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Extrapolation Method for Fuzzy Differential Inclusions

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Abstract
In this paper, a numerical method for solving 'fuzzy differential inclusions' is considered. The *fuzzy reachable* set can be approximated by the proposed method with complete error analysis which is discussed in detail. Moreover, the extrapolation method is employed for increasing the accuracy of the approximate solution. The method is illustrated by solving some linear and nonlinear fuzzy initial value problems.
Keywords: Fuzzy differential inclusions, Euler's method, Extrapolation method, Fuzzy initial value problem.

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1 Introduction

The topics of numerical methods for solving fuzzy differential equations have been rapidly growing in recent years. The concept of fuzzy derivative was first introduced by S. L. Chang and L.A. Zadeh [4]. It was followed up by D. Dubois and H. Prade [8], who defined and used the extension principle. Other methods have been discussed by M. L Puri and D. A. Ralescu [15] and R. Goetschel and W. Voxman [10]. The fuzzy differential equation and the initial value problem were regularly treated by O. Kaleva [12, 13], by S. Seikkala [16]," and by other researchers". Hüllermeier [11] suggested a different formulation of the FIVP (fuzzy initial value proble), based on a family of differential inclusions at each r-level, $0 \leq r \leq 1$,

$$x'(t) \in [f(t, x(t))]^r, \quad x(0) \in [x_0]^r,$$

where $[f(\cdot,\cdot)]^r : [0,T] \times R^n \to \kappa_c^n$, and κ_c^n is the space of nonempty convex compact subsets of R^n. The numerical methods for solving fuzzy differential equations are introduced in [1, 2, 3]. The paper is organized as follows:

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In section 2, some basic definitions and results on fuzzy numbers along with a definition of a fuzzy derivative, discussed by D. Vorobiev and S. Seikkala [17] and Phil Diamond in [6], are given. In section 3, we define the problem which is a fuzzy initial value problem. The numerical method for fuzzy differential inclusions is discussed in section 4. And the extrapolation method is discussed in section 5. The proposed algorithm is illustrated by solving some examples in section 6, and conclusions are in provided section 7.

2 Preliminaries

Denote by κ^n the set of all nonempty compact subset of R^n and by κ_c^n the subset of κ^n consisting of nonempty convex compact sets. Recall that

$$\rho(x, A) = \min_{a\in A} \|x - a\|$$

is the distance of a point $x \in \mathrm{R}^n$ from $A \in \kappa^n$ and that the *Hausdorff separation* $\rho(A, B)$ of $A, B \in \kappa^n$ is defined as

$$\rho(A, B) = \max_{a\in A} \rho(a, B).$$

Note that the notation is consistent since $\rho(a, B) = \rho(\{a\}, B)$. Now, ρ is not a metric. In fact, $\rho(A, B) = 0$ if and only if $A \subseteq B$. The *Hausdorff metric* d_H on κ^n is defined by

$$d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\}$$

and (κ^n, d_H) is a complete metric space. An open ϵ-neighborhood of $\mathrm{A} \in \kappa^n$ is the set

$$\mathrm{N}(\mathrm{A}, \epsilon) = \{x \in \mathrm{R}^n : \rho(x, \mathrm{A}) < \epsilon\} = \mathrm{A} + \epsilon\mathrm{B}^n,$$

where B^n is the open unit ball in R^n. A mapping $F : \mathrm{R}^n \to \kappa^n$ is *upper semicontinuous* (usc) at x_0 if for all $\epsilon > 0$ there exits $\delta = \delta(\epsilon, x_0)$ such that

$$F(x) \subset N(F(x_0), \epsilon)) = F(x_0 + \epsilon B^n)$$

for all $x \in N(x_0, \delta)$. Let D^n denote the set of upper semicontinuous normal fuzzy sets on R^n with the property of compact support. That is, $u \in D^n$, then $u : \mathrm{R}^n \to [0, 1]$ is usc , supp(u)= $\overline{\{x \in \mathrm{R}^n : u(x) > 0\}}$ is compact and there exists at least one $\xi \in supp(u)$ for which $u(\xi) = 1$. The r-level set of u, $0 < r \leq 1$, is

$$[u]^r = \{x \in \mathrm{R}^n : u(x) \geq r\}.$$

Clearly, for $\alpha \leq \beta, [u]^\alpha \supseteq [u]^\beta$. The level sets are nonempty from normality and compact by usc and compact support. The metric d_H is defined on D^n as

$$d_\infty(u, v) = \sup\{d_H([u]^r, [v]^r) : 0 \leq r \leq 1\}, \quad u, v \in D^n$$

and (D^n, d_∞) is a complete metric space. Denote by E^n the subset of fuzzy convex elements of D^n. The metric space (E^n, d_∞) is also complete, [7]. Let $I = [0, T]$ be a finite interval, $y_0 \in R^n$, and G be a map from $I \times R^n$ into the set of all subsets of R^n; one must find an absolutely continuous function $x(\cdot)$ on I such that

$$\begin{cases} x'(t) & \in G(t, x(t)); \qquad for \;\; almost \;\; all \;\; t \in I, \\ x(0) & = y_0 \in Y_0 \subset R^n. \end{cases} \tag{2.1}$$

