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Approximating the Fuzzy Solution of the Non-linear Fuzzy Volterra Integro-differential Equation Using Fixed Point Theorems

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Abstract

In this paper, the fuzzy solution of non-linear fuzzy Volterra integro-differential equation (NFIDE) is approximated. To do this, we define a sequence of fuzzy functions which approximate the fuzzy solution of this type of equations and finally, we estimate upper bound of the error.

Keywords : Fuzzy integro-differential equation; Fixed point; fuzzy numbers.

1 Introduction

The application of the theory of fuzzy functions has recently increased, due to the industrial interest in fuzzy control. The fuzzy differential and integral equations are important parts of the fuzzy analysis theory and they hold the theoretical as well as applicational value in control theory.

The fuzzy mapping function was introduced by Cheng and Zadeh [7]. Later, Dubois and Prade [9] presented an elementary fuzzy calculus based on the extension principle [22]. Puri and Ralescu [20] suggested two definitions for fuzzy function. The concept of integration of fuzzy functions was first introduced by Dubois and prade [9].
Park et al. [17] considered the existence of solution of fuzzy integral equation in Banach space. Park and Jeong [18, 19] studied the existence of solution of fuzzy integral equations of the form

$$x(t) = f(t) + \int_0^t k(t, s, x(s))ds, \quad t \geq 0$$

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where f and x are fuzzy-valued functions ($f, x : (a,b) \to E$ where E is the set of all fuzzy numbers) and k is a crisp function on real numbers. Alternative approaches were later suggested by Goetschel and Voxman [11], Kaleva [13], Matloka [15], Nanda [16] and others, while Goetschel and Voxman [11] and later Matloka [15] preferred a Riemann integral type approach, Kaleva [13] chose to define the integral of fuzzy function, using the Lebesgue type concept for integration. A Numerical method was introduced by Allahviranloo and Otadi [3] for solving fuzzy integrals.
The existence and uniqueness of solutions of fuzzy Volterra integro-differential equations of the second kind using strongly generalized differentiability were discussed by Hajighasemi et al. in [12].
This paper is organized as follows: In Section 2, the basic concept, fuzzy number, Hausdorff metric, generalized H-differentiability and fuzzy ranking are brought. In Section 3, the main section of the paper, some fixed point theorems for approximating the fuzzy solution of non-linear fuzzy Volterra integro-differential equation (NFIDE) are proved and error bound is obtained. Finally, the conclusion is drawn in Section 4.

2 Basic concepts

A nonempty subset A of R is called convex if and only if $(1-k)x + ky \in A$ for every $x, y \in A$ and $k \in [0,1]$. By $p_k(R)$, we denote the family of all nonempty compact convex subsets of R.
There are various definitions for the concept of fuzzy numbers ([8, 10]).

Definition 2.1. *A fuzzy number is a function $u : R \to [0,1]$ satisfying the following properties:*
(i) u is normal, i.e. $\exists x_0 \in R$ with $u(x_0) = 1$,
(ii) u is a convex fuzzy set (i.e. $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\} \forall x, y \in R, \lambda \in [0,1]$),
(iii) u is upper semi-continuous on R,
(iv) $\overline{\{x \in R : u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

The set of all fuzzy real numbers is denoted by E. Obviously $R \subset E$. Here $R \subset E$ is understood as $R = \{\chi_x : \chi \text{ is usual real number}\}$. For $0 < r \leq 1$, denote $[u]_r = \{x \in R; u(x) \geq r\}$ and $[u]_0 = \overline{\{x \in R; u(x) > 0\}}$. Then it is well-known that for any $r \in [0,1]$, $[u]_r$ is a bounded closed interval. For $u, v \in E$, and $\lambda \in R$, where sum $u + v$ and the product $\lambda.u$ are defined by $[u+v]_r = [u]_r + [v]_r$, $[\lambda.u]_r = \lambda[u]_r$, $\forall r \in [0,1]$, where $[u]_r + [v]_r = \{x + y : x \in [u]_r, y \in [v]_r\}$ means the conventional addition of two intervals (subsets) of R and $\lambda[u]_r = \{\lambda x : x \in [u]_r\}$ means the conventional product between a scalar and a subset of R(see e.g. [8, 21].

