\right)$ for all $\psi \in S_1(0)$.

Consequently

$$p = \max_{u \in [Y(T)]^1} \min_{\psi \in S_1(0)} C(u, \psi) + C\left([S_k]^1, -\psi\right) \geqslant 0.$$

From $[u(T, w^*)]^1 \cap [S_k]^1 \neq \emptyset$ we have $q(T, \psi) = C\left([u(T, w^*)]^1, \psi\right) + C\left([S_k]^1, -\psi\right) \geqslant 0$ for all $\psi \in S_1(0)$.

From [21] we have that the function $q(T, \psi)$ is continuous on $R_+ \times S_1(0)$.

If $q(T,\psi) > 0$ for all $\psi \in S_1(0)$ then we have $q^0(T) = \min\limits_{\psi \in S_1(0)} q(T,\psi) \geqslant \gamma > 0$. Hence there exists $\tau < T$ such that $q^0(\tau) > 0$. Consequently we have,

$$C\left([u(\tau,w^*)]^1,\psi\right) + C\left([S_k]^1,-\psi\right) > 0$$

for all $\psi \in S_1(0)$, i.e. $[u(\tau,w^*)]^1 \cap [S_k]^1 = \emptyset$.

It contradicts that T is optimal time.

If $p > 0$,

$$\max_{u\in[Y(T)]^1} \min_{\psi \in S_1(0)} C(u,\psi) + C\left([S_k]^1,-\psi\right) = C\left(\tilde{u},\tilde{\psi}\right) + C\left([S_k]^1,-\tilde{\psi}\right)$$

and $[u(T,w^*)]^1 \neq \tilde{u}$, than we have a contradiction. Hence there exist $\tilde{\psi} \in S_1(0)$ such that:

$$C\left([u(T,w^*)]^1,\tilde{\psi}\right) = \max_{u\in[Y(T)]^1} C\left(u,\tilde{\psi}\right),$$

$$C\left([u(T,w^*)]^1,\tilde{\psi}\right) = -C\left([S_k]^1,-\tilde{\psi}\right).$$

Consequently

$$\left(\int_0^T \Phi(T)\,\Phi^{-1}(s)\,B(s)\,w^*(s)ds,\tilde{\psi}\right) = \max_{w(\cdot)\in LW}\left(\int_0^T \Phi(T)\,\Phi^{-1}(s)\,B(s)\,w(s)ds,\tilde{\psi}\right).$$

Then:

$$\left(\Phi(T)\,\Phi^{-1}(s)\,B(s)\,w^*(s),\tilde{\psi}\right) = \max_{w(\cdot)\in LW}\left(\Phi(T)\,\Phi^{-1}(s)\,B(s)\,w(s),\tilde{\psi}\right)$$

for almost everywhere $s \in [0,T]$. If $\psi(t) = \left(\Phi(T)\,\Phi^{-1}(t)\right)^T\tilde{\psi}\Big/\left\|\left(\Phi(T)\,\Phi^{-1}(t)\right)^T\tilde{\psi}\right\|$, then the theorem is proved. □

Theorem 4.2. (Sufficient optimal condition). *Let the conditions A1-A3,A5,B1-B4 are true and* $w^*(\cdot)$ *is the possible control. Let the following conditions*

1) *The pair* $(w^*(\cdot),u(\cdot,w^*))$ *satisfies the maximum principle on* $[0,T]$;

2) *For all* $t \in [0,T)$

$$C\left([u(t,w^*)]^1,\psi(t)\right) < -C\left([S_k]^1 - \psi(t)\right) \;;$$

are true.

Then the control $w^*(\cdot)$ *is optimal.*

Proof. Let $w(\cdot) \in LW$ is any possible control on $[0,t_1]$, $t_1 < T$. From the maximum condition of the Definition 2.7 we have

$$C\left([u(t,w)]^1,\psi(t)\right) \leqslant C\left([u(t,w^*)]^1,\psi(t)\right), \tag{4.7}$$

for all $t \in [0,t_1]$.

Then $C\left([u(t,w)]^1,\psi(t)\right) < -C\left([S_k]^1,-\psi(t)\right)$ for all $t \in [0,t_1]$, i.e. $[u(t,w)]^1 \cap [S_k]^1 = \emptyset$ for all $t \in [0,t_1]$.

