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Research Article

Fuzzy Local Fractional Differential Equations

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Abstract
In this paper, we propose a new type of fuzzy fractional differential equations under fuzzy Kolwankar- Gangal local fractional derivatives (for short, fuzzy KG-LFD) using fuzzy Riemann-Liouville differentiability. Then, we prove some basic results in this area like, the relation between different types of fuzzy KG-LFD and their r-cut representations, composition of the fuzzy KG-LFD and the fuzzy local fractional integral. Also, an application is provided in details such that the explicit solutions are expressed through Mittag-Leffler function.

Keywords : Fuzzy Riemann-Liouville differentiability; Fuzzy Kolwankar-Gangal local fractional diferentiability; Fuzzy local fractional differential equations (FLFDEs); Mittag-Leffler function; Fuzzy-valued function.

1 Introduction

In the last decade, fractional calculus attracted a huge number of physicists and mathematicians. Fractional differential and integral equations play an important roles in the modeling of real problems in scientific fields and engineering, as shown in [1, 35, 27, 31]. Firstly, the fractional derivative has been introduced by Riemann-Liouvill, wellknown as Riemann-Liouvill derivative defined by $\frac{d^\beta f(x)}{[d(x-a)]^\beta} = \frac{1}{\Gamma(1-\beta)} \int_a^x \frac{f(t)dt}{(x-t)^\beta}), 0 < \beta < 1$. For more details see [6, 8, 16, 26, 27, 29, 31]. The local fractional integrals and derivatives are appeared in many mathematical and physical problems and have plenty of usages. One of the most important usages of them is to analyze fractal sets, because fractals are irregular and nowhere differentiable functions [20]. Based on the local fractional calculus, and we can easily use the properties local fractional integrals and derivatives to analyze them. This was introduced by Kolwankar and Gangal local fractional derivative KG-LFD [20, 21] in 1996 so that to investigate the local behavior of fractional differential equations and then was studied by many researches [9, 12, 16, 15]. LFD operators engender a new kind of differential equations, referred as local fractional differential equations (LFDEs)

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different from the conventional fractional differential equations(FDEs). The (LFDEs) involving LFD has been used in modeling phenomena involving fractal time. LFD therefore provide a much needed tool for calculus of fractal space-time [9]. Recently, Agarwal, Arshad and Allahviranloo[6, 8, 2] proposed on the fractional differential equations with uncertainty. They considered Riemann-Liouville differentiability to solve (FFDEs). In this paper, concentrate on local fractional derivatives which were defined by KG-LFD. We first define local fractional H-differentiability which is a direct extension of local fractional derivatives KG-LFD with using hukuhara difference. So that to drive such concept which is constructed based on the combination of strongly generalized differentiability [11] and local fractional derivative [20, 22, 23].
At the end, we try to solve fuzzy local fractional differential equations (FLFDEs) under local fractional H-differentiability.
This paper is organized as follows:
In section 2, we recall some basic notions of fuzzy number. In section 3, Riemann-Liouville H-differentiability is introduced. We introduce of local fractional H-differentiability is based on the Riemann-Liouville H-differentiability for a fuzzy-valued function of a single variable and some of its properties are considered, are given in section 4. In section 5, the solutions (FLFDEs) under local fractional H-differentiability and An applications to (FLFDEs) by Mittag-Leffer function in fractal space is also given. Finally, conclusion and future research are drawn in section 6.

2 Preliminaries

We now recall some definitions needed through the paper. The basic definition of fuzzy numbers is given in [40]. By R, we denote the set of all real numbers. A fuzzy number is a mapping $u : R \to [0, 1]$ with the following properties:
(a) u is upper semi-continuous,
(b) u is fuzzy convex, i.e., $u(\lambda x + (1 - \lambda)y) \geq min\{u(x), u(y)\}$ for all $x, y \in R, \lambda \in [0, 1]$,
(c) u is normal, i.e., $\exists x_0 \in R$ for which $u(x_0) = 1$,
(d) supp $u = \{x \in R \mid u(x) > 0\}$ is the support of the u, and its closure cl(supp u) is compact. Let E be the set of all fuzzy number on R. The r-level set of a fuzzy number $u \in E$, $0 \leq r \leq 1$, denoted by $[u]_r$, is defined as

$$[u]_r = \begin{cases} \{x \in R \mid u(x) \geq r\} & if \quad 0 < r \leq 1 \\ cl(supp\ u) & if \quad r = 0 \end{cases}$$

It is clear that the r-level set of a fuzzy number is a closed and bounded interval $[\underline{u}(r), \overline{u}(r)]$, where $\underline{u}(r)$ denotes the left-hand endpoint of $[u]_r$ and $\overline{u}(r)$ denotes the right-hand endpoint of $[u]_r$. Since each $y \in R$ can be regarded as a fuzzy number $\widetilde{y}$ defined by

$$\widetilde{y}(t) = \begin{cases} 1 & if \quad t = y \\ 0 & if \quad t \neq y \end{cases}$$

Definition 2.1. *A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$, $0 \leq r \leq 1$, which satisfy the following requirements:*

1. *$\underline{u}(r)$ is a bounded non-decreasing left continuous function in $(0, 1]$, and right continuous at 0,*

