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Research Article

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New iterative method for solving linear Fredholm fuzzy integral equations of the second kind

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Abstract

Fuzzy integral equations play major roles in various areas, therefore a new method for finding a solution of the Fredholm fuzzy integral equation is presented. This method converts the fuzzy integral equation into linear system by using the Taylor series. For this scope, first the Taylor expansion of unknown function is substituted in parametric form of the given equation. Then we differentiate both sides of the resulting integral equation and also approximate the Taylor expansion by a suitable truncation limit. This work yields a linear system in crisp case. Now the solution of this system yields unknown Taylor coefficients of the solution functions. The proposed method is illustrated by several examples with computer simulations.

Keywords : Fredholm fuzzy integral equations; Taylor series; Convergence analysis; Approximate solutions.

1 Introduction

Since many mathematical formulations of physical phenomena contain fuzzy integral equations and these equations are very useful for solving many problems in several applied fields like mathematical physics and engineering, therefor various approaches for solving these problems have been proposed. Also these equations usually can not be solved explicitly, so it is required to obtain the approximate solutions. There are numerous numerical methods which have been focusing on the solution of integral equations. For example, Tricomi in his book [23], introduced the classical method of successive approximations for nonlinear integral equations. Variational iteration method [13] and Adomian decomposition method [3] are effective and convenient for solving integral equations. Also the Homotopy analysis method (HAM) was proposed by Liao [15, 16, 17, 18] and then has been applied in [1, 5, 8]. Moreover, some different valid methods for solving this kind of equations have been developed in the last years. First time, Taylor expansion approach was presented for solution of integral equations by Kanwal and Liu in [12] and then has been extended in [10, 19, 20, 21, 25, 26]. In this paper we want to propose a new numerical approach to approximate the solution of the linear Fredholm fuzzy integral equation. This method converts the given fuzzy equation that supposedly has an unique fuzzy solution, into a crisp linear system. For this scope, first the Taylor expansion of unknown function is substituted in parametric form of the present fuzzy equation. Then we differentiate both sides of the resulting integral equations of the fuzzy equation N times and also approximate the Taylor expansion by a

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suitable truncation limit. This work yields a linear system in crisp case, such that the solution of the linear system yields the unknown Taylor coefficients of the solution function. An interesting feature of this method is that we can get an approximate of the Taylor expansion in arbitrary point to any desired degree of accuracy. Here is an outline of the paper.
In Section 2, the basic notations and definitions of integral equation and Taylor polynomial method are briefly presented . Section 3 describes how to find a approximate solution of the given Fredholm fuzzy integral equation by using the proposed approach. Finally in Section 4, we apply the proposed method on some examples to show the simplicity and efficiency of the method.

2 Preliminaries

This section briefly deals with the foundation of fuzzy numbers and integral equations which are used in the next sections. We started by defining the fuzzy number.

Definition 2.1 *A fuzzy number is a fuzzy set* $u : \mathbb{R}^1 \to I = [0,1]$ *which satisfies*

i u *is upper semicontinuous,*

ii $u(x) = 0$ *outside some interval* $[a,d]$,

iii *There are real numbers* $b, c : a \le b \le c \le d$ *for which:*

1. $u(x)$ *is monotonically increasing on* $[a,b]$,
2. $u(x)$ *is monotonically decreasing on* $[c,d]$,
3. $u(x) = 1, b \le x \le c$.

The set of all fuzzy numbers (as given by definition 2.1) is denoted by E^1 [7, 21].

Definition 2.2 *A fuzzy number* v *is a pair* $(\underline{v}, \overline{v})$ *functions* $\underline{v}(r), \overline{v}(r) : 0 \le r \le 1$. *which satisfy the following requirements:*

i $\underline{v}(r)$ *is a bounded monotonically increasing left continuous function,*

ii $\overline{v}(r)$ *is a bounded monotonically decreasing left continuous function,*

iii $\underline{v}(r) \le \overline{v}(r)$*:* $0 \le r \le 1$.

A popular fuzzy number is the triangular fuzzy number $v = (a, b, c)$ with membership function,

$$\mu_v(x) = \begin{cases} \frac{x-a}{b-a}, & a \le x \le b \\ \frac{c-x}{c-b}, & b \le x \le c \\ 0\ , & otherwise, \end{cases}$$

where $a \le b \le c$. Its parametric form is:

$[v]_l^\alpha = a + (b-a)\alpha$ and $[v]_u^\alpha = c - (c-b)\alpha$, $0 \le \alpha \le 1$.