Recall that a continuous function $x : I \to Y \subseteq R^n$ is said to be *absolutely continuous* if there exists a locally integrable function ν such that

$$\int_t^s \nu(\alpha)d\alpha = x(s) - x(t)$$

for all $t, s \in I$. The differential inclusion (2.1) is said to have a solution $x(t)$ on I. If $x(\cdot)$ is absolutely continuous, $x(0) = y_0$ and $x(\cdot)$ satisfies the inclusion a.e. on I. Let $\sum(y_0, \tau)$, be the reachable set, that is,

$$\sum(y_0, \tau) := \{x : I \to R^n | x \textit{ is the solution of } (2.1)\} \subset \mathrm{C}(I),$$

and $A(y_0, \tau) = \{x(\tau) : x(.) \in \sum(y_0, \tau)\}$ be the attainable set that is, the set of all points $x(\cdot)$ that constitute the ends of the trajectories of (2.1). Obviously, $A(y_0, \tau), 0 < \tau \leq T$ is a compact subset of R^n. As a rule, the set $\sum(y_0, \tau)$ consists of more than one element, i.e., we have a bundle of trajectories. We use a finite difference scheme together with suitable selection procedures resulting in a sequence of grid reachable sets $\sum(y_0, t_0), \sum(y_0, t_1), \ldots, \sum(y_0, T)$, on a uniform grid $0 = t_0 \leq t_1 \leq \ldots \leq t_N = T$ with step size $h = \frac{T - t_0}{N} = t_i - t_{i-1},\ i = 1, \ldots, N$. The technique based on the midpoint approximation and Euler's method is used for (2.1) as follows,

$$\begin{aligned} &y_0 \in Y_0 \subset R^n, \\ &y_{i+1} \in y_i + hf(t_i, y_i; r), \quad i = 0, \ldots, N-1. \end{aligned} \tag{2.2}$$

$$y_{i+1} \in y_{i-1} + 2hf(t_i, y_i), \quad i = 0, \ldots, N-1. \tag{2.3}$$

In all cases, for any random y_0 selections from Y_0, the asymptotic expansion of the result $x(t, h) \in R^n$ is of the form

$$x(t, h) = y(t, h) + \sum_{i=1}^{m} e_i(t)h^{\beta_i} + h^{\beta_{m+1}}\alpha_{m+1}(h), \quad 0 < \beta_1 < \beta_2 < \ldots < \beta_{m+1}, \tag{2.4}$$

where the expansion $\beta_i,\ i = 1, 2, \ldots, m$, need not be integers. The coefficient $e_i(t) \in R^n$ is related to the derivatives of the solution $x(t, h)$ and independent of h. The function $\alpha_{m+1}(h)$ is bounded for $h \to 0$, and $x(t, h) = lim_{h \to 0} y(t, h)$ is the exact solution of the problem at hand. This technique requires two starting values, because both $y_0, y_1 \in R^n$ needed before the first midpoint approximation $y_2 \in R^n$, can be determined. As usual, we use the initial condition for $y_0 = x(0) \in Y_0$. To determine the second starting value, y_1, we apply Euler's method. This produces an approximation $y(t, h)$ to $x(t, h)$ that is of the form

$$x(t, h) = y(t, h) + \sum_{i=1}^{m} e_i(t)h^{2i} + h^{2m+2}\alpha_{2m+2}(h). \tag{2.5}$$

Definition 2.1. *The fuzzy number $X \in E^n$ is called pyramidal fuzzy number if its r-level sets are n-dimensional rectangles for $0 \leq r \leq 1$.*

3 A fuzzy initial value problem

Let $U \in E^n$, then $U = (u_1, \ldots, u_n)^T$ where $u_i \in E = E^1, i = 1, \ldots, n$. The parametric form of u_i is $u_i(r) = (\underline{u_i}(r), \overline{u_i}(r)),\ r \in [0,1]$, where $\underline{u_i}(r) = m_i + \alpha_i(r-1)$ is an increasing function and $\overline{u_i}(r) = m_i + \gamma_i(1-r)$ is an decreasing function where m_i is the core and α_i, γ_i are the left and right spreads, respectively, in $(m_i, \alpha_i, \gamma_i)$. Also $\underline{u_i}(1) = \overline{u_i}(1)$. The parametric form of inclusion triangles have the following increasing and decreasing functions, respectively,

$$\begin{aligned} \underline{u_i}(r,\beta) &= m_i + \alpha_i(1-\beta)(r-1), \\ \overline{u_i}(r,\beta) &= m_i + \gamma_i(1-\beta)(1-r),\ r,\beta \in [0,1]. \end{aligned} \tag{3.6}$$

The convex hull (C.H.) of the corners can be obtained with any variation of $r \in [0,1]$ for any $\beta \in [0,1]$. Then, the components of a fuzzy triangular number $U \in E^n$ are as follows

$$\begin{aligned} U_\beta &= (u_{1,\beta}, \ldots, u_{n,\beta}) \\ &= (C.H.(A_1, B_1, C_1), \ldots, C.H.(A_n, B_n, C_n))^T,\ 0 \le \beta \le 1. \end{aligned}$$

where

$$u_{i,\beta} = C.H.(A_i, B_i, C_i) = \bigcup_{0 \le \beta \le 1} \{(\underline{u_i}(r,\beta), \overline{u_i}(r,\beta)) \mid 0 \le r \le 1\}.$$