2. $\overline{u}(r)$ *is a bounded non-increasing left continuous function in* $(0,1]$, *and right continuous at 0,*

3. $\underline{u}(r) \leq \overline{u}(r),\ 0 \leq r \leq 1.$

For arbitrary $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and $k > 0$ we define addition $u + v$, subtraction $u \ominus v$ and scalar multiplication by k as (See [28, 40])

$$u+v = (\underline{u}(r)+\underline{v}(r), \overline{u}(r)+\overline{v}(r)), u-v = (\underline{u}(r)-\overline{v}(r), \overline{u}(r)-\underline{v}(r)), k\odot u = \begin{cases} (k\underline{u}, k\overline{u}), & k \geq 0, \\ (k\overline{u}, k\underline{u}), & k < 0. \end{cases}$$

Lemma 2.1. *[30]. If* $u \in E$ *then the following properties hold:*
(i) $[u]^{r_2} \subset [u]^{r_1}$ *if* $0 < r_1 \leq r_2 \leq 1$*;*
(ii) $\{r_n\} \subset (0,1]$ *is a nondecreasing sequence which converges to* r *then* $[u]^r = \bigcap_{n\geq 1} [u]^{r_n}$ *(i.e,* u_1^r *and* u_2^r *are left-continuous with respect to* r*).*
Conversely, if $A_r = \{[u_1^r, u_2^r]; r \in (0,1]\}$ *is a family of closed real intervals verifying (i) and (ii), then* $\{A_r\}$ *defined a fuzzy number* $u \in E$ *such that* $[u]^r = A_r$.

The Hausdorff distance between fuzzy numbers given by $d : E \times E \longrightarrow R_+ \bigcup\{0\}$,

$$d(u,v) = \sup_{r\in[0,1]} \max\left\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\right\},$$

where $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r)) \subset R$ is utilized in [10]. Then, it is easy to see that d is a metric in E and has the following properties (See [32])

(1) $d(u \oplus w, v \oplus w) = d(u,v), \quad \forall u, v, w \in E,$
(2) $d(k \odot u, k \odot v) = |k| d(u,v), \quad \forall k \in R, u, v \in E,$
(3) $d(u \oplus v, w \oplus e) \leq d(u,w) + d(v,e), \quad \forall u, v, w, e \in E,$
(4) (d, E) is a complete metric space.

Definition 2.2. . *Let* $x, y \in E$. *If there exists* $z \in E$ *such that* $x = y + z$, *then* z *is called the H-difference of x,y and it is denoted* $x \ominus y$. *In what follows, we fixed* $I = (a,b)$, *for* $a, b \in R$, $a < b$.

3 Fuzzy Riemann-Liouville differentiability

In this section, we introduce our definition of fuzzy Riemann-Liouville integrals and derivatives under Hukuhara difference. We try to produce such definitions and statements similar to the non-fractional one in fuzzy context [10].
We denote $C^F[a,b]$ as the space of all continuous fuzzy-valued functions on $[a,b]$. Also, we denote the space of all Lebesque integrable fuzzy-valued functions on the bounded interval $[a,b] \subset R$ by $L^F[a,b]$. Now, we define the fuzzy Riemann-Liouville integral of fuzzy-valued function as follows:

Definition 3.1. *[2]. Let* $f \in C^F[a,b] \cap L^F[a,b]$. *The fuzzy Riemann-Liouville integral of fuzzy-valued function* f *is defined as following:*

$$(I_{a^+}^{\beta} f)(x) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\beta}}, \quad x > a, \quad 0 < \beta \leq 1. \tag{3.1}$$

Let us consider the r-cut representation of fuzzy-valued function f as $f(x;r) = [\underline{f}(x;r), \overline{f}(x;r)]$, for $0 \leq r \leq 1$, then we can indicate the fuzzy Riemann-Liouville integral of fuzzy-valued function f based on the lower and upper functions is indicated as follows:

Theorem 3.1. *[2]. Let $f \in C^F[a,b] \cap L^F[a,b]$ is a fuzzy-valued function. The fuzzy Riemann-Liouville integral of a fuzzy-valued function f can be expressed as follows:*

$$(I^{\beta}_{a^+} f)(x;r) = \left[(I^{\beta}_{a^+} \underline{f})(x;r), (I^{\beta}_{a^+} \overline{f})(x;r)\right] \ , \ 0 \leq r \leq 1 \tag{3.2}$$

where

$$(I^{\beta}_{a^+} \underline{f})(x;r) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{\underline{f}(t;r)dt}{(x-t)^{1-\beta}}, \tag{3.3}$$

$$(I^{\beta}_{a^+} \overline{f})(x;r) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{\overline{f}(t;r)dt}{(x-t)^{1-\beta}}. \tag{3.4}$$

Now, we define the fuzzy Riemann-Liouville fractional derivatives about order $0 < \beta < 1$ for fuzzy-valued function f.