2.1 Operation on fuzzy numbers

We briefly mention fuzzy number operations defined by the Zadeh extension principle [26, 27].

$\mu_{A+B}(z) = max\{\mu_A(x) \wedge \mu_B(y) | \ z = x + y\}$,
$\mu_{f(Net)}(z) = max\{\mu_A(x) \wedge \mu_B(y) | \ z = xy\}$,

where A and B are fuzzy numbers $\mu_*(.)$ denotes the membership function of each fuzzy number, $\wedge$ is the minimum operator and $f(x) = x$ is the activation function for output unit of our fuzzy neural network.
The above operations on fuzzy numbers are numerically performed on level sets (i.e. α-cuts). For $0 < \alpha \le 1$, a α-level set of a fuzzy number A is defined as:

$[A]^\alpha = \{x | \ \mu_A(x) \ge \alpha, \ x \in \mathbb{R}\}$,
$[A]^\alpha = [[A]_l^\alpha, [A]_u^\alpha]$,

where $[A]_l^\alpha$ and $[A]_u^\alpha$ are the lower and the upper limits of the α-level set $[A]^\alpha$, respectively.

From interval arithmetic [2], the above operations on fuzzy numbers are written for the α-level sets as follows:

$$[A]^\alpha + [B]^\alpha = [[A]_l^\alpha, [A]_u^\alpha] + [[B]_l^\alpha, [B]_u^\alpha] = [[A]_l^\alpha + [B]_l^\alpha, [A]_u^\alpha + [B]_u^\alpha], \tag{2.1}$$

$$f([Net]^\alpha) = f([Net]_l^\alpha, [Net]_u^\alpha]) = [f([Net]_l^\alpha), f([Net]_u^\alpha)],$$

$$\begin{aligned} k[A]^\alpha &= k[[A]_l^\alpha, [A]_u^\alpha] = [k[A]_l^\alpha, k[A]_u^\alpha], \quad if\ k \ge 0, \\ k[A]^\alpha &= k[[A]_l^\alpha, [A]_u^\alpha] = [k[A]_u^\alpha, k[A]_l^\alpha], \quad if\ k < 0. \end{aligned} \tag{2.2}$$

For arbitrary $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$ we define addition $(u+v)$ and multiplication by k as [7, 21]:

$\overline{(u+v)}(r) = \overline{u}(r) + \overline{v}(r),$
$\underline{(u+v)}(r) = \underline{u}(r) + \underline{v}(r),$

$$\overline{(ku)}(r) = k.\overline{u}(r), \ \underline{(kv)}(r) = k.\underline{u}(r), \quad if \ k \geq 0,$$
$$\overline{(ku)}(r) = k.\underline{u}(r), \ \underline{(kv)}(r) = k.\overline{u}(r), \quad if \ k < 0.$$

Definition 2.3 *For arbitrary fuzzy numbers* $u, v \in \mathrm{E}^1$ *the quantity*

$$D(u,v) = \sup_{0 \leq r \leq 1} \{max[|\underline{u}(r) - \underline{v}(r)|\ , |\overline{u}(r) - \overline{v}(r)|]\}$$

is the distance between u and v. It is shown that (E^1, D) is a complete metric space [22].

Definition 2.4 *Let* $f : [a,b] \rightarrow \mathrm{E}^1$. *For each partition* $P = \{t_0, t_1, ..., t_n\}$ *of* $[a,b]$ *and for arbitrary* $\xi_i \in [t_{i-1}, t_i]$ $(1 \leq i \leq n)$, *suppose*

$$R_P = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}),$$

$$\Delta := max\{|t_i - t_{i-1}|,\ i = 1, ..., n\}.$$

The definite integral of $f(t)$ over $[a,b]$ is

$$\int_a^b f(t)dt = \lim_{\Delta \to 0} R_P$$

provided that this limit exists in the metric D. If the fuzzy function $f(t)$ is continuous in the metric D, its definite integral exists [7]. Also,

$$\overline{(\int_a^b f(t,r)\ dt)} = \int_a^b \overline{f}(t,r)dt,$$
$$\underline{(\int_a^b f(t,r)\ dt)} = \int_a^b \underline{f}(t,r)dt.$$

More details about properties of the fuzzy integral are given in [7, 11].

2.2 Integral equation

The basic definition of integral equation is given in [10].

