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A new iterative with memory class for solving nonlinear equations

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Abstract

In this work we develop a new optimal without memory class for approximating a simple root of a nonlinear equation. This class includes three parameters. Therefore, we try to derive some with memory methods so that the convergence order increases as high as possible. Some numerical examples are also presented.

Keywords : Multi-step methods; Nonlinear equations; Optimal order; Methods with memory; Kung-Traub's conjecture.

1 Introduction

Construction and development of optimal iterative without memory methods for solving nonlinear equations have been considered after the Kung and Traub seminal paper [3]. On the other hand, it is possible to derive with memory methods in which they could have even better efficiencies. To this purpose, one should apply some parameters in the construction without memory method in such a way that they could be accelerated during the iterative process to increase the convergence order without any new functional evaluations.

Inspired and motivated by with memory notion, some one, two, and three accelerator methods have been used in the literature [4]. To the best of our knowledge, there are only three kind of these works which dealing with three accelerators [4, 5, 7]. Soleymani et al's work is as follows [7]:

$$\begin{cases} x_0, \gamma_0, q_0, t_0, \quad \text{are given suitably.} \\ \gamma_n = -\frac{1}{N_3'(x_n)}, q_n = -\frac{N_4''(w_n)}{2N_4'(w_n)}, \\ t_n = \frac{1}{6}N_4'''(w_n), n = 1, 2, 3, \dots, \\ w_n = x_n + \gamma_n f(x_n), n = 0, 1, 2, \dots, \\ y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]+q_n f(w_n)}, \\ x_{n+1} = y_n - \left(1 + \frac{f(y_n)}{f(x_n)}\right) \\ \qquad \times \frac{f(y_n)}{f[y_n,w_n]+q_n f(w_n)+t_n(y_n-x_n)(y_n-w_n)}, \end{cases} \tag{1.1}$$

where $N_3(t)$ is the third degree interpolation passing through the four nodes $x_{n-1}, w_{n-1}, y_{n-1}, x_n$, and $N_4(t)$ is the fourth-degree interpolation passing through the latest five nodes $x_{n-1}, w_{n-1}, y_{n-1}, x_n, w_n$.

In this work, based on Soleymani et al's [7], we try to introduce a new class of this method. In other word, our class includes Soleymani et al's method [7] if we change one of provided conditions. To this end, we first develop a new iterative without memory class in which it is optimal in the sense of Kung and Traub. Some concrete methods are given. Then, similar to the mentioned

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work above, we produce with memory methods having three accelerators. As the main contribution of this work, it is worth mentioning that this is the first class with the mentioned features, i.e., including three accelerators.

2 Developing an optimal without memory class with three parameters

Here, we are focusing on developing a new optimal without memory class. We wish that this class could be extended to with memory. Also, we want that this class could be extended to the with memory method (1.1) if $h''(0) = 0$. So, we suggest the following scheme (for the sake of simplicity, iteration index is dropped):

$$\begin{cases} w = x + \gamma f(x), \\ y = x - \frac{f(x)}{f[x,w]+qf(w)}, \\ s = \frac{f(y)}{f(x)}; \\ \hat{x} = y - h(s)\frac{f(y)}{f[y,w]+qf(w)+t(y-x)(y-w)}. \end{cases} \quad (2.2)$$

This class uses three functional evaluations per cycle; however, its convergence order has not been determined. It depends on how the weight function $h(s)$ is chosen. In the following theorem, we address the conditions in which the suggested class (2.2) obtains the convergence order four, i.e., an optimal without memory class.

Theorem 2.1 *Let the initial guess x_0 is close enough to the simple root α for the nonlinear equation $f(x) = 0$. If $h(s)$ is a differentiable function that satisfies the conditions $h(0) = h'(0) = h''(0) = 1$, then the iterative class (2.2) is optimal and has the following error equation*

$$e_{n+1} = Ae_n^4 + O(e_n^5), \quad (2.3)$$

where

$$\begin{aligned} A =& \frac{1}{f'(\alpha)}\Big((1+\gamma f'(\alpha))^2 (c_2+q)(c_2 \\ &\times f'(\alpha)(c_2(\gamma f'(\alpha)+5)+2q \\ &\times (\gamma f'(\alpha)+3)) + f'(\alpha)q^2(\gamma f'(\alpha) \\ &+1) + 2t - 2c_3 f'(\alpha))\Big), \end{aligned}$$

$c_k = \frac{f^{(k)}(\alpha)}{f'(\alpha)}, k = 2, 3,$ *and* $e_n = x_n - \alpha$.