Definition 3.2. *[2]. Let $f \in C^F[a,b] \cap L^F[a,b]$, $x_0 \in (a,b)$ and Then :*
$\Phi(x) = \frac{1}{\Gamma(1-\beta)} \int_a^x \frac{f(t)dt}{(x-t)^{\beta}}$. We say f is Riemann-Liouville H-differentiable about order $0 < \beta < 1$ at x_0,if there exists an element $({}^{RL}D^{\beta}_{a^+} f)(x_0) \in E$, such that for $h > 0$ sufficiently small

$$(i) \quad ({}^{RL}D^{\beta}_{a^+} f)(x_0) = \lim_{h \longrightarrow 0^+} \frac{\Phi(x_0+h) \ominus \Phi(x_0)}{h} = \lim_{h \longrightarrow 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0-h)}{h},$$

or

$$(ii) \quad ({}^{RL}D^{\beta}_{a^+} f)(x_0) = \lim_{h \longrightarrow 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0+h)}{-h} = \lim_{h \longrightarrow 0^+} \frac{\Phi(x_0-h) \ominus \Phi(x_0)}{-h},$$

or

$$(iii) \quad ({}^{RL}D^{\beta}_{a^+} f)(x_0) = \lim_{h \longrightarrow 0^+} \frac{\Phi(x_0+h) \ominus \Phi(x_0)}{h} = \lim_{h \longrightarrow 0^+} \frac{\Phi(x_0-h) \ominus \Phi(x_0)}{-h},$$

or

$$(i\nu) \quad ({}^{RL}D^{\beta}_{a^+} f)(x_0) = \lim_{h \longrightarrow 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0+h)}{-h} = \lim_{h \longrightarrow 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0-h)}{h},$$

For sake of simplicity, we say that the fuzzy-valued function f is ${}^{RL}[(1)-\beta]$- differentiable if it is differentiable as in the Definition 3.2 case (i), and f is ${}^{RL}[(2)-\beta]$ differentiable if it is differentiable as in the Definition 3.2 case (ii) and so on for the other cases.

Theorem 3.2. *[2]. Let $f \in C^F[a,b] \cap L^F[a,b]$, x_0 in (a,b) and $0 < \beta < 1$. Then:*

(i) Let us consider f is ${}^{RL}[(1)-\beta]$-differentiable fuzzy-valued function, then:

$$\left({}^{RL}D^{\beta}_{a^+}f\right)(x_0,r) = \left[({}^{RL}D^{\beta}_{a^+}\underline{f})(x_0,r), ({}^{RL}D^{\beta}_{a^+}\overline{f})(x_0,r)\right], \quad 0 \le r \le 1,$$

(ii) Let us consider f is ${}^{RL}[(2)-\beta]$-differentiable fuzzy-valued function, then:

$$\left({}^{RL}D^{\beta}_{a^+}f\right)(x_0,r) = \left[({}^{RL}D^{\beta}_{a^+}\overline{f})(x_0,r), ({}^{RL}D^{\beta}_{a^+}\underline{f})(x_0,r)\right], \quad 0 \le r \le 1,$$

where

$$({}^{RL}D^{\beta}_{a^+}\underline{f})(x_0,r) = \left[\frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\int_a^x \frac{\underline{f}(t;r)dt}{(x-t)^{\beta}}\right]_{x=x_0},$$

$$({}^{RL}D^{\beta}_{a^+}\overline{f})(x_0,r) = \left[\frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\int_a^x \frac{\overline{f}(t;r)dt}{(x-t)^{\beta}}\right]_{x=x_0}.$$

Theorem 3.3. *[2]. Let $f \in C^F[a,b] \cap L^F[a,b]$ be a Riemann-Liouville H-differentiable of order $0 < \beta \le 1$ on each point $x \in (a,b)$ in the sense of Definition 3.2 case (iii) or case (iv). Then $({}^{RL}D^{\beta}_{a^+}f)(x) \in R$ for all $x \in (a,b)$.*

4 Fuzzy Local fractional differentiability

In this section, we introduce our definition of fuzzy local fractional differentiability is based on the Riemann-Liouville differentiability for a fuzzy-valued functions.

Definition 4.1. *Let us consider the fuzzy-valued function $f : [a,b] \to E$, then the fuzzy local fractional integrable of f of order β on the interval $[a,b]$ is denoted by $\left({}_aI^{\beta}_b f\right)(x)$ and defined as follows:*

$$({}_aI^{\beta}_b f)(x) = \frac{1}{\Gamma(1+\beta)}\int_a^b f(t) \odot (dt)^{\beta}. \tag{4.5}$$

Theorem 4.1. *. Let $f \in C^F_{\beta}[a,b] \cap L^F[a,b]$ is a fuzzy-valued function. The fuzzy local fractional integral of a fuzzy-valued function f can be expressed as follows:*

$$\left(I^{\beta}_a f\right)(x;r) = \left[(I^{\beta}_a \underline{f})(x;r), (I^{\beta}_a \overline{f})(x;r)\right], \quad 0 \le r \le 1, \tag{4.6}$$

where

$$\left(I^{\beta}_a \underline{f}\right)(x;r) = \frac{1}{\Gamma(1+\beta)}\int_a^x \underline{f}(t;r)(dt)^{\beta},$$

$$\left(I_a^{\beta}\overline{f}\right)(x;r)=\frac{1}{\Gamma(1+\beta)}\int_a^x \overline{f}(t;r)(dt)^{\beta}.$$