Now, let $f : \mathrm{I} \times \mathrm{E}^n \to E^n$ and consider the fuzzy initial value problem (FIVP)

$$\begin{cases} x'(t) &= f(t, x(t)), \quad t \in I = [0,T], \\ x(0) &= Y_0 \in E^n, \end{cases} \tag{3.7}$$

interpreted as a family of differential inclusions. Set

$$\bigcup_{0 \le \beta \le 1} \{(\underline{f}(t,x,r,\beta), \overline{f}(t,x,r,\beta)) \mid 0 \le r \le 1\} := F(t,x,\beta)$$

and

$$\bigcup_{0 \le \beta \le 1} \{(\underline{Y_0}(r,\beta)), \overline{Y_0}(r,\beta) \mid 0 \le r \le 1\} := Y_0(\beta)$$

and identify the FIVP with the family of differential inclusions

$$\begin{cases} x'_\beta(t) &\in F(t, x_\beta(t); \beta), \quad t \in I = [0,T], \\ x_\beta(0) &= Y_0 \in Y_0(\beta), \quad 0 \le \beta \le 1 \end{cases} \tag{3.8}$$

where $F : \Omega \times [0,1] \to \kappa_c^n$ and Ω is an open subset of $I \times E^n$ containing $(0, Y_0(\beta)),\ \beta \in [0,1]$. Denote the set of all solutions of (3.7) on I by $\sum_\beta(Y_0(\beta), T)$ and the attainable set by $A_\beta(Y_0(\beta), T) = \{x(T) : x(\cdot) \in \sum_\beta(Y_0(\beta), T)\}$, where $\exists f(random) \in F(t, x; \beta)$ and $\exists Y_0(random) \in Y_0(\beta)$ such that

$$\begin{cases} x'_\beta(t) &= f(t, x_\beta(t)), \quad t \in I = [0,T], \\ x_\beta(0) &= Y_0 \in E^n,\ \textit{almost every where} \end{cases} \tag{3.9}$$

that can be solved as β-levels. Suppose that $g(T, k, Y_0)$ is the solution of (3.8) at the end point, then

$$\sum_\beta(.,.) = \bigcup_{0 \le \beta \le 1} \{g(t, k, Y_0) \mid k \in k(\beta),\ Y_0 \in Y_0(\beta)\}.$$

Let

$$Z_T(R^n) = \{x(.) \in C([0,T]; R^n) : x'(.) \in L^\infty([0,T]; R^n)\}.$$

These sets are not generally convex; they are acyclic which are stronger than simply connected, [9].

Theorem 3.1. *[6], Let $Y_0 \in E^n$ and let Ω be an open set in $R \times R^n$ containing $\{0\} \times supp(Y_0)$. Suppose that $f : \Omega \to E^n$ is usc and $F(t,x;\beta) \in \kappa_c^n$ for all $(t,x,\beta) \in R^{n+1} \times [0,1]$. Let the boundedness assumption hold for all $y_0 \in supp(Y_0)$ and the inclusion*

$$x'(t) \in F(t,x;0), \;\; x(0) \in supp(Y_0).$$

Then, the attainable sets $A_\beta(Y_0(\beta),T), \beta \in [0,1],$, of the family of inclusions (3.7) are the level sets of a fuzzy set $A(Y_0,T) \in D^n$. The solution sets $\sum_\beta(Y_0(\beta),T)$ of (3.7) are the level sets of a fuzzy set $\sum(Y_0,T)$ included in $Z_T(R^n)$.

4 Euler‘s method

Let $x_\beta(t_i) \approx y_i(\beta)$ for all $\beta \in [0,1]$, then Euler‘s method for approximating the reachable set of problem (3.9) is proposed as follows:

$$\begin{aligned} &y_{0,\beta} \in Y_0(\beta) \subset R^n, \\ &y_{i+1,\beta} = \textstyle\bigcup_{s_\beta \in Y_i(\beta)} (s_\beta + hf(t_i, s_\beta; \beta)), \quad i = 0, \ldots, N-1, \quad \beta \in [0,1], \end{aligned} \tag{4.10}$$

where

$$\bigcup_{0 \le \beta \le 1} \{(\underline{Y_i}(r,\beta)), \overline{Y_i}(r,\beta) \mid 0 \le r \le 1\} := Y_i(\beta)$$

and

$$f(t_i, s_\beta; \beta)) \in \bigcup_{0 \le \beta \le 1} \{(\underline{f}(t, s_\beta, r, \beta), \overline{f}(t, s_\beta, r, \beta)) \mid 0 \le r \le 1\} = F(t, s_\beta, \beta).$$

Lemma 4.1. *Let a sequence of numbers $\{W_n\}_{n=0}^N$ satisfy*

$$|W_{n+1}| \le A|W_n| + B, \quad 0 \le n \le N-1,$$

for some given positive constants A and B. Then

$$|W_n| \le A^n |W_0| + B\frac{A^n - 1}{A - 1}, \quad 0 \le n \le N.$$

Proof: See [14].