Definition 2.5 *The Fredholm integral equation of the second kind is*

$$F(t) = f(t) + \lambda(ku)(t), \tag{2.3}$$

where

$$(ku)(t) = \int_a^b k(s,t)F(s)ds.$$

In Eq. (2.1), $k(s,t)$ is an arbitrary kernel function over the square $a \leq s, t \leq b$ and $f(t)$ is a function of $t : a \leq t \leq b$. If the kernel function satisfies $k(s,t) = 0,\ s > t$, we obtain the Volterra integral equation

$$F(t) = f(t) + \lambda \int_a^t k(s,t)F(s)ds. \tag{2.4}$$

In addition, if $f(t)$ be a crisp function then the solution of above equation crisp as well. Also if $f(t)$ be a fuzzy function we have Fredholm fuzzy integral equation of the second kind which may only process fuzzy solutions. Sufficient conditions for the existence equation of the second kind, where $f(t)$ is a fuzzy function, are given in [4, 6]. Now let $(\underline{f}(t,r), \overline{f}(t,r))$ and $(\underline{F}(t,r), \overline{F}(t,r))$ $(0 \leq r \leq 1;\ a \leq t \leq b)$ be parametric form of $f(t)$ and $F(t)$ respectively. In order to design a numerical scheme for solving Eq. (2.3), we write the parametric form of the given fuzzy integral equation as follows:

$$\begin{cases} \overline{F}(t,r) = \overline{f}(t,r) + \lambda \int_a^b \overline{U}(s,r)ds \\ \\ \underline{F}(t,r) = \underline{f}(t,r) + \lambda \int_a^b \underline{U}(s,r)ds \end{cases}, \tag{2.5}$$

where

$$\overline{U}(s,r) = \begin{cases} k(s,t)\overline{F}(s,r) & ,k(s,t) \geq 0 \\ k(s,t)\underline{F}(s,r) & ,k(s,t) < 0 \end{cases},$$

and

$$\underline{U}(s,r) = \begin{cases} k(s,t)\underline{F}(s,r) & ,k(s,t) \geq 0 \\ k(s,t)\overline{F}(s,r) & ,k(s,t) < 0 \end{cases}.$$

Suppose that in Eq. (2.5) the functions $\overline{f}(t,r)$, $\underline{f}(t,r)$ and the kernel $k(s,t)$ are given and assumed to be sufficiently differentiable with respect to all their arguments on the interval $a \leq t, s \leq b$. Also, $F(t,r) = [\underline{F}(t,r), \overline{F}(t,r)]$ is the solution where to be determined.

2.3 Taylor series

Let us first recall the basic principles of the Taylor polynomial method for solving Fredholm fuzzy integral equations (2.3). Since these results are the key for our problems therefore we explain them. Without loss of generality, we assume that

$$\begin{cases} \lambda.k(s,t) \geq 0 & ,a \leq s \leq c \\ \lambda.k(s,t) < 0 & ,c \leq s \leq b \end{cases}.$$

Therefore Eq. (2.5) is transformed to following form:

$$\begin{cases} \overline{F}(t,r) = \overline{f}(t,r) + \lambda \int_a^c k(s,t)\overline{F}(s,r)ds + \\ \lambda \int_c^b k(s,t)\underline{F}(s,r)ds \\ \underline{F}(t,r) = \underline{f}(t,r) + \lambda \int_a^c k(s,t)\underline{F}(s,r)ds + \\ \lambda \int_c^b k(s,t)\overline{F}(s,r)ds \end{cases}. \tag{2.6}$$

Now we want to obtain the numerical solution of the above system in the form of

$$\overline{F}_N(t,r) = \sum_{i=0}^{N} \left(\frac{1}{i!} \cdot \frac{\partial^{(i)}\overline{F}(t,r)}{\partial t^i}|_{t=z} .(t-z)^i\right), \tag{2.7}$$

$a \leq t, z \leq b, 0 \leq r \leq 1,$

$$\underline{F}_N(t,r) = \sum_{i=0}^{N} \left(\frac{1}{i!} \cdot \frac{\partial^{(i)}\underline{F}(t,r)}{\partial t^i}|_{t=z} .(t-z)^i\right),$$