Obviously the pair $(T,w^*(\cdot))$ is optimal and The proof is complated. □

Remark 4.1. *If $q : [0,T] \to E^n$, then*

$$[u(t,w)]^1 \neq \Phi(t)[u_0]^1 + \Phi(t)\int_0^t \Phi^{-1}(s)B(s)w(s)ds + \Phi(t)\int_0^T \Phi^{-1}(s)[q(s)]^1 ds$$

and the theorems 4.1 and 4.2 are not true.

Example 4.1. *Consider the following control linear fuzzy differential equation*

$$u' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u + w + f, \quad u(0) = 0,$$

where $u \in E^2$ is the state; $w = (w_1, w_2)^T \in W = S_1(0)$ is the control; $f \in E^2$ is the fuzzy set, where

$$\vartheta(f) = \begin{cases} 1 - 4f_1^2 - 9f_2^2, & 4f_1^2 + 9f_2^2 \leqslant 1 \\ 0, & 4f_1^2 + 9f_2^2 > 1 \end{cases}.$$

Consider the following optimal control problem: it is necessary to find the minimal time T and the control $w^*(\cdot) \in LW$ such that the fuzzy solution of system satisfies the condition:

$$u(T, w^*) \cap S_k \neq \emptyset,$$

where $S_k \in E^2$ is the terminal set such that

$$\sigma(x) = \begin{cases} \sqrt{1 - (x_1 - 2\pi)^2 - (x_2 - 1)^2}, & x \in Q,\ x_2 \geqslant 1 \\ \sqrt{1 - (x_1 - 2\pi)^2}, & x \in Q,\ -1 < x_2 < 1 \\ \sqrt{1 - (x_1 - 2\pi)^2 - (x_2 + 1)^2}, & x \in Q,\ x \leqslant -1 \\ 0, & x \notin Q \end{cases},$$

$$Q = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2 \ : \ \begin{array}{l} 2\pi - 1 \leqslant x_1 \leqslant 2\pi + 1 \\ \sqrt{1 - (x_1 - 2\pi)^2} - 1 \leqslant x_2 \leqslant \\ \leqslant \sqrt{1 - (x_1 - 2\pi)^2} + 1 \end{array} \right\}.$$

Obviously, the optimal pair $T = 2\pi$ and $w^*(t) = (\cos(t), -\sin(t))$ satisfy of the conditions the theorem 4.1:

1) $(w^*(t), \psi(t)) = C(W, \psi(t))$ *for almost everywhere $t \in [0, 2\pi]$;*

2) $C\left([u(T, w^*)]^1, \psi(T)\right) = -C\left([S_k]^1, -\psi(T)\right)$,

where $\psi(t) = (\cos(t), -\sin(t))^T$ for almost everywhere $t \in [0, 2\pi]$, $[x(T, w^*)]^1 = (T\cos(T), -T\sin(T))^T = (2\pi, 0)^T$, $[S_k]^1 = \left\{(x_1, x_2)^T \ : \ x_1 = 2\pi,\ -1 \leqslant x_2 \leqslant 1\right\}$.

5 Conclusions

In this article we considered the some properties of the fuzzy solution of the control linear fuzzy differential equations and research the optimal time problems for it, when $u(T, w^*) \cap S_k \neq \emptyset$.

We did not consider the optimal time problems of the fuzzy solution of the control linear fuzzy differential equations, when $u(T, w^*) \subset S_k$ or $u(T, w^*) \supset S_k$ and the optimal time problems of the fuzzy solution of the control nonlinear fuzzy differential equations. These cases require a separate study.

References

[1] T. Allahviranloo, S. Abbasbandy, N. Ahmady, E. Ahmady, Improved predictorcorrector method for solving fuzzy initial value problems, Information Sciences 179 (2009) 945-955.

[2] T. Allahviranloo, E. Ahmady, A. Ahmady, N-th fuzzy differential equations , Information Sciences 178 (2008) 1309-1324.

[3] J.-P. Aubin, Fuzzy differential inclusions, Probl. Control Inf. Theory 19 (1) (1990) 55-67.

[4] V.A. Baidosov, Differential inclusions with fuzzy right-hand side, Soviet Mathematics 40 (3) (1990) 567-569.

[5] V.A. Baidosov, Fuzzy differential inclusions, J. of Appl. Math. and Mechan. 54 (1) (1990) 8-13.

[6] B. Bede, T. Gnana Bhaskar, V. Lakshmikantham, Perspectives of fuzzy initial value problems, Communications in Applied Analysis 11 (2007) 339- 358.