Proof.
Let $e_n = x_n - \alpha, e_{n,w} = w_n - \alpha$, and $e_{n,y} = y_n - \alpha$. Since $f(\alpha) = 0 \neq f'(\alpha)$, hence $f(x_n) = f'(\alpha)(e_n + c_2 e_n^2 + c_3 e_n^3)$. After some algebraic calculations, we have

$$e_{n,w} = (1+\gamma f'(\alpha))e_n + O(e_n^2), \quad (2.4)$$

$$e_{n,y} = (1+\gamma f'(\alpha))(c_2+q)e_n^2 + O(e_n^3), \quad (2.5)$$

and

$$e_{n+1} = (1-h(0))(1+\gamma f'(\alpha))(c_2+q)e_n^2 + O(e_n^3). \quad (2.6)$$

To obtain a higher method, we need $h(0) = 1$. Then,

$$\begin{aligned} e_{n+1} =& -(1-h'(0))(1+\gamma f'(\alpha))(c_2+q)^2 e_n^3 \\ &+ O(e_n^4). \end{aligned} \quad (2.7)$$

If $h'(0) = 1$, then one obtains an optimal method. Also, setting $h''(0) = 1$ leads to the desired result.

Remark 2.1 *It should be noted that the third condition, i.e. $h''(0) = 1$, can be replaced by $|h''(0)| < \infty$. We choose $h''(0) = 1$ to provide as simple as possible the error equation. Soleymani et al's method [7] is obtained if we consider $h''(0) = 0$.*

Some concrete weight functions that satisfy the conditions in Theorem 2.1 are

$$\begin{aligned} h_1(s) &= 1 + s + \frac{s2}{2}, \quad h_2(s) = e^s, \\ h_3(s) &= \frac{1}{1 - s - \frac{1}{2}s^2} \\ h_4(s) &= \frac{2+s}{2-s}. \end{aligned} \quad (2.8)$$

Therefore, if one uses any of these weight functions instead of the generic weight function $h(s)$ in (2.2), a new optimal without memory method is obtained.

3 Developing tre-accelerators class of with memory methods

In the preceding section, we introduced an optimal without memory class of iterative methods. Also, some concretes were derived. This class uses only three function evaluations per full cycle. By looking at its error equation (2.3), it is still possible to vanish the coefficient of e_n^4. There are some possibilities for this task. We leave the details, and one can consult the references. We just have to point out the key ideas: if we wish to increase the convergence order of the class (2.2), first we should consider it as with memory. In other words, similar to the method (1.1), or those given in [4, 5], all the three parameters in the derived methods of the class (2.2) need to be updated in each iteration. We need [4]:

Lemma 3.1 *If* $\gamma_n = -1/N_3'(x_n)$ *and* $p_n = -N_4''(w_n)/(2N_4'(w_n))$ *and* $\lambda_n = N_4'''(w_n)/6$, $n = 1, 2, \ldots,$ *then the estimates*

$$1 + \gamma_n f'(\alpha) \sim e_{n-1,y}\, e_{n-1,w}\, e_{n-1} \tag{3.9}$$

$$c_2 + q_n \sim e_{n-1,y}\, e_{n-1,w}\, e_{n-1}, \tag{3.10}$$

and

$$t_n - f'(\alpha)c_3 \sim e_{n-1,w}\, e_{n-1}, \tag{3.11}$$

hold.