Another definition for a fuzzy number is as follows:

Definition 2.2. *An arbitrary fuzzy number in the parametric form is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, which satisfy the following requirements:*

1. *$\underline{u}(r)$ is a bounded left-continuous non-decreasing function over $[0,1]$.*
2. *$\overline{u}(r)$ is a bounded left-continuous non-increasing function over $[0,1]$.*
3. *$\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.*

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \leq r \leq 1$. We recall that for $a < b < c$, $a, b, c \in R$, the triangular fuzzy number $u = (a, b, c)$ determined by a, b, c is given such that $\underline{u}(r) = a + (b - c)r$ and $\overline{u}(r) = c - (c - b)r$ are the endpoints of the r-level sets, for all $r \in [0, 1]$. Here $\underline{u}(r) = \overline{u}(r) = b$ and it is denoted by $[u]_1$. For arbitrary $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ we define addition and multiplication by k as

1. $\underline{(u+v)}(r) = (\underline{u}(r) + \underline{v}(r))$,
2. $\overline{(u+v)}(r) = (\overline{u}(r) + \overline{v}(r))$,
3. $\underline{(ku)}(r) = k\underline{u}(r), \overline{(ku)}(r) = k\overline{u}(r), \quad k \geq 0$,
4. $\underline{(ku)}(r) = k\overline{u}(r), \overline{(ku)}(r) = k\underline{u}(r), \quad k < 0$.

In this paper, we represent an arbitrary fuzzy number with compact support by a pair of functions $(\underline{u}(r), \overline{u}(r)), 0 \leq r \leq 1$. Also, we use the Hausdorff distance between fuzzy numbers. This fuzzy number space as shown in [4] can be embedded into Banach space $B = \overline{c}[0,1] \times \overline{c}[0,1]$ where the metric is usually defined as follows: Let E be the set of all upper semicontinuous normal convex fuzzy numbers with bounded $r-$level sets. Since the $r-$cuts of fuzzy numbers are always closed and bounded, the intervals are written as $u[r] = [\underline{u}(r), \overline{u}(r)]$, for all r. We denote by ω the set of all nonempty compact subsets of R and by ω_c the subsets of ω consisting of nonempty convex compact sets. Recall that

$$\rho(x, A) = \min_{a \in A} \|x - a\|$$

is the distance of a point $x \in$ R from $A \in \omega$ and the *Hausdorff separation* $\rho(A, B)$ of $A, B \in \omega$ is defined as

$$\rho(A, B) = \max_{a \in A} \rho(a, B).$$

Note that the notation is consistent, since $\rho(a, B) = \rho(\{a\}, B)$. Now, ρ is not a metric. In fact, $\rho(A, B) = 0$ if and only if $A \subseteq B$. The *Hausdorff metric* d_H on ω is defined by

$$d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\}.$$

The metric d_∞ is defined on E as

$$d_\infty(u, v) = \sup\{d_H(u[r], v[r]) : 0 \leq r \leq 1\}, \quad u, v \in E.$$

for arbitrary $(u, v) \in \overline{c}[0,1] \times \overline{c}[0,1]$. The following properties are well-known. (see e.g. [10, 21])

(i) $d_\infty(u + w, v + w) = d_\infty(u, v), \quad \forall u, v, w \in E$,

(ii) $d_\infty(k.u, k.v) = |k| d_\infty(u, v), \quad \forall k \in R, u, v \in E$,

(iii) $d_\infty(u + v, w + e) \leq d_\infty(u, w) + d_\infty(v, e), \quad \forall u, v, w, e \in E$,

(iv) $d_\infty(u, v) = d_\infty(v, u), \quad \forall u, v \in E$.