Theorem 4.1. *Let $F \in C^2(\Omega)$ in (4.10) be a compact convex-valued mapping that satisfies the Lipschitz condition in x with the Lipschitz constant $L > 0$ and x_β be a solution of (3.7), then $lim_{h \to 0} y_N(\beta) = x_\beta(T)$, for any $\beta \in [0,1]$.*

Proof: Let

$$x_\beta(t_{i+1}) = \bigcup_{\bar{x}_\beta \in X_i(\beta)} (\bar{x}_\beta + hf(t_i, \bar{x}_\beta; \beta)),\ a.e.$$

and

$$y_{i+1}(\beta) = \bigcup_{\bar{y}_\beta \in Y_i(\beta)} (\bar{y}_\beta + hf(t_i, \bar{y}_\beta; \beta) + O(h^2)),$$

it is sufficient to prove $lim_{h\to 0} \parallel \bar{y}_{i+1}(\beta) - \bar{x}_\beta(t_{i+1}) \parallel = 0, i = 0, \ldots, N-1, \beta \in [0,1]$. Since

$$\bar{y}_{i+1}(\beta) = \bar{y}_i(\beta) + hf(t_i, \bar{y}_i(\beta); \beta) + O(h^2)$$

and

$$\bar{x}_\beta(t_{i+1}) \approx \bar{x}_\beta(t_i) + hf(t_i, \bar{x}_\beta(t_i); \beta)$$

then

$$\parallel \bar{y}_{i+1}(\beta) - \bar{x}_\beta(t_{i+1}) \parallel \leq \parallel \bar{y}_i(\beta) - \bar{x}_\beta(t_i) \parallel (1 + Lh) + O(h^2).$$

By using Lemma (4.1) for all t_i in particular at T,

$$\parallel \bar{y}_N(\beta) - \bar{x}_\beta(T) \parallel \leq \frac{1}{L} O(h)[e^{LT} - 1],$$

by which the proof is completed.

5 Extrapolation Method

Let $\sum_\beta(Y_0(\beta), t)$ be the reachable set formed on the exact solutions and $\sum'_\beta(Y_0(\beta), t)$ be the reachable set formed on the approximate solutions at t. Assume that we have a fixed step size h and that we wish to approximate $x(t_1; \beta) = x(h; \beta) \in \sum_\beta(Y_0(\beta), t_1)$, with $t_0 = 0$ for all $\beta \in [0,1]$. For the first extrapolation step, we set $h_0 = \frac{h}{2}$ and use Euler's method with random $y_0 \in Y_0(\beta)$ to approximate $x(h_0; \beta) = x(\frac{h}{2}; \beta)$ as

$$y_1(\beta) \in y_0(\beta) + h_0 f(a, y_0; \beta), \quad \forall \beta \in [0,1]. \tag{5.11}$$

We then apply the midpoint approximatin with $t_0 = a$ and $t_1 = h_0 = \frac{h}{2}$ to produce a first approximation to $x(h; \beta) = x(2h_0; \beta) \in \sum_r(Y_0(\beta), t_2)$, as $y_2(\beta) \in y_0(\beta) + 2h_0 f(h_0, y_1; \beta)$. The endpoint correction is applied to obtain the final approximation of $x(h; \beta)$ for all $\beta \in [0,1]$ with the step size h_0. This results in the approximation of $O(h_0^2)$ to $x(t_1; \beta)$

$$x_1^1(\beta) \in \frac{1}{2}[y_2(\beta) + y_1(\beta) + h_0 f(2h_0, y_2; \beta)], \quad \forall \beta \in [0,1].$$

We save the approximation $x_1^1(\beta) \in \sum_\beta(Y_0(\beta), t_1)$ and discard the intermediate results $y_1(\beta) \in \sum'_\beta(Y_0(\beta), t_1)$ and $y_2(\beta) \in \sum'_(Y_0(\beta), t_2)$ for all $\beta \in [0,1]$. To obtain the next approximation, $x_2^1(\beta)$ to $x(t_1; \beta)$, we set $h_1 = \frac{h}{4}$ and use initial value of Euler's method to achieve an approximation to $x(h_1; \beta) = x(\frac{h}{4}; \beta)$ that we will call $y_1(\beta) \in \sum'_\beta(Y_0(\beta), t_1)$

$$y_1(\beta) \in y_0(\beta) + h_1 f(a, y_0; \beta), \quad \forall \beta \in [0,1]. \tag{5.12}$$

Next, we produce approximations $y_2(\beta) \in \sum'_\beta(Y_0(\beta), t_2)$ to $x(2h_1; \beta) = x(\frac{h}{2}; \beta)$ and $y_3(\beta) \in \sum'_\beta(Y_0(\beta), t_3)$ to $x(3h_1; \beta) = x(\frac{3h}{4}; \beta)$ given by

$$y_2(\beta) \in y_0(\beta) + 2h_1 f(h_1, y_1(\beta); \beta), \quad \forall \beta \in [0,1],$$

and

$$y_3(\beta) \in y_1(\beta) + 2h_1 f(2h_1, y_2(\beta); \beta), \quad \forall \beta \in [0,1].$$

Then, we produce the approximation $y_4(\beta) \in \sum'_\beta(Y_0(\beta), t_4)$ to $x(4h_1;\beta) = x(t_1;\beta)$ given by

$$y_4(\beta) \in y_2(\beta) + 2h_1 f(3h_1, y_3(\beta); \beta), \quad \forall \beta \in [0,1].$$

The endpoint correction is now applied to $y_3(\beta) \in \sum'_\beta(Y_0(\beta), t_3)$ and $y_4(\beta) \in \sum'_\beta(Y_0(\beta), t_4)$ to produce the improved $O(h_1^2)$ approximation to $x(t_1;\beta)$

$$x_2^1(\beta) \in \frac{1}{2}[y_4(\beta) + y_3(\beta) + h_1 f(4h_1, y_4(\beta); \beta)], \quad \forall \beta \in [0,1].$$