Proof. *since* $f(t)\in E$ *then for* $(0<\beta<1)$, *we set*

$$A_r =: \frac{1}{\Gamma(1+\beta)}\left[\int_a^x \underline{f}(t;r)(dt)^{\beta}, \int_a^x \overline{f}(t;r)(dt)^{\beta}\right],$$

for $r_1\leq r_2$, *we have that* $\underline{f}(t;r_1)\leq \underline{f}(t;r_2)$ *and* $\overline{f}(t;r_1)\geq \overline{f}(t;r_2)$. *It follows that* $A_{r_1}\supseteq A_{r_2}$. *Since*

$$\underline{f}(t;0)\leq \underline{f}(t;r_n)\leq \underline{f}(t;1),$$

we have

$$\begin{cases} |\underline{f}(t;r_n)|\leq \max\{|\underline{f}(t;0)|,|\underline{f}(t;1)|=: g_1(t)\\ |\overline{f}(t;r_n)|\leq \max\{|\overline{f}(t;0)|,|\overline{f}(t;1)|=: g_2(t)\end{cases}$$

for $r_n\in(0,1]$ *and* $i=1,2$ *obviously,* g_i *is Lebesgue integrable on* $[a,x]$. *Therefore, if* $r_n\uparrow r$ *then by the Lebesgue's Dominated Convergence Theorem, we have*

$$\begin{cases} \lim_{n\longrightarrow\infty}\frac{1}{\Gamma(1+\beta)}\int_a^x \underline{f}(t;r_n)(dt)^{\beta}=\frac{1}{\Gamma(1+\beta)}\int_a^x \underline{f}(t;r)(dt)^{\beta},\\ \lim_{n\longrightarrow\infty}\frac{1}{\Gamma(1+\beta)}\int_a^x \overline{f}(t;r_n)(dt)^{\beta}=\frac{1}{\Gamma(1+\beta)}\int_a^x \overline{f}(t;r)(dt)^{\beta}.\end{cases}$$

From Lemma 2.1, the proof is complete. □

Definition 4.2. *Let* $f:(a,b)\to E$, $f\in C^F[a,b]\cap L^F[a,b]$, $x_0\in(a,b)$ *and Then :* $\Phi(x)=\frac{1}{\Gamma(1-\beta)}\int_{x_0}^x \frac{f(t)\ominus f(x_0)dt}{(x-t)^{\beta}}$. *we say that* f *the local fractional H-differentiable* (LFD) *order* β, $(0<\beta<1)$ *at* x_0, *if there exists an element* $(\mathcal{D}^{\beta}f)(x)|_{x=x_0}\in E$, *such that for* $h>0$ *sufficiently small*

$$(i)\ (\mathcal{D}^{\beta}f)(x_0)=\lim_{x\longrightarrow x_0^+}\left(\lim_{h\longrightarrow 0^+}\frac{\Phi(x_0+h)\ominus\Phi(x_0)}{h}\right)=\lim_{x\longrightarrow x_0^+}\left(\lim_{h\longrightarrow 0^+}\frac{\Phi(x_0)\ominus\Phi(x_0-h)}{h}\right),$$

or

$$(ii)\ (\mathcal{D}^{\beta}f)(x_0)=\lim_{x\longrightarrow x_0^+}\left(\lim_{h\longrightarrow 0^+}\frac{\Phi(x_0)\ominus\Phi(x_0+h)}{-h}\right)=\lim_{x\longrightarrow x_0^+}\left(\lim_{h\longrightarrow 0^+}\frac{\Phi(x_0-h)\ominus\Phi(x_0)}{-h}\right),$$

or

$$(iii)\ (\mathcal{D}^{\beta}f)(x_0)=\lim_{x\longrightarrow x_0^+}\left(\lim_{h\longrightarrow 0^+}\frac{\Phi(x_0+h)\ominus\Phi(x_0)}{h}\right)=\lim_{x\longrightarrow x_0^+}\left(\lim_{h\longrightarrow 0^+}\frac{\Phi(x_0-h)\ominus\Phi(x_0)}{-h}\right),$$

or

$$(i\nu)\ (\mathcal{D}^{\beta}f)(x_0)=\lim_{x\longrightarrow x_0^+}\left(\lim_{h\longrightarrow 0^+}\frac{\Phi(x_0)\ominus\Phi(x_0+h)}{-h}\right)=\lim_{x\longrightarrow x_0^+}\left(\lim_{h\longrightarrow 0^+}\frac{\Phi(x_0)\ominus\Phi(x_0-h)}{h}\right),$$

provided that the Hukuhara differences exist.
For sake of simplicity, we say that the fuzzy-valued function f is ${}^{LF}[(1)-\beta]$- differentiable if it is differentiable as in the Definition 4.2 case (i), and f is ${}^{LF}[(2)-\beta]$ differentiable if it is differentiable as in the Definition 4.2 case (ii) and so on for the other cases.