[7] M. Ghanbari, Numerical solution of fuzzy initial value problems under generalized differentiability by HPM, Int. J. Industrial Mathematics 1 (1) (2009) 19-39

[8] E. Hullermeier, An approach to modelling and simulation of uncertain dynamical systems, Internat. J. Uncertain. Fuzziness Knowledge-Based Systems 5 (2) (1997) 117-137.

[9] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems 24 (3) (1987) 301-317.

[10] O. Kaleva, The Cauchy problem for fuzzy differential equations, Fuzzy Sets and Systems 35 (3) (1990) 389-396.

[11] O. Kaleva, The Peano theorem for fuzzy differential equations revisited, Fuzzy Sets and Systems 98 (1) (1998) 147-148.

[12] O. Kaleva, A note on fuzzy differential equations, Nonlinear Anal. 64 (5) (2006) 895-900.

[13] T.A. Komleva, A.V. Plotnikov, N.V. Skripnik, Differential equations with set-valued solutions, Ukrainian Mathematical Journal.(Springer New York) 60 (10) (2008) 1540-1556.

[14] T.A. Komleva, L.I. Plotnikova, A.V. Plotnikov, Averaging of the fuzzy dierential equations. Work of the Odessa Polytehnikal University 27 (1) (2007) 185-190.

[15] Y. C. Kwun, D. G. Park, Optimal control problem for fuzzy differential equations, in Proceedings of the Korea-Vietnam Joint Seminar, (1998) 103114.

[16] V. Lakshmikantham, T. Gnana Bhaskar, J. Vasundhara Devi, Theory of set differential equations in metric spaces, Cambridge Scientific Publishers, Cambridge, (2006).

[17] V. Lakshmikantham, R. N. Mohapatra, Theory of fuzzy differential equations and inclusions, Series in Mathematical Analysis and Applications, 6. Taylor & Francis, Ltd., London, (2003).

[18] I.V. Molchanyuk, A.V. Plotnikov, Linear control systems with a fuzzy parameter, Nonlinear Oscil. (N. Y.) 9 (1) (2006) 59-64.

[19] J.Y. Park, H.K. Han, Existence and uniqueness theorem for a solution of fuzzy differential equations, Int. J. Math. Math. Sci. 22 (2) (1999) 271-279.

[20] J.Y. Park, H.K. Han, Fuzzy differential equations, Fuzzy Sets and Systems 110 (1) (2000) 69-77.

[21] A.V. Plotnikov, N.V. Skripnik, Differential equations with "clear" and fuzzy multivalued right-hand sides. Asymptotics Methods, AstroPrint, Odessa, (2009).

[22] A.V. Plotnikov, N.V. Skripnik, The generalized solutions of the fuzzy differential inclusions, International J. of Pure and Appl. Math. 56 (2) (2009) 165-172.

[23] N.D. Phu, T.T. Tung, Some properties of sheaf-solutions of sheaf fuzzy control problems, Electron. J. Differential Equations 108 (2006) 8 pp. (electronic) http:www.ejde.math.txstate.edu

[24] N.D. Phu, T.T. Tung, Some results on sheaf-solutions of sheaf set control problems, Nonlinear Anal. 67 (5) (2007) 1309-1315.

[25] N.D. Phu, T.T. Tung, Existence of solutions of fuzzy control differential equations, J. Sci. Tech. Devel. 10 (5) (2007) 5-12.

[26] M.L. Puri, D.A. Ralescu, Fuzzy random variables, J. Math. Anal. Appl. (114) (1986) 409-422.

[27] S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems 24 (3) (1987) 319-330.

[28] N.V. Skripnik, Existence of classic solutions of the fuzzy differential inclusions, Ukr. math. bull. (5) (2008) 244-257.

[29] D. Vorobiev, S. Seikkala, Towards the theory of fuzzy differential equations, Fuzzy Sets and Systems 125 (2) (2002) 231-237.

[30] V.S. Vasil'kovskaya, A. V. Plotnikov, Integrodifferential systems with fuzzy noise, Ukrainian Math. J. 59 (10) (2007) 1482-1492.

[31] C. Wu, S. Song, E. Stanley Lee, Approximate solutions, existence and uniqueness of the Cauchy problem of fuzzy differential equations, J. Math. Anal. Appl. 202 (1996) 629-644.

[32] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338-353.