The approximation to $x(h;\beta)$ has the properties

$$x(h;\beta) \in x_1^1(\beta) + e_1(\beta)(\frac{h}{2})^2 + e_2(\beta)(\frac{h}{2})^4 + \ldots = x_1^1(\beta) + e_1(\beta)\frac{h^2}{4} + e_2(\beta)\frac{h^4}{16} + \ldots, \quad (5.13)$$

and

$$x(h;\beta) \in x_2^1(\beta) + e_1(\beta)(\frac{h}{4})^2 + e_2(\beta)(\frac{h}{4})^4 + \ldots = x_2^1(\beta) + e_1(\beta)\frac{h^2}{16} + e_2(\beta)\frac{h^4}{256} + \ldots, \quad (5.14)$$

for all $\beta \in [0,1]$ where $e : [0,1] \to R^n$ and $x : [0,T] \times [0,1] \to R^n$. We can eliminate the $O(h^2)$ portion of this truncation error by averaging these two formulas appropriately, for any $\beta \in [0,1]$. Specifically, if we subtract (5.13) from 4 times (5.14) and divide the result by 3, we have

$$x(h;\beta) \in x_2^1(\beta) + \frac{1}{3}(x_2^1(\beta) - x_1^1(\beta)) - e_2(\beta)\frac{h^4}{64} + \ldots.$$

So the approximation

$$x_2^2(\beta) \in x_2^1(\beta) + \frac{1}{3}(x_2^1(\beta) - x_1^1(\beta))$$

for all $\beta \in [0,1]$ has error of order $O(h^4)$. Continuing in this manner, we next let $h_2 = \frac{h}{6}$ and apply Euler's method once followed b. Then we use the endpoint correction to determine the h^2 approximation, $x_3^1(\beta)$, to $x(h;\beta)$, which can be averaged with $x_2^1(\beta)$ to produce a second $O(h^4)$ approximation that we denote by $x_3^2(\beta)$. Then $x_3^2(\beta)$ and $x_2^2(\beta)$ are averaged to eliminate the $O(h^4)$ error terms and produce an approximation with error of order $O(h^6)$. Higher-order formulas are generated by continuing the process. The error is controlled by requiring that the approximations $x_1^1(\beta), x_2^2(\beta), \ldots$ be computed until $|x_i^i(\beta) - x_{i-1}^{i-1}(\beta)|$ is less than a given tolerance. If $x_i^i(\beta)$ is found to be acceptable, then $y_1(\beta)$ is set to $x_i^i(\beta)$ and computations begin again to determine $y_2(\beta)$, which will approximate $x(t_2;\beta) = x(2h;\beta)$, for any $\beta \in [0,1]$. The process is repeated until the approximation $y_N(\beta) \in A'_\beta(Y_0(\beta), t_N)$ to $x(b;\beta) \in A_\beta(Y_0(\beta), t_N)$, for all $\beta \in [0,1]$, is determined.

6 Examples

Example 6.1. *Consider the following fuzzy differential inclusions with constant coefficients*

$$\begin{cases} \frac{dx_1(t;\beta)}{dt} \in 3x_1(t;\beta) - 2x_2(t;\beta), & 0 \le t \le 0.3, \\ \frac{dx_2(t;\beta)}{dt} \in 2x_1(t;\beta) - x_2(t;\beta). & \end{cases} \quad (6.15)$$

As an initial value for the fuzzy initial value problem (6.15), we take a number $Y_0 \in E^2$ *such that*

$Y_0(\beta) = \{(x_1(0;\beta), x_2(0;\beta)) \in R^2 : x_1(0;\beta) \in [\beta-1, 1-\beta]\ ,\ x_2(0;\beta) \in [.5+.5\beta, 1.5-.5\beta]\}$, $\beta \in [0,1]$,

where

$$\sum_\beta (Y_0(\beta), t) = \begin{pmatrix} x_1(t;\beta) \\ x_2(t;\beta) \end{pmatrix} = \begin{pmatrix} e^t[x_1(0;\beta) + 2t(x_1(0;\beta) - x_2(0;\beta))] \\ e^t[x_2(0;\beta) + 2t(x_1(0;\beta) - x_2(0;\beta))] \end{pmatrix}, \tag{6.16}$$

in which $(x_1(0;\beta), x_2(0;\beta)) \in Y_0(\beta)$. Fig. 1., Fig. 2. and Fig. 3. show the plan of $\beta-$ level sets of $\bigcup_\beta \sum_\beta (Y_0(\beta), T)$ (pyramidal) and its approximation with Euler's method with $h = 0.3$ and the extrapolation method, respectively, for $\beta \in \{0, 0.1, \ldots, 1\}$.