$a \leq t, z \leq b,\ 0 \leq r \leq 1,$

which are the Taylor expansions of degree N at $t = z$ for the unknown functions $\overline{F}(t,r)$ and $\underline{F}(t,r)$, respectively. For this scope, we differentiate each equation of system (2.6) N times with respect to t and get

$$\frac{\partial^{(j)}\overline{F}(t,r)}{\partial t^j} = \frac{\partial^{(j)}\overline{f}(t,r)}{\partial t^j} + \lambda \int_a^c \frac{\partial^{(j)}k(s,t)}{\partial t^j} .\overline{F}(s,r)ds \tag{2.8}$$

$$+ \lambda \int_c^b \frac{\partial^{(j)}k(s,t)}{\partial t^j} .\underline{F}(s,r)ds,$$

$$\frac{\partial^{(j)}\underline{F}(t,r)}{\partial t^j} = \frac{\partial^{(j)}\underline{f}(t,r)}{\partial t^j} + \lambda \int_a^c \frac{\partial^{(j)}k(s,t)}{\partial t^j} .\underline{F}(s,r)ds$$

$$+ \lambda \int_c^b \frac{\partial^{(j)}k(s,t)}{\partial t^j} .\overline{F}(s,r)ds,\ j = 0, ..., N.$$

For brevity, we define below symbols as,

$$\overline{F}^{(i)}(z,r) := \frac{\partial^{(i)}\overline{F}(t,r)}{\partial t^i}|_{t=z}$$

and

$$\underline{F}^{(i)}(z,r) := \frac{\partial^{(i)}\underline{F}(t,r)}{\partial t^i}|_{t=z}.$$

The aim of this paper is determining of the coefficients $\overline{F}^{(i)}(z,r)$ and $\underline{F}^{(i)}(z,r)$, $(i = 0, ..., N)$ in Eq. (2.7). For this aim, we expanded $\overline{F}(s,r)$ and $\underline{F}(s,r)$ in Taylor series at $z = a$ and substituted it's N-th truncation in (2.8). Now we can write:

$$\overline{F}^{(j)}(a,r) = \frac{\partial^{(j)}\overline{f}(t,r)}{\partial t^j}|_{t=a} + \sum_{i=0}^{N} w_{j,i}. \tag{2.9}$$

$$\overline{F}^{(i)}(a,r) + \sum_{i=0}^{N} w'_{j,i} \cdot \underline{F}^{(i)}(a,r),$$

$$\underline{F}^{(j)}(a,r) = \frac{\partial^{(j)}\underline{f}(t,r)}{\partial t^j}|_{t=a} + \sum_{i=0}^{N} w_{j,i} \cdot$$

$$\underline{F}^{(i)}(a,r) + \sum_{i=0}^{N} w'_{j,i} \cdot \overline{F}^{(i)}(a,r),$$

where

$$w_{j,i} = \frac{\lambda}{i!} \int_a^c \frac{\partial^{(j)}k(s,t)}{\partial t^j}|_{t=a} .(s-a)^i ds,$$

and

$$w'_{j,i} = \frac{\lambda}{i!} \int_c^b \frac{\partial^{(j)}k(s,t)}{\partial t^j}|_{t=a} .(s-a)^i ds.$$

We now write the matrix form of expression (2.9) as

$$WY = E, \tag{2.10}$$

where

$$Y = [\underline{F}(a,r), ..., \underline{F}^{(N)}(a,r), \overline{F}(a,r), ..., \overline{F}^{(N)}(a,r)]' = \begin{bmatrix} \underline{Y} \\ \overline{Y} \end{bmatrix},$$

$$E = [-\underline{f}(a,r), ..., -\frac{\partial^{(N)}\underline{f}(t,r)}{\partial t^N}|_{t=a}, -$$

$$\overline{f}(a,r), ..., -\frac{\partial^{(N)}\overline{f}(t,r)}{\partial t^N}|_{t=a}]' = \begin{bmatrix} \underline{E} \\ \overline{E} \end{bmatrix},$$

$$W = \begin{bmatrix} W_{1,1} & W_{1,2} \\ W_{2,1} & W_{2,2} \end{bmatrix},$$

whose elements are defined by

3 Convergence analysis

In this section we proved that the above numerical method convergence to the exact solution of Eq. (2.5).