Theorem 2.1. *(i) If we define $\widetilde{0} = \chi_0$, then $\widetilde{0} \in E$ is a neutral element with respect to addition, i.e. $u + \widetilde{0} = \widetilde{0} + u = u$, for all $u \in E$.*

(ii) *With respect to* $\widetilde{0}$, *none of* $u \in E \setminus R$, *has opposite in* E.

(iii) *For any* $a, b \in R$ *with* $a, b \geq 0$ *or* $a, b \leq 0$ *and any* $u \in E$, *we have* $(a+b).u = a.u+b.u$; *however, this relation dose not necessarily hold for any* $a, b \in R$, *in general.*

(iv) *For any* $\lambda \in R$ *and any* $u, v \in E$, *we have* $\lambda.(u + v) = \lambda.u + \lambda.v$.

(v) *For any* $\lambda, \mu \in R$ *and any* $u \in E$, *we have* $\lambda.(\mu.u) = (\lambda.\mu).u$.*(see [21])*

Remark 2.1. $d_\infty(u, 0) = d_\infty(0, u) = \parallel u \parallel$.

Definition 2.3. *Consider* $x, y \in E$. *If there exists* $z \in E$ *such that* $x = y + z$, *then* z *is called the H-difference of* x *and* y *and it is denoted by* $x \ominus y$.

In this paper, the sign "$\ominus$" always stands for H-difference and note that $x \ominus y \neq x + (-y)$. Let us recall the definition of strongly generalized differentiability introduced in [4].

Lemma 2.1. *,[4]. Let* $u, v \in E$ *be such that* $u(1) - \underline{u}(0) > 0$, $\overline{u}(0) - u(1) > 0$ *and* $len(v) = (\overline{v}(0) - \underline{v}(0)) \leq \min\{u(1) - \underline{u}(0), \overline{u}(0) - u(1)\}$. *Then the H-difference* $u \ominus v$ *exists.*

Definition 2.4. *,[5]. Let* $f : (a, b) \to E$ *and* $x_0 \in (a, b)$. *We say that* f *is strongly generalized differentiable at* x_0 *(Bede-Gal differentiability), if there exists an element* $f'(x_0) \in E$, *such that*

(i) *For all* $h > 0$ *sufficiently small,* $\exists f(x_0 + h) \ominus f(x_0)$, $\exists f(x_0) \ominus f(x_0 - h)$ *and the limits(in the metric* d_∞*)*

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0)$$

or

(ii) *For all* $h > 0$ *sufficiently small,* $\exists f(x_0) \ominus f(x_0 + h)$, $\exists f(x_0 - h) \ominus f(x_0)$ *and the limits(in the metric* d_∞*)*

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0)$$

or

(iii) *For all* $h > 0$ *sufficiently small,* $\exists f(x_0 + h) \ominus f(x_0)$, $\exists f(x_0 - h) \ominus f(x_0)$ *and the limits(in the metric* d_∞*)*

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0)$$

or

(iv) *For all* $h > 0$ *sufficiently small,* $\exists f(x_0) \ominus f(x_0 + h)$, $\exists f(x_0) \ominus f(x_0 - h)$ *and the limits(in the metric* d_∞*)*

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0)$$

(h and $-h$ at denominators mean $\frac{1}{h}$ and $\frac{-1}{h}$, respectively)

Proposition 2.1. *,[8]. If $f : (a,b) \to E$ is a continuous fuzzy valued function then $g(x) = \int_a^x f(t)d$ is differentiable with derivative $g'(x) = f(x)$.*

Lemma 2.2. *,[4]. For $x_0 \in R$, the fuzzy differential equation $y' = f(x,y)$, $y(x_0) = y_0 \in E$ where $f : R \times E \longrightarrow E$ is supposed to be continuous, is equivalent to one of the integral equations:*