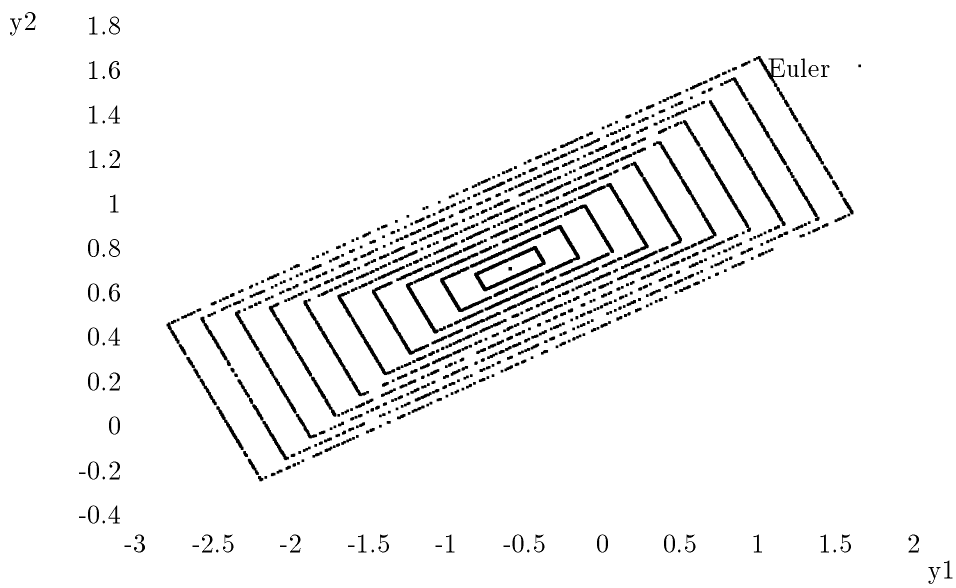

Fig. 1.

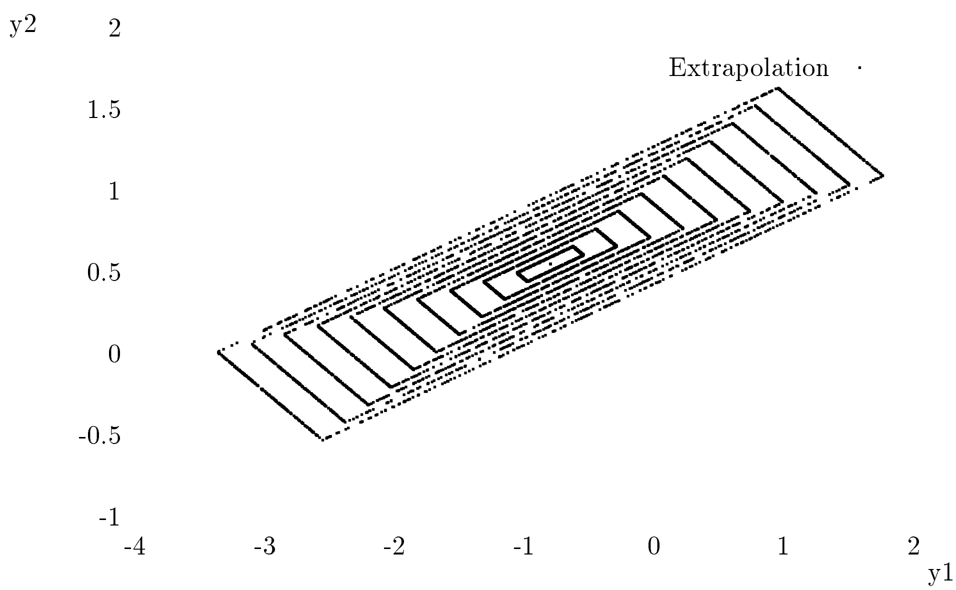

Fig. 2.

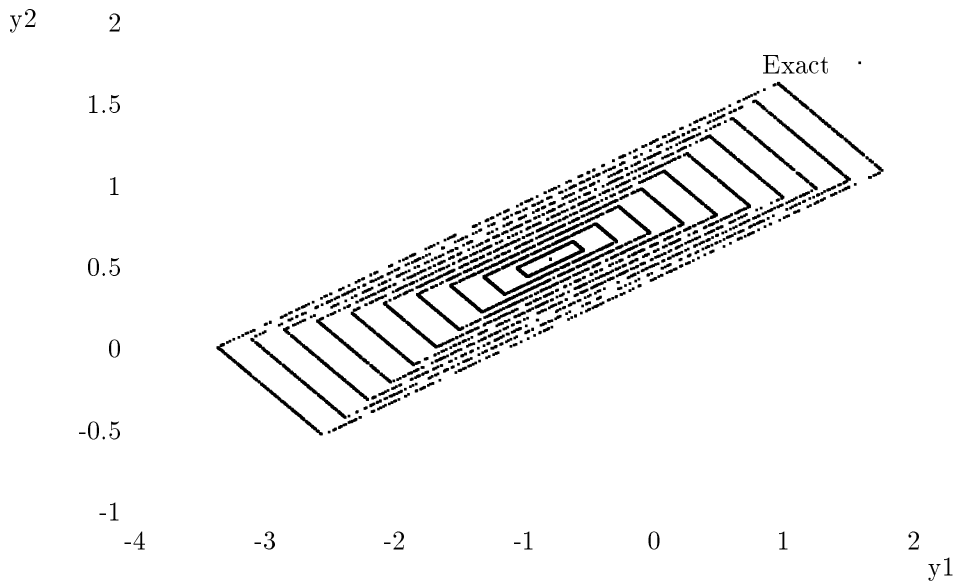

Fig. 3.

Let $C = \bigcup_\beta \sum_\beta (Y_0(\beta), T)$ and B be the approximation of C which is obtained by numerical methods. In Table 1, we compare $d_H(C, B)$ for the extrapolation method with $\epsilon = 0.001$ and Euler's method.