Theorem 3.1 *Let $\underline{F}_N(t)$ and $\overline{F}_N(t)$ be polynomial solution of (2.5) such that it's coefficients produced by solving linear system (2.10). Then these polynomials converges to the exact solution of the fuzzy Fredholm integral equation (2.3), when $N \longrightarrow +\infty$.*

Proof. First consider the Eq. (2.3) in the form

$$F(t) = f(t) + \lambda \int_a^t k(s,t)F(s)ds. \qquad (3.11)$$

If the series (2.7) converges to $\overline{F}(t,r)$ and $\underline{F}(t,r)$ respectively, then we can write:

$$\begin{cases} \overline{F}_N(t,r) = \overline{f}(t,r) + \lambda \int_a^c k(s,t)\overline{F}_N(s,r)ds + \\ \lambda \int_c^b k(s,t)\underline{F}_N(s,r)ds \\ \\ \underline{F}_N(t,r) = \underline{f}(t,r) + \lambda \int_a^c k(s,t)\underline{F}_N(s,r)ds + \\ \lambda \int_c^b k(s,t)\overline{F}_N(s,r)ds \end{cases}. \qquad (3.12)$$

and it holds that

$$\overline{F}(t,r) =$$

$$\lim_{N\to\infty} \overline{F}_N(t,r), \ and\ \underline{F}(t) = \lim_{N\to\infty} \underline{F}_N(t,r).$$

We defined the error function $e_N(t,r)$ by subtracting Eqs. (3.12)-(2.6) as follows:

$$e_N(t,r) = \overline{e}_N(t,r) + \underline{e}_N(t,r), \qquad (3.13)$$

where

$$\overline{e}_N(t,r) = (\overline{F}(t,r) - \overline{F}_N(t,r))$$

$$+\lambda \int_a^c k(s,t)(\overline{F}(s,r) - \overline{F}_N(s,r))ds +$$

$$\lambda \int_c^b k(s,t)(\underline{F}(s,r) - \underline{F}_N(s,r))ds,$$

and

$$\underline{e}_N(t,r) = (\underline{F}(t,r) - \underline{F}_N(t,r))$$

$$+\lambda \int_a^c k(s,t)(\underline{F}(s,r) - \underline{F}_N(s,r))ds +$$

$$\lambda \int_c^b k(s,t)(\overline{F}(s,r) - \overline{F}_N(s,r))ds.$$

We must prove when

$N \longrightarrow +\infty$, the error function $e_N(t)$ becomes to zero. Hence we proceed as follows:

$$\|e_N\| = \|\overline{e}_N + \underline{e}_N\| \leq \|(\overline{F}(t,r)$$

$$-\overline{F}_N(t,r))\| + \|(\underline{F}(t,r) - \underline{F}_N(t,r))\|$$

$$+|\lambda| \int_a^b \|k\| \ (\|\overline{F}(s,r) - \overline{F}_N(s,r)\| +$$

$$\|\underline{F}(s,r) - \underline{F}_N(s,r)\|)dt.$$

Since $\|k\|$ is bounded, therefore $\|(\overline{F}(s,r) - \overline{F}_N(s,r))\| \rightarrow 0$ and $\|(\underline{F}(s,r) - \underline{F}_N(s,r))\| \longrightarrow 0$ imply that $\|e_N\| \longrightarrow 0$ and proof is completed.□

4 Numerical examples

In this section, we present three examples of linear Fredholm fuzzy integral equations and results will be compared with the exact solutions.

Example 4.1 *Consider the following Fredholm integral equation with:*

$$\overline{f}(t,r) = \frac{5r(2t-1)^2}{24} - t(r-2) + \frac{(2t-1)^2(r-2)}{24},$$

$$\underline{f}(t,r) = rt - \frac{r(2t-1)^2}{24} - \frac{5(2t-1)^2(r-2)}{24},$$

and kernel

$$k(s,t) = (2t-1)^2(1-2s), \ 0 \leq s,t \leq 1,$$

and $a = 0$, $b = 1$, $\lambda = 1$. The exact solution in this case is given by

$$\overline{F}(t,r) = (2-r)t \ and\ \underline{F}(t,r) = rt.$$

$$W_{1,1}=W_{2,2}=\begin{array}{ccccc} w_{0,0}-1 & w_{0,1} & \cdots & w_{0,N-1} & w_{0,N} \\ w_{1,0} & w_{1,1}-1 & \cdots & w_{1,N-1} & w_{1,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{N-1,0} & w_{N-1,1} & \cdots & w_{N-1,N-1}-1 & w_{N-1,N} \\ w_{N,0} & w_{N,1} & \cdots & w_{N,N-1} & w_{N,N}-1 \end{array}$$

$$W_{1,2}=W_{2,1}=\begin{array}{ccccc} w'_{0,0} & w'_{0,1} & \cdots & w'_{0,N-1} & w'_{0,N} \\ w'_{1,0} & w'_{1,1} & \cdots & w'_{1,N-1} & w'_{1,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w'_{N-1,0} & w'_{N-1,1} & \cdots & w'_{N-1,N-1} & w'_{N-1,N} \\ w'_{N,0} & w'_{N,1} & \cdots & w'_{N,N-1} & w'_{N,N} \end{array}$$