Remark 4.1. *One can easily verify that the fuzzy local derivative can be expressed by fuzzy Riemann-Liouville derivative as follows:*

$$\left(\mathcal{D}^{\beta}f\right)(x_0)=\lim_{x\longrightarrow x_0^+}\left({}^{RL}D^{\beta}_{x_0^+}\{f(x)\ominus f(x_0)\}\right)$$

Theorem 4.2. . *Let $f\in C^F[a,b]\cap L^F[a,b]$, x_0 in (a,b) and $0<\beta<1$. Then:*

(i) Let us consider f is ${}^{LF}[(1)-\beta]$-differentiable fuzzy-valued function at point x_0, then:

$$\left(\mathcal{D}^{\beta}f\right)(x_0,r)=\left[(\mathcal{D}^{\beta}\underline{f})(x_0,r),(\mathcal{D}^{\beta}\overline{f})(x_0,r)\right],\quad 0\le r\le 1,$$

(ii) Let us consider f is ${}^{LF}[(2)-\beta]$-differentiable fuzzy-valued function at point x_0, then:

$$\left(\mathcal{D}^{\beta}f\right)(x_0,r)=\left[(\mathcal{D}^{\beta}\overline{f})(x_0,r),(\mathcal{D}^{\beta}\underline{f})(x_0,r)\right],\quad 0\le r\le 1,$$

where

$$\left(\mathcal{D}^{\beta}\underline{f}\right)(x_0,r)=\lim_{x\longrightarrow x_0^+}\frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\int_{x_0}^{x}\frac{\underline{f}(t;r)-\underline{f}(x_0;r)dt}{(x-t)^{\beta}},$$

$$\left(\mathcal{D}^{\beta}\overline{f}\right)(x_0,r)=\lim_{x\longrightarrow x_0^+}\frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\int_{x_0}^{x}\frac{\overline{f}(t;r)-\overline{f}(x_0;r)dt}{(x-t)^{\beta}}.$$

Proof. *Let us consider f ${}^{RL}[(1)-\beta]$-differentiable and $x_0\in(a,b)$, then we have:*
$\left(\mathcal{D}^{\beta}f\right)(x_0,r)=\lim_{x\longrightarrow x_0^+}{}^{RL}D^{\beta}_{x_0^+}([f(x,r)\ominus f(x_0,r)])$
$=\lim_{x\longrightarrow x_0^+}\left({}^{RL}D^{\beta}_{x}[\underline{f}(x,r)-\underline{f}(x_0,r),\overline{f}(x,r)-\overline{f}(x_0,r)]\right)$
By linear property ${}^{RL}D^{\beta}_{x_0^+}$ and passing to the $\lim_{x\longrightarrow x_0^+}$, we get:
$\left(\mathcal{D}^{\beta}f\right)(x_0,r)=\left[\lim_{x\longrightarrow x_0^+}({}^{RL}D^{\beta}_{x}(\underline{f}(x,r)-\underline{f}(x_0,r))),\lim_{x\longrightarrow x_0^+}({}^{RL}D^{\beta}_{x}(\overline{f}(x,r)-\overline{f}(x_0,r)))\right]$
Consequently, we get

$$\left(\mathcal{D}^{\beta}f\right)(x_0,r)=\left[(\mathcal{D}^{\beta}\underline{f})(x_0,r),(\mathcal{D}^{\beta}\overline{f})(x_0,r)\right],$$

which completes the proof of this part. Now, Let us consider f is ${}^{RL}[(2)-\beta]$-differentiable, using definition of fuzzy KG-LFD we have:

$$\left(\mathcal{D}^{\beta} f\right)(x_0,r) = \lim_{x\longrightarrow x_0^+} {}^{RL}D^{\beta}_{x_0^+}\left([f(x,r)\ominus f(x_0,r)]\right)$$
$$= \lim_{x\longrightarrow x_0^+} \left({}^{RL}D^{\beta}_{x}[\underline{f}(x,r)-\underline{f}(x_0,r), \overline{f}(x,r)-\overline{f}(x_0,r)]\right)$$

By linear property ${}^{RL}D^{\beta}_{x_0^+}$ type of differentiability of f and passing to the $\lim_{x\longrightarrow x_0^+}$, we get: $\left(\mathcal{D}^{\beta} f\right)(x_0,r)=\Big[\lim_{x\longrightarrow x_0^+}({}^{RL}D^{\beta}_{x}(\overline{f}(x,r)-\overline{f}(x_0,r))), \lim_{x\longrightarrow x_0^+}({}^{RL}D^{\beta}_{x}(\underline{f}(x,r)-\underline{f}(x_0,r)))\Big]$

Consequently, we get

$$\left(\mathcal{D}^{\beta} f\right)(x_0,r) = \left[(\mathcal{D}^{\beta}\overline{f})(x_0,r), (\mathcal{D}^{\beta}\underline{f})(x_0,r)\right],$$

which completes the proof. □
Now, we state some useful results about the reminder type of fuzzy KG-LFD differentiability. The proof is completely similar to the obtained result by Bede et al. [10] and Salahshour et al. [34], so the proof is omitted.