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt, \quad \forall x \in [x_0, x_1]$$

or

$$y_0 = y(x) + (-1). \int_{x_0}^{x} f(t, y(t))dt, \quad \forall x \in [x_0, x_1]$$

on some interval $(x_0, x_1) \subset R$, depending on the strong differentiability considered, (i) or (ii), respectively.
Here the equivalence between two equations means that any solution of an equation is a solution for the other one,too

Remark 2.2. *,[4]. In the case of strongly generalized differentiability, to the fuzzy differential equation $y' = f(x,y)$ we may attach two different integral equations, while in the case of differentiability in the sense of the Definition of H-differentiable, we may attach only one. The second integral equation in Lemma (2.2) can be written in the form $y(x) = y_0 \ominus (-1). \int_{x_0}^{x} f(t, y(t))dt$.*

3 Main results

In this section, we are going to approximate the fuzzy solution of non-linear fuzzy Volterra integro-differential equation.

Consider the non-linear fuzzy Volterra integro-differential equation (NFIDE)

$$\begin{cases} y'(t) = f(t, y(t)) + \int_0^t K(t, s, y(s))ds, & t \in I = [0,1] \\ y(0) = y_0 \in E \end{cases} \tag{3.1}$$

where $K : I \times I \times E \to E$, $f : I \times E \to E$ are continuous fuzzy valued functions satisfying a Lipschitz condition with respect to the last variables, i.e.,

There exist $L_f \geq 0$ and $L_K \geq 0$ such that

$$\begin{aligned} d_\infty(f(t, y_1), f(t, y_2)) &\leq L_f d_\infty(y_1, y_2) \\ d_\infty(K(t, s, y_1), K(t, s, y_2)) &\leq L_K d_\infty(y_1, y_2) \end{aligned} \tag{3.2}$$

for $t, s \in I$ and for $y_1, y_2 \in E$ and also, f is differentiable respect to t and K is differentiable respect to t, s.

We will check that a fuzzy valued function $z : I \to E$ is a fuzzy solution of (3.1) if and only if it is a fixed point of the operator $T : C(E) \to C(E)$ given by following formulas:

(a) $Ty(t) = y_0 + \int_0^t f(u, y(u))du + \int_0^t \int_0^u K(u, s, y(s))dsdu$

where y is $(i)-$differentiable.

(b) $Ty(t) = y_0 \ominus (-1) \int_0^t f(u, y(u))du \ominus (-1) \int_0^t \int_0^u K(u, s, y(s))dsdu$

where y is $(ii)-$differentiable.

where T is an operator of the complete fuzzy metric space (d_∞, E).

Now, we are going to show that the operator T satisfies the hypothesis of the Banach fixed point theorem and thus the sequence $\{T^m(z_0)\}_{m\in N}$ convergence to the fuzzy solution z of (3.1) for any $z_0 \in C(E)$.

Using the following Lemma, we show that the operator T^n also satisfies a suitable Lipschitz condition.

Lemma 3.1. *For any $p, q \in C(E)$ and $n \in N$, we have*

$$d_\infty(T^n p, T^n q) \leq \frac{L^n}{n!} d_\infty(p, q)$$

where $L := L_f + L_K$

Proof: The proof is clear.
According to the fixed point theorem and Lemma (3.1), T has a unique fixed point z and

$$for\ all \quad z_0 \in C(E) \quad and \quad m \geq 1, \quad d_\infty(z, T^m z_0) \leq \sum_{k=m}^{\infty} \frac{L^k}{k!} d_\infty(Tz_0, z_0) \tag{3.3}$$