In this example we assume that $z=0$. Using Eqs. (2.9)-(2.10), we calculated the coefficients matrix as following:

With using of above matrix, we can rewrite the linear system (2.10) as follows:

$$W\begin{bmatrix}\underline{F}(0,r)\\ \underline{F}'(0,r)\\ \underline{F}''(0,r)\\ \overline{F}(0,r)\\ \overline{F}'(0,r)\\ \overline{F}''(0,r)\end{bmatrix}=\begin{bmatrix}0.2500\ r-0.4166\\ 1.6667-2.0000\ r\\ 2.0000\ r-3.3333\\ 0.0833-0.2500\ r\\ 2.0000\ r-2.3333\\ 0.6666-2.0000\ r\end{bmatrix}$$

The vector solution of above linear system is:

$$\begin{bmatrix}\underline{F}(0,r)\\ \underline{F}'(0,r)\\ \underline{F}''(0,r)\\ \overline{F}(0,r)\\ \overline{F}'(0,r)\\ \overline{F}''(0,r)\end{bmatrix}=\begin{bmatrix}0\\ r\\ 0\\ 0\\ 2-r\\ 0\end{bmatrix}$$

Approximate solution and exact solution are compared in Fig. 1 for $r=0,0.5,1$.

After propagating this solution in Eq. (2.7), the calculated solution is equal to exact solution. In other words, with using of this method we can find the analytical solution for this kind of equation, if the exact solution of given problem be a polynomial.

Example 4.2 *Let fuzzy integral equation*

$\overline{f}(t,r)=$

$3(r^2-2)(3t^2+2)-t^3(3r^2-6)-r(r^4+2)(9t^2-10),$

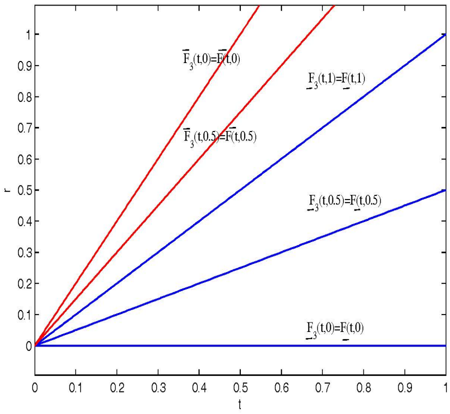

Figure 1: Comparison the exact and approximated solutions for Example 4.1.

$\underline{f}(t,r)=$

$3(r^2-2)(9t^2-10)+t^3(r^5+2r)-r(r^4+2)(3t^2+2),$

$$k(s,t)=3(2-s^2+t^2),\ \ 0\le s,t\le 2,$$

with the exact solution $\overline{F}(t,r)=t^3(6-3r^2)$, $\underline{F}(t,r)=t^3(r^5+2)$ and $a=0$, $b=2$, $\lambda=1$. By using Eqs. (2.9)-(2.10), we worked as following:

$$W=\begin{bmatrix}W_{1,1} & W_{1,2}\\ W_{2,1} & W_{2,2}\end{bmatrix},$$

$W_{1,1}=$

$$\begin{bmatrix}4.6569 & 3.0000 & 1.1314 & 0.33333 & 0.0808\\ 0.0000 & -1.0000 & 0.0000 & 0.0000 & 0.0000\\ 8.4853 & 6.0000 & 1.8284 & 1.0000 & 0.2828\\ 0.0000 & 0.0000 & 0.0000 & -1.0000 & 0.0000\\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -1.0000\end{bmatrix},$$

W=

-0.7500	0.0416	0.0052	-0.2500	-0.2083	-0.0885
-1.0000	-1.1667	-0.0208	1.0000	0.8333	0.3541
2.0000	0.3333	-0.9583	-2.0000	-1.6667	-0.7083
-0.2500	-0.2083	-0.0885	-0.7500	0.0416	0.0052
1.0000	0.8333	0.3541	-1.0000	-1.1667	-0.0208
-2.0000	-1.6667	-0.7083	2.0000	0.3333	-0.9583