Theorem 4.3. *Let $f \in C^F[a,b] \cap L^F[a,b]$ be a local fractional H-differentiable of order $0<\beta\leq 1$ on each point $x\in(a,b)$ in the sense of Definition 4.2 case (iii) or case (iv). Then $(\mathcal{D}^{\beta} f)(x)\in R$ for all $x\in(a,b)$.*

Remark 4.2. *For case $\beta=1$, the fuzzy KG-LFD reduces to the generalized differentiability [10].*

Definition 4.3. *If there exists the relation*

$$d(f(x),f(x_0)) < \varepsilon^{\beta}, \tag{4.7}$$

with $|x-x_0|<\delta$, for $\exists\delta>0$ and $\varepsilon,\delta\in R$. Then fuzzy valued $f(x)$ is called fuzzy local fractional continuous on the interval (a,b), denoted by

$$f(x)\in C^F_{\beta}(a,b).$$

Lemma 4.1. *If $0<\beta<1$ and $f(x)\in C^F_{\beta}[a,b]\cap L^F[a,b]$, then the following equality hold almost every where on $[a,b]$ for case ${}^{LF}[(1)-\beta]$-differentiability:*

$$\left(\mathcal{D}^{\beta} I^{\beta}_{a} f\right)(x;r) = \left[\underline{f}(x;r), \overline{f}(x;r)\right] \quad 0\leq r\leq 1, \tag{4.8}$$

and the following equality hold almost every where on $[a,b]$ for case of ${}^{LF}[(2)-\beta]$-differentiability:

$$\left(\mathcal{D}^{\beta} I^{\beta}_{a} f\right)(x;r) = \left[\overline{f}(x;r), \underline{f}(x;r)\right] \quad 0\leq r\leq 1. \tag{4.9}$$

Proof. Using Definition 4.2 and Theorem 4.2, we obtain:

$$\left(I_a^\beta f\right)(x;r) = \left[I_a^\beta \underline{f}(x;r), I_a^\beta \overline{f}(x;r)\right] \quad 0 \le r \le 1.$$

Using linear property $\mathcal{D}^\beta$ for case ${}^{LF}[(1)-\beta]$-differentiability we have

$$\mathcal{D}^\beta\left(I_a^\beta f(x;r)\right) = \left[\mathcal{D}^\beta I_a^\beta \underline{f}(x;r), \mathcal{D}^\beta I_a^\beta \overline{f}(x;r)\right]$$

$$= \left[\underline{f}(x;r), \overline{f}(x;r)\right]$$

and for case ${}^{LF}[(2)-\beta]$-differentiability:

$$\mathcal{D}^\beta\left(I_a^\beta f(x;r)\right) = \left[\mathcal{D}^\beta I_a^\beta \overline{f}(x;r), \mathcal{D}^\beta I_a^\beta \underline{f}(x;r)\right]$$

$$= \left[\overline{f}(x;r), \underline{f}(x;r)\right]$$

Lemma 4.2. *Let $f(x) \in C_\beta^F[a,b] \cap L^F(a,b)$ and $0<\beta<1$, then we have*

$$\left({}_0I_a^\beta \mathcal{D}^\beta f\right)(x) = f(x) \ominus f(0),$$

for case ${}^{LF}[(1)-\beta]$-differentiability and we have

$$\left({}_0I_a^\beta \mathcal{D}^\beta f\right)(x) = -(f(0)) \ominus (-f(x))$$

,
for case ${}^{LF}[(2)-\beta]$-differentiability, and provided that the mentioned Hukuhara differences exist. Also, $-f(x) = [-\overline{f}(x;r), -\underline{f}(x;r)]$.

Proof. Indeed, we have by direct computation for case of ${}^{LF}[(1)-\beta]$-differentiability:

$$\left(I_a^\beta \mathcal{D}^\beta f\right)(x;r) = \left[(I_a^\beta \mathcal{D}^\beta \underline{f})(x;r), (I_a^\beta \mathcal{D}^\beta \overline{f})(x;r))\right] = \left[\underline{f}(x;r) - \underline{f}(x_0;r), \overline{f}(x;r) - \overline{f}(x_0;r)\right],$$

and for ${}^{LF}[(2)-\beta]$-differentiability:

$$\left(I_a^\beta \mathcal{D}^\beta f\right)(x;r) = \left[(I_a^\beta \mathcal{D}^\beta \overline{f})(x;r), (I_a^\beta \mathcal{D}^\beta \underline{f})(x;r))\right] = \left[\overline{f}(x;r) - \overline{f}(x_0;r), \underline{f}(x;r) - \underline{f}(x_0;r)\right],$$

for all $0 \le r \le 1$ which complete the proofs. □

5 Result on the fuzzy local fractional calculus

Definition 5.1. *A fuzzy-valued function $f(x)$ is called a function of exponent β, $(0<\beta<1)$, which satisfy Höltder function of exponent β,if for $x,y \in R$ such that*

$$d(f(x),f(y)) < C|x-y|^\beta, \quad for \ \ any \ \ x,y \in R, \tag{5.10}$$

where C is real constant.

Definition 5.2. *A functionf* $: R \to E, x \to f(x)$, *is called continuous of order* β, $(0 < \beta < 1)$ *or* β*-continuous on the interval* $[a, b]$, *if*

$$d(f(x), f(y)) = O((x-y)^{\beta}), \quad for \ \ any \ \ x, y \in [a, b], \tag{5.11}$$