We define a sequence of the crisp functions $\{b_n\}_{n\in N}$ and we can say, for every continuous fuzzy valued function x there exists a sequence of fuzzy numbers $\{\beta_n\}_{n\in N}$ such that $x = \sum_{n\in N} \beta_n b_n$. We now define a sequence of projections $\{P_n\}_{n\in N}$ as follows:

$$P_n\left(\sum_{k\in N} \beta_k b_k\right) = \sum_{k=1}^{n} \beta_k b_k$$

and for all fuzzy valued functions x, $\{P_n\}$ satisfies the following interpolation property:

$$P_n(x)(t_i) = x(t_i), \quad t_i \in I, \quad 1 \leq i \leq n \tag{3.4}$$

We can use fuzzy lagrange interpolation [14] and fuzzy spline interpolation [2]. Also, we define a sequence of projections $\{Q_n\}_{n\in N}$ where for any two-variable fuzzy valued function x, it satisfies the following interpolation property:

$$Q_{n^2}(x)(t_i, t_j) = x(t_i, t_j), \quad t_i, t_j \in I, \quad 1 \leq i, j \leq n \tag{3.5}$$

We can use fuzzy shepard interpolation that is introduced in [6]. so, it is not difficult to check that this projections satisfy similar to the ones for the one-dimensional case.
Let T_n be the set $\{t_1, \ldots, t_n\}$ ordered in an increasing way for $n \leq 2$, so we can say T_n is a partition of $[0, 1]$. Let ΔT_n denote the norm of the partition T_n.
The next remark is concluded easily from Eqs. (3.4) and (3.5) and the mean-value theorem for one and two variables:

1. If x is a differentiable one-variable fuzzy valued function and $n \geq 2$, then

$$d_\infty(x, P_n(x)) \leq 2\|x\|\Delta T_n \tag{3.6}$$

2. If x is a differentiable two-variable fuzzy valued function and $n \geq 2$, then

$$d_\infty(x, Q_{n^2}(x)) \leq 4 \max\left\{\left\|\frac{\partial x}{\partial s}\right\|, \left\|\frac{\partial x}{\partial t}\right\|\right\} \Delta T_n \tag{3.7}$$

We are now going to approximate the unique fixed point of the non-linear operator $T : C(E) \to C(E)$ given by Eqs. (a) and (b). To this end, we will define the approximating sequence described above.

Theorem 3.1. *Let K, f, z_0 are continuous and differentiable fuzzy valued functions and $m \in N$. Let $\{\varepsilon_1, \ldots, \varepsilon_m\}$ be a set of positive numbers and define, for $k \in \{1, \ldots, m\}$ and $0 \leq t, s \leq 1$, the functions*

$$\begin{aligned}
&\varphi_{k-1}(t) := f(t, z_{k-1}(t)),\\
&\rho_{k-1}(t, s) := K(t, s, z_{k-1}(s)),\\
&z_k(t) := y_0 + \textstyle\int_0^t P_{m_k}(\varphi_{k-1}(u))du + \int_0^t \int_0^u Q_{n_{k^2}}(\rho_{k-1}(u, s))dsdu,\\
&\qquad z \text{ is (i)-}differentiable,\\
&z_k(t) := y_0 \ominus (-1) \textstyle\int_0^t P_{m_k}(\varphi_{k-1}(u))du \ominus (-1) \int_0^t \int_0^u Q_{n_{k^2}}(\rho_{k-1}(u, s))dsdu, ,\\
&\qquad z \text{ is (ii)-}differentiable,
\end{aligned} \tag{3.8}$$

where

(i) $\Delta T_{m_k} \leq \frac{\varepsilon_k}{4\|\varphi'_{k-1}\|}, \quad m_k \in N$

(ii) $\Delta T_{n_k} \leq \frac{\varepsilon_k}{4M_{k-1}}, \quad n_k \in N$
where

$$M_{k-1} := \max\left\{\left\|\frac{\partial \rho_{k-1}}{\partial t}\right\|, \left\|\frac{\partial \rho_{k-1}}{\partial s}\right\|\right\} \tag{3.9}$$