$W_{1,2} =$

$$\begin{bmatrix} -1.656 & -3.000 & -2.731 & -1.666 & -0.766 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 3.514 & 6.000 & 5.171 & 3.000 & 1.372 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \end{bmatrix}$$

Now we can write system (2.10) as follows:

$$W \begin{bmatrix} \underline{F}(0,r) \\ \underline{F}'(0,r) \\ \underline{F}''(0,r) \\ \underline{F}'''(0,r) \\ \underline{F}^{(4)}(0,r) \\ \overline{F}(0,r) \\ \overline{F}'(0,r) \\ \overline{F}''(0,r) \\ \overline{F}'''(0,r) \\ \overline{F}^{(4)}(0,r) \end{bmatrix} = \begin{bmatrix} 2r(r^4+2)+30r^2-60 \\ 0 \\ 6r(r^4+2)-54r^2+108 \\ -6r^5-12r \\ 0 \\ 12-6r^2-10r(r^4+2) \\ 0 \\ 18r(r^4+2)-18r^2+36 \\ 18r^2-36 \\ 0 \end{bmatrix}$$

The vector solution of the above linear system is:

$$\begin{bmatrix} \underline{F}(0,r) \\ \underline{F}'(0,r) \\ \underline{F}''(0,r) \\ \underline{F}'''(0,r) \\ \underline{F}^{(4)}(0,r) \\ \overline{F}(0,r) \\ \overline{F}'(0,r) \\ \overline{F}''(0,r) \\ \overline{F}'''(0,r) \\ \overline{F}^{(4)}(0,r) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6r(r^4+2) \\ 0 \\ 0 \\ 0 \\ 0 \\ -18(r^2-2) \\ 0 \end{bmatrix}$$

Approximate solution and exact solution are compared in Fig. 2 for $r = 0, 0.5, 1$.

As showed, the exact solution and approximate solution are equal.

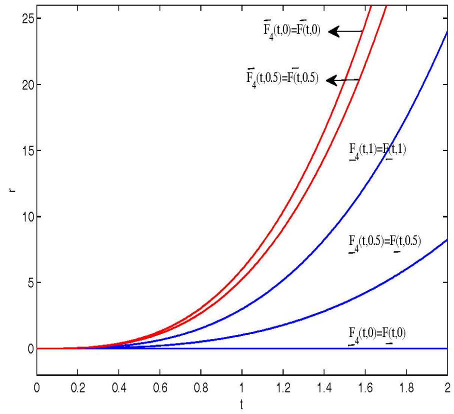

Figure 2: Comparison the exact and approximated solutions for Example 4.2.

Example 4.3 *Let fuzzy integral equation*

$$\overline{f}(t,r) = sin(\frac{t}{2})((\frac{13}{15})(r^2+r) + (\frac{2}{15})(4-r^3-r)),$$

$$\underline{f}(t,r) = sin(\frac{t}{2})((\frac{2}{15})(r^2+r) + (\frac{13}{15})(4-r^3-r)),$$

$$k(s,t) = 0.1sin(s)sin(\frac{t}{2}), \ 0 \le s,t \le 2\Pi,$$

with the exact solution $\overline{F}(t,r) = (4-r^3-r)sin(\frac{t}{2})$, $\underline{F}(t,r) = (r^2+r)sin(\frac{t}{2})$ *and* $a = 0$, $b = 2\pi$, $\lambda = 1$.

Similarly by using Eqs. (2.9)-(2.10), we calculated the coefficients matrix W in $z = 0$ as following:

$$W = \begin{bmatrix} W_{1,1} & W_{1,2} \\ W_{2,1} & W_{2,2} \end{bmatrix},$$

$W_{1,1} =$

$$\begin{bmatrix} -1.00 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.100 & -0.84 & 0.146 & 0.101 & 0.056 & 0.026 \\ 0.000 & 0.000 & -1.00 & 0.000 & 0.000 & 0.000 \\ -0.02 & -0.03 & -0.03 & -1.02 & -0.01 & -0.00 \\ 0.000 & 0.000 & 0.000 & 0.000 & -1.00 & 0.000 \\ 0.006 & 0.009 & 0.009 & 0.006 & 0.003 & -0.99 \end{bmatrix},$$

$$W_{1,2} =$$

$$\begin{bmatrix} 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ -0.10 & -0.47 & -1.13 & -1.85 & -2.31 & -2.35 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.025 & 0.117 & 0.283 & 0.463 & 0.579 & 0.588 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ -0.00 & -0.02 & -0.07 & -0.11 & -0.14 & -0.14 \end{bmatrix}.$$