Table 1
The distance between the reachable set and its approximations

Euler	*Extra.*
0.7226	1.7251e-005

Example 6.2. *Consider the following fuzzy differential inclusions*

$$\begin{aligned} x_1'(t;\beta) \in &\quad -x_2(t;\beta) + 0.1x_1(t;\beta)(9 - x_1(t;\beta)^2 - x_2(t;\beta)^2) + w(\beta) \\ x_2'(t;\beta) \in &\quad -x_1(t;\beta) + 0.1x_2(t;\beta)(9 - x_1(t;\beta)^2 - x_2(t;\beta)^2) + w(\beta) \end{aligned}$$

We take a number $Y_0 \in E^2$ *such that*

$$Y_0(\beta) = \{(x_1(0;\beta), x_2(0;\beta)) \in R^2 : x_1(0;\beta) \in [\beta - 1, 1 - \beta]\ ,\ x_2(0;\beta) \in [.5 + .5\beta, 1.5 - .5\beta]\}$$

and $w(\beta) \in [\beta - 1, 1 - \beta], \beta \in [0, 1]$.

Then

$$f_1(t, x_\beta(t); \beta) = \begin{pmatrix} -x_2(t;\beta) + 0.1x_1(t;\beta)(9 - x_1(t;\beta)^2 - x_2(t;\beta)^2) + \beta - 1 \\ -x_1(t;\beta) + 0.1x_2(t;\beta)(9 - x_1(t;\beta)^2 - x_2(t;\beta)^2) + \beta - 1 \end{pmatrix}$$

and

$$f_2(t, x_\beta(t); \beta) = \begin{pmatrix} -x_2(t;\beta) + 0.1x_1(t;\beta)(9 - x_1(t;\beta)^2 - x_2(t;\beta)^2) + 1 - \beta \\ -x_1(t;\beta) + 0.1x_2(t;\beta)(9 - x_1(t;\beta)^2 - x_2(t;\beta)^2) + 1 - \beta \end{pmatrix}.$$

The Hausdorff distance between the solutions obtained the extrapolation method and Euler's method for $h = 0.5$ and $h = 0.25$ with $\epsilon = e - 5$ are $7.1801e - 008$ and 0.3202, respectively, as, are shown in Fig. 4. and Fig. 5.

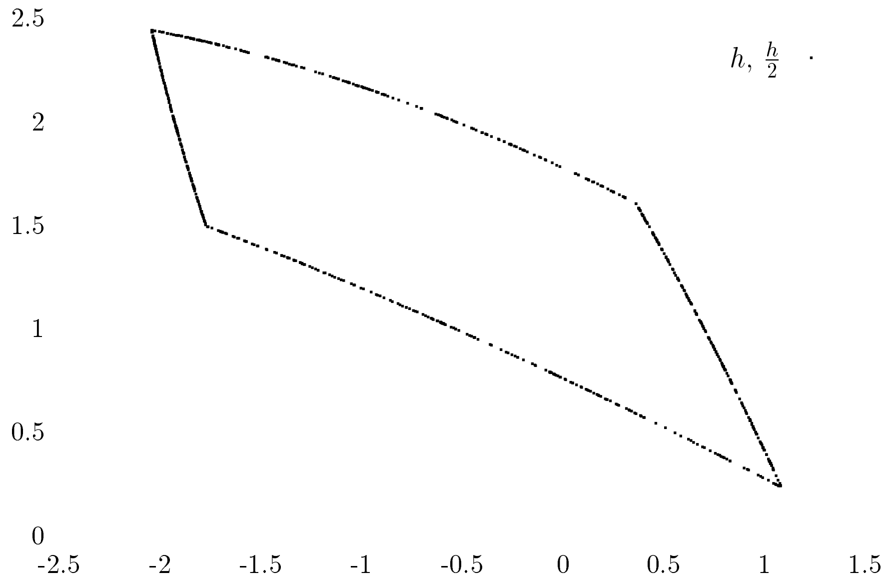

Fig. 4. Extrapolation method results with h and $\frac{h}{2}$.

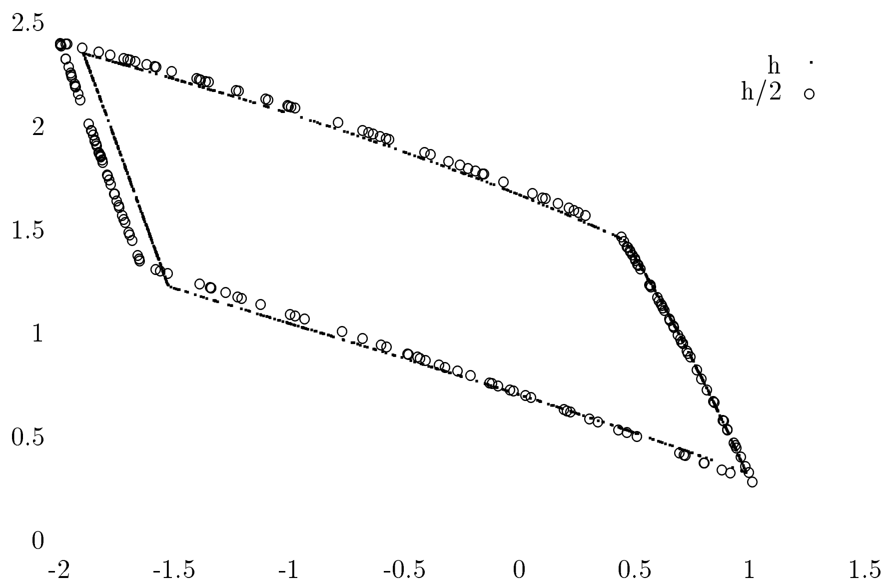

Fig. 5. Euler's method results with h and $\frac{h}{2}$.