Then, for all $k \in \{1, \ldots, m\}$, it is satisfies that

$$d_\infty(Tz_{k-1}, z_k) \leq \varepsilon_k \tag{3.10}$$

Proof: Since, we just need an upper bound of the derivation, so, the kind of differentiability of φ_{k-1} and ρ_{k-1} is not important. Using condition (i) and applying Eq. (3.6), following inequality is obtained

$$d_\infty(\varphi_{k-1}, P_{m_{k-1}}(\varphi_{k-1})) \leq \frac{\varepsilon_k}{2}, \quad k \in \{1, \ldots, m\} \tag{3.11}$$

and also, from condition (ii) and Eq. (3.7) we have

$$d_\infty(\rho_{k-1}, P_{n^2_{k-1}}(\rho_{k-1})) \leq \varepsilon_k, \quad k \in \{1, \ldots, m\} \tag{3.12}$$

Without loss of generality, we consider z is $(ii)-$differentiable so, we derive for all $t \in I$

$$\begin{aligned} d_\infty(Tz_{k-1}(t), z_k(t)) &= d_\infty(y_0 \ominus (-1)\int_0^t f(u, z_{k-1}(u))du \ominus (-1)\int_0^t\int_0^u K(u,s,z_{k-1}(s))dsdu, \\ &\qquad y_0 \ominus (-1)\int_0^t P_{m_k}(\varphi_{k-1}(u))du \ominus (-1)\int_0^t\int_0^u Q_{n_{k^2}}(\rho_{k-1}(u,s))dsdu \\ &= d_\infty(y_0 \ominus (-1)\int_0^t \varphi_{k-1}(u)du \ominus (-1)\int_0^t\int_0^u \rho_{k-1}(u,s)dsdu, \\ &\qquad y_0 \ominus (-1)\int_0^t P_{m_k}(\varphi_{k-1}(u))du \ominus (-1)\int_0^t\int_0^u Q_{n_{k^2}}(\rho_{k-1}(u,s))dsdu \\ &\le \int_0^t d_\infty(\varphi_{k-1}(u), P_{m_k}(\varphi_{k-1}(u))du \\ &+ \int_0^t\int_0^u d_\infty(\rho_{k-1}(u,s)Q_{n_{k^2}}(\rho_{k-1}(u,s))dsdu \\ &\le \varepsilon_k \end{aligned}$$

and therefore Eq. (3.10) is derived.
In order to establish the fact that the sequence defined in Theorem (3.1) approximates the fuzzy solution of NFIDE, the following lemma is brought.

Lemma 3.2. *Let $m \in N$ and $\{z_0, z_1, \ldots, z_m\}$ be any subset of continuous fuzzy valued functions on I. Then*

$$d_\infty(z, z_m) \le \sum_{k=m}^{\infty} \frac{L^k}{k!} d_\infty(Tz_0, z_0) + \sum_{k=1}^{m} d_\infty(Tz_{k-1}, z_k) \tag{3.13}$$

where z is the fixed point of operator T and $L = L_f + L_K$

Proof: From Lemma (3.1), we have

$$d_\infty(T^{m-k+1}z_{k-1}, T^{m-k}z_k) \le \frac{L^{m-k}}{(m-k)!} d_\infty(Tz_{k-1}, z_k), \quad k \in \{1, \ldots, m\} \tag{3.14}$$

and so,

$$\sum_{k=1}^{m} d_\infty(T^{m-k+1}z_{k-1}, T^{m-k}z_k) \le \sum_{k=1}^{m} \frac{L^{m-k}}{(m-k)!} d_\infty(Tz_{k-1}, z_k) \tag{3.15}$$

By applying Eq. (3.3) to $d_\infty(z, z_m)$ and taking into account that

$$d_\infty(z, z_m) \le d_\infty(z, T^m z_0) + \sum_{k=1}^{m} d_\infty(T^{m-k+1}z_{k-1}, T^{m-k}z_k) \tag{3.16}$$

then the proof is complete.

4 Conclusion

In this work, we defined a sequence of fuzzy functions for approximating the fuzzy solution of NFIDE and we proved some theorems in detail and we presented that if z is the exact solution NFIDE (3.1), then for the sequence of approximating functions $\{z_m\}_{m\ge 0}$ the error is given by Eq. (3.13).

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