Now we van write: (2.10) as follows:

$$W \begin{bmatrix} \underline{F}(0,r) \\ \underline{F}'(0,r) \\ \underline{F}''(0,r) \\ \underline{F}'''(0,r) \\ \underline{F}^{(4)}(0,r) \\ \underline{F}^{(5)}(0,r) \\ \overline{F}(0,r) \\ \overline{F}'(0,r) \\ \overline{F}''(0,r) \\ \overline{F}'''(0,r) \\ \overline{F}^{(4)}(0,r) \\ \overline{F}^{(5)}(0,r) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{r^3}{15} - \frac{13r^2}{30} - \frac{11r}{30}0 - \frac{4}{15} \\ 0 \\ -\frac{r^3}{60} + \frac{13r^2}{120} + \frac{11r}{120} + \frac{1}{15} \\ 0 \\ \frac{r^3}{240} - \frac{13r^2}{480} - \frac{11r}{480} - \frac{1}{60} \\ 0 \\ \frac{13r^3}{30} - \frac{r^2}{15} + \frac{11r}{30} - \frac{26}{15} \\ 0 \\ -\frac{13r^3}{120} + \frac{r^2}{60} - \frac{11r}{120} + \frac{13}{30} \\ 0 \\ \frac{13r^3}{480} - \frac{r^2}{240} + \frac{11r}{480} - \frac{13}{120} \end{bmatrix}$$

The vector solution of above linear system is:

$$\begin{bmatrix} \underline{F}(0,r) \\ \underline{F}'(0,r) \\ \underline{F}''(0,r) \\ \underline{F}'''(0,r) \\ \underline{F}^{(4)}(0,r) \\ \underline{F}^{(5)}(0,r) \\ \overline{F}(0,r) \\ \overline{F}'(0,r) \\ \overline{F}''(0,r) \\ \overline{F}'''(0,r) \\ \overline{F}^{(4)}(0,r) \\ \overline{F}^{(5)}(0,r) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0.0127\ r^3 + 0.5023\ r^2 + 0.5151\ r - 0.0511 \\ 0 \\ -0.0031\ r^3 - 0.1255\ r^2 - 0.1287\ r + 0.0127 \\ 0 \\ 0.0007\ r^3 + 0.0313\ r^2 + 0.03219\ r - 0.0031 \\ 0 \\ -0.5023\ r^3 - 0.0127\ r^2 - 0.5151\ r + 2.0093 \\ 0 \\ 0.1255\ r^3 + 0.0031\ r^2 + 0.1287\ r - 0.5023 \\ 0 \\ -0.0313\ r^3 - 0.0007\ r^2 - 0.0321\ r + 0.1255 \end{bmatrix}$$

Fig. 3 shows the accuracy of the solution functions. The differences between 6-th truncation limits of Taylor series with exact solution are quite noticeable.

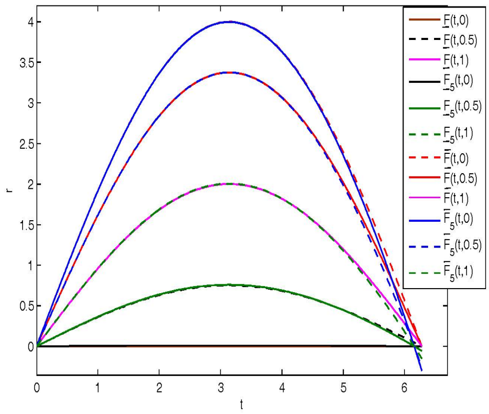

Figure 3: Comparison the exact and approximated solutions for Example 4.3.

From this example, we can conclude that to get the best approximating solution for unknown functions, the truncation limit N must be chosen large enough.

5 Conclusion

In some cases an analytical solution can not be found for integral equation, therefore numerical methods have been applied. In this paper we have worked out a computational method for the approximate solution of the linear Fredholm fuzzy integral equations of the second kind. the presented course in this study is a method for computing unknown Taylor coefficients of the solution functions. Consider that to get the best approximating solution of the present fuzzy equation, the truncation limit N must be chosen large enough. An interesting feature of this method is finding the analytical solution for given equation, if the exact solution was a polynomial of degree N or less than N. The analyzed examples illustrate the ability and reliability of the present method.

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