Proposition 5.1. *Assume that the continuous function* $f(x)^{((k+1)\beta)} \in C_{\beta}^{F}(a, b))$, $f : R \to E$, *and* $f^{(k\beta)}$ *is local fractional* H*-differentiability of order* $k\beta$ *near the point* $x = x_0$, *for any positive integer* k *and any* β, $0 < \beta < 1$, *then the following equality holds, which is*

if $f(x)$ *is* ${}^{LF}[(1) - \beta]$*-differentiability then:*

$$f(x; r) = \left[\sum_{k=0}^{n} \frac{\underline{f}^{(k\beta)}(x_0; r)}{\Gamma(1+\beta k)}(x-x_0)^{k\beta}, \sum_{k=0}^{n} \frac{\overline{f}^{(k\beta)}(x_0; r)}{\Gamma(1+\beta k)}(x-x_0)^{k\beta}\right]$$

or

if $f(x)$ *is* ${}^{LF}[(2) - \beta]$*-differentiability then:*

$$f(x; r) = [f_1(x, r), f_2(x, r)],$$

where

$$f_1(x; r) = \sum_{k=1,even}^{n} \frac{\underline{f}^{(k\beta)}(x_0; r)}{\Gamma(1+\beta k)}(x-x_0)^{k\beta} + \sum_{k=1,odd}^{n} \frac{\overline{f}^{(k\beta)}(x_0; r)}{\Gamma(1+\beta k)}(x-x_0)^{k\beta},$$

$$f_2(x; r) = \sum_{k=1,even}^{n} \frac{\overline{f}^{(k\beta)}(x_0; r)}{\Gamma(1+\beta k)}(x-x_0)^{k\beta} + \sum_{k=1,odd}^{n} \frac{\underline{f}^{(k\beta)}(x_0; r)}{\Gamma(1+\beta k)}(x-x_0)^{k\beta}.$$

Remark 5.1. *Let* $E_{\beta} : R \to E$, $x \to E_{\beta}(x)$, *denote a continuously function, which is so-called the Mittag-Leffler function*

$$E_{\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+\beta k)}, \quad 0 < \beta < 1.$$

As further result of the above formula, in fractional space defined by the expression

$$E_{\beta}(x^{\beta}) = \sum_{k=0}^{\infty} \frac{x^{\beta k}}{\Gamma(1+\beta k)}, \quad 0 < \beta < 1,$$

one has a continuously non-differentiable function.

6 FLFDEs under local fractional H-differentiability

Here, we will investigate the integral form of the original (FLFDE) under local fractional H-differentiability.First we consider the following fuzzy local fractional differential equation (FLFDE):

$$\begin{cases} (\mathcal{D}^{\beta}y)(x) = f[x, y(x)] \\ y(x_0) = y_0 \in E \end{cases} \tag{6.12}$$

where $f(x) \in C_{\beta}^{F}[a,b] \cap L^{F}[a,b]$ continuous fuzzy-valued function and $x_0 \in [a,b]$.

Lemma 6.1. *Let $0 < \beta < 1$ and $x_0 \in R$. the fuzzy local fractional differential equation (FLFDE)* (6.12) *is equivalent to one of the following integral equations:*

$$y(x) = y(x_0) + \frac{1}{\Gamma(1+\beta)} \int_{x_0}^{x} f(t, y(t))(dt)^{\beta}, \quad x \in [x_0, b], \tag{6.13}$$

if $y(t)$ is ${}^{LF}[(1)-\beta]$-differentiable, and

$$y(x) = y(x_0) \ominus \frac{1}{\Gamma(1+\beta)}(-1) \int_{x_0}^{x} f(t, y(t))(dt)^{\beta}, \quad x \in [x_0, b], \tag{6.14}$$

if $y(t)$ is ${}^{LF}[(2)-\beta]$-differentiable, provided that $\ominus$ exists.

Proof. Let us consider $y(t)$ is ${}^{LF}[(1)-\beta]$-differentiable and using Lemma 4.2 and Definition 4.1, we have

$$I_a^{\beta}(\mathcal{D}^{\beta}y)(x) = I_a^{\beta}\left(f[x, y(x)]\right).$$

Therefore,

$$y(x) \ominus y(x_0) = \frac{1}{\Gamma(1+\beta)} \int_a^x f(t, y(t))(dt)^{\beta}.$$

Then, we conclude that

$$y(x) = y(x_0) + \frac{1}{\Gamma(1+\beta)} \int_a^x f(t, y(t))(dt)^{\beta}.$$

If $y(t)$ is ${}^{LF}[(2)-\beta]$-differentiable, and using Lemma 4.2 and Definition 4.1, we get

$$I_a^{\beta}(\mathcal{D}^{\beta}y)(x) = I_a^{\beta}\left(f[x, y(x)]\right),$$

which leads to derive

$$(-y(x_0)) \ominus (-y(x)) = \frac{1}{\Gamma(1+\beta)} \int_a^x f(t, y(t))(dt)^{\beta}.$$

Finally, we obtain

$$y(x) = y(x_0) \ominus \frac{1}{\Gamma(1+\beta)}(-1)\int_a^x f(t, y(t))(dt)^\beta,$$

which complete the proofs. □

7 An application

Now, we drive the solutions to the fuzzy local fractional differential equations under local fractional H-differentiability in fractal space according to the related Volterra integral equation proposed in Lemma 6.1. To this end, consider The relaxation equation

$$\begin{cases} \frac{d^\beta y(x)}{dx^\beta} = \lambda c^\beta y(x), \quad c > 0,\ x > 0,\ 0 < \beta < 1, \\ y(x_0) = y_0 \in E \end{cases} \tag{7.15}$$