Example 6.3. *Consider the system* $x'(t) = AX(t) + F(t)$ *where* $A \in E^{n\times n}$, $F : [0,T] \to E^n$, $X_0 \in E^n$,

$$A = \begin{pmatrix} (1.5, 0.5, .5), (1.0, 0.5, 0.5) \\ (1.0, 0.5, 0.5), (2.5, 0.5, 0.5) \end{pmatrix}, \tag{6.17}$$

$$F(t) = \begin{pmatrix} (3+4t)/2\ ,\ (1+2t)/2\ ,\ (1+2t)/2 \\ \\ (e^{-t}+e^{t})/2\ ,\ (e^{t}-e^{-t})/2\ ,\ (e^{t}-e^{-t})/2 \end{pmatrix}, \tag{6.18}$$

and

$$X_0 = \begin{pmatrix} (1.5, 0.5, 0.5) \\ (0.5, 0.5, 0.5) \end{pmatrix} \tag{6.19}$$

The exact solutions for $\beta = 0$ are $\sum_\beta(t) = (\underline{\sum}_\beta(t), \overline{\sum}_\beta(t))$ where

$$\underline{\sum}_0 = \begin{pmatrix} -2.5308 - 1.1430t + 0.0826e^{-t} + 3.1007e^{\underline{k}t} + 0.3476e^{\underline{v}t} \\ 0.2040 + 0.3734t + 0.3479e^{-t} - 0.5517e^{\underline{k}t} - 0.0002e^{\underline{v}t} \end{pmatrix} \tag{6.20}$$

and

$$\overline{\sum}_0 = \begin{pmatrix} -3.9942 - 2.3797t - 6.0022e^{t} + 10.4599e^{\overline{k}t} + 1.5363e^{\overline{v}t} \\ 2.3996 + 1.1998t + 4.0022e^{t} - 7.5399e^{\overline{k}t} + 2.1380e^{\overline{v}t} \end{pmatrix} \tag{6.21}$$

in which $\underline{k} = 0.7929,\ \underline{v} = 2.2071,\ \overline{k} = 0.9189,\ \overline{v} = 4.0811$. The numerical solution for the family of inclusions

$$x'_\beta(t) \in A_\beta X_\beta(t) + F_\beta(t),\ X_\beta(0) \in X_0(\beta),$$

have the Hausdorff distance from the exact solution at the endpoint of interval $[0, 0.01]$, by using Euler's method and the extrapolation method.

Table 2
The distance between the reachable set and its approximations

h	*Extrapolation*	*Euler*
0.01	0.0606	0.1603
0.001	1.5900e-004	0.0097

Example 6.4. *[5] Consider the FIVP in E^1, $x' = -x, x(0) = X_0$, where X_0 is a symmetric triangular fuzzy number with support [-1,1].*

Since $-x_\beta = \{-x_\beta\}$ is a singleton set in κ_c^1, it is interpreted as a family of differential inclusions

$$x'_\beta(t) = -x_\beta(t),\ x_\beta(0) \in X_\beta = (1-\beta)[-1, 1],\ \beta \in [0, 1],$$

which has the solution set $\sum_\beta(X_\beta, t)$ on $[0, t]$ comprising the functions

$$x_\beta(t) = x_\beta(0)e^{-t},\ x_\beta(0) \in X_\beta.$$

Now, we demonstrate the reachable set and its approximation at $t = 1$ with $h = 0.1$ and $\epsilon = e - 5$ in Fig. 6.

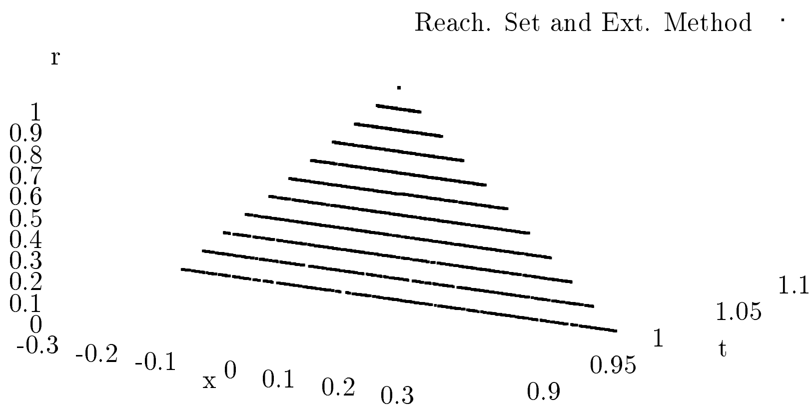

Fig. 6. $t = 1$.

Table 3
The distance between the reachable set and its approximations

Euler	*Extra.*
0.0209	4.0449e-004

7 Conclusion

The method presented in this paper for the approximation of the reachable set, is based on pyramidal fuzzy numbers. The $\beta-$level sets of these fuzzy numbers are $n-$dimensional rectangles. The convex hull of the corners of rectangles or the set of all points on them (in $n-$dimensional space) can form the reachable set. In this paper, all points ($n-$dimensional vectors) belonging to the reachable set are approximated by using Euler's method. This can be improved by the extrapolation method.

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