We solve this FLFDEs according to two following cases
Case I. Let us consider $\lambda \geq 0$. Then, using ${}^{LF}[(1)-\beta]$-differentiability and applying Eq. (6.13), we get the solution as follows:

$$\begin{cases} \underline{y}(x;r) = \underline{y}(x_0;r)E_\beta[c^\beta t^\beta], \\ \overline{y}(x;r) = \overline{y}(x_0;r)E_\beta[c^\beta t^\beta]. \end{cases} \tag{7.16}$$

Case II. Let us consider $\lambda < 0$. Then, using ${}^{LF}[(2)-\beta]$-differentiability and applying Eq. (6.14), we get the solution as follows:

$$\begin{cases} \underline{y}(x;r) = \underline{y}(x_0;r)E_\beta[-c^\beta t^\beta], \\ \overline{y}(x;r) = \overline{y}(x_0;r)E_\beta[-c^\beta t^\beta]. \end{cases} \tag{7.17}$$

Where E_β is the classical Mittag-Leffler function with fractal dimension β defined by

$$E_\beta(x^\beta) = \sum_{k=0}^{\infty} \frac{x^{\beta k}}{\Gamma(1+\beta k)}.$$

Now, we denote the solution (7.16) by $y_1(x;r)$ and the solution (7.17) by $y_2(x;r)$. So, using obtained results we have:

$$\begin{cases} y_1(x;r) = mE_\beta[c^\beta t^\beta], \\ y_2(x;r) = mE_\beta[-c^\beta t^\beta]. \end{cases} \tag{7.18}$$

Taking initial value condition into account in (7.18), we obtain the solutions of equation (7.15), which is

$$\begin{cases} y_1(x;r) = y_0E_\beta[c^\beta t^\beta], \\ y_2(x;r) = y_0E_\beta[-c^\beta t^\beta]. \end{cases} \tag{7.19}$$

Given any point $x = x_0$, we have the fuzzy local fractional Taylor expansion of $y(x)$ in the following form:

if $y(x)$ $^{LF}[(1)-\beta]$-differentiability then:

$$E_\beta(c^\beta x^\beta; r) = \left[\sum_{k=0}^{n} \frac{c^{k\beta}\underline{E}_\beta(c^\beta x_0^\beta)(x-x_0)^{\beta k}}{\Gamma(1+k\beta)}, \sum_{k=0}^{n} \frac{c^{k\beta}\overline{E}_\beta(c^\beta x_0^\beta)(x-x_0)^{\beta k}}{\Gamma(1+k\beta)}\right], \tag{7.20}$$

and if $y(x)$ $^{LF}[(2)-\beta]$-differentiability then:

$$E_\beta(-c^\beta x^\beta; r) = \left[E_{\beta,1}(-c^\beta x^\beta; r), E_{\beta,2}(-c^\beta x^\beta; r)\right], \tag{7.21}$$

$$E_{\beta,1}(-c^\beta x^\beta; r) = \left[\sum_{k=0,keven}^{n} \frac{c^{k\beta}\underline{E}_\beta(-c^\beta x_0^\beta)(x-x_0)^{\beta k}}{\Gamma(1+k\beta)} + \sum_{k=0,kodd}^{n} (-1)^k \frac{c^{k\beta}\overline{E}_\beta(-c^\beta x_0^\beta)(x-x_0)^{\beta k}}{\Gamma(1+k\beta)}\right], \tag{7.22}$$

$$E_{\beta,2}(-c^\beta x^\beta; r) = \left[\sum_{k=0,keven}^{n} \frac{c^{k\beta}\overline{E}_\beta(-c^\beta x_0^\beta)(x-x_0)^{\beta k}}{\Gamma(1+k\beta)} + \sum_{k=0,kodd}^{n} (-1)^k \frac{c^{k\beta}\underline{E}_\beta(-c^\beta x_0^\beta)(x-x_0)^{\beta k}}{\Gamma(1+k\beta)}\right], \tag{7.23}$$

where $E_{\beta,1}$ and $E_{\beta,2}$ are the lower and upper functions, and we always get the Hölder relation

$$d(E_\beta(c^\beta x_1^\beta), E_\beta(c^\beta x_2^\beta)) \leq m|x_1 - x_2|^\beta, \tag{7.24}$$

$$d(E_\beta(-c^\beta x_1^\beta), E_\beta(-c^\beta x_2^\beta)) \leq m|x_1 - x_2|^\beta. \tag{7.25}$$

For any $x_1, x_2 > 0$. Using Eqs. (7.20), (7.22), (7.23) and Eqs. (7.24), (7.25), the solutions of fuzzy the local fractional equation are continuous and the fractal property, is valid.

8 Conclusion and future research

In this paper, we introduce the fuzzy local fractional derivatives and related fuzzy local fractional differential equations about order $0 < \beta < 1$. Also, some useful and new results are derived like as the relation between each type of fuzzy local fractional differentiability and their r-cuts, the composition of the fuzzy KG-LFD and related fuzzy integrals. Moreover, as an application, we have solved some examples. Indeed, the solutions of FLFDEs under local fractional H-differentiability is expressed using Mittag-Leffler function.
For future research, we will solve mentioned problems using some well-known analytic methods like as fuzzy fractional Laplace transform method.

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