To summarise, we have

Theorem 3.1 *Under the given conditions in Theorem 2.1, the following with memory class has convergence order at least 7.24:*

$$\begin{cases} x_0, \gamma_0, q_0, t_0, \quad \text{are given suitably.} \\ \gamma_n = -\frac{1}{N_3'(x_n)}, q_n = -\frac{N_4''(w_n)}{2N_4'(w_n)}, \\ t_n = \frac{1}{6}N_4'''(w_n), n = 1, 2, 3, \ldots, \\ w_n = x_n + \gamma_n f(x_n), n = 0, 1, 2, \ldots, \\ y_n = x_n - \frac{f(x_n)}{f[x_n, w_n] + q_n f(w_n)}, \\ s_n = \frac{f(y_n)}{f(x_n)} \\ x_{n+1} = y_n - h(s_n) \\ \qquad \times \frac{f(y_n)}{f[y_n, w_n] + q_n f(w_n) + t_n(y_n - x_n)(y_n - w_n)}, \end{cases} \tag{3.12}$$

where $N_3(t)$ is the third degree interpolation passing through the four nodes $x_{n-1}, w_{n-1}, y_{n-1}, x_n$, and $N_4(t)$ is the fourth-degree interpolation passing through the latest five nodes $x_{n-1}, w_{n-1}, y_{n-1}, x_n, w_n$.

Proof. If we suppose that $\{x_n\}$ has convergence order R, then

$$e_{n+1} \sim e_n^R \sim e_{n-1}^{R^2}, \qquad e_n = x_n - \alpha. \tag{3.13}$$

Also, if the sequences $\{w_n\}$, and $\{y_n\}$ have convergence order R_1, and R_2, respectively, then

$$e_{n,w} \sim e_n^{R_1} = (e_{n-1}^R)^{R_1} = e_{n-1}^{RR_1}, \tag{3.14}$$

$$e_{n,y} \sim e_n^{R_2} = (e_{n-1}^R)^{R_2} = e_{n-1}^{RR_2}. \tag{3.15}$$

On the other hand, considering Lemma 3.1, and using it to Equations (2.4), (2.5), and (2.3), then

$$e_{n+1} \sim e_{n-1}^{4R+3R_2+4R_1+4}, \tag{3.16}$$

$$e_{n,w} \sim e_{n-1}^{R+R_2+R_1+1}, \tag{3.17}$$

and

$$e_{y,n} \sim e_{n-1}^{2R+2R_2+2R_1+2}. \tag{3.18}$$

Comparing the right hand sides of Eqs. (3.13)-(3.16), (3.14)-(3.17), and (3.15)-(3.18), we conclude

$$\begin{cases} R^2 - (4R + 3R_2 + 4R_1 + 4) = 0, \\ RR_1 - (R + R_2 + R_1 + 1) = 0, \\ RR_2 - (2R + 2R_2 + 2R_1 + 2) = 0. \end{cases} \tag{3.19}$$

This system has the solution $R = 7.24$, $R_2 = 3.89$, and $R_1 = 1.94$.

Remark 3.1 *The with memory class (3.12) has efficiency index* $7.24^{1/3} \approx 1.93$, *while the optimal without memory class (2.2) has efficiency index* $4^{1/3} \approx 1.59$. *We recall that efficiency index is defined by* $E(p, n) = p^{1/n}$, *where* p *is the convergence order, and* n *is the functional evaluations in each iteration.*

In the following, we represent some of the with memory methods derived from the mentioned class (3.12):

Method 1

$$\begin{cases} x_0, \gamma_0, q_0, t_0, \quad \text{are given suitably.} \\ \gamma_n = -\frac{1}{N_3'(x_n)}, q_n = -\frac{N_4''(w_n)}{2N_4'(w_n)}, \\ t_n = \frac{1}{6}N_4'''(w_n), n = 1, 2, 3, \ldots, \\ w_n = x_n + \gamma_n f(x_n), n = 0, 1, 2, \ldots, \\ y_n = x_n - \frac{f(x_n)}{f[x_n, w_n] + q_n f(w_n)}, \\ s_n = \frac{f(y_n)}{f(x_n)} \\ x_{n+1} = y_n - (1 + s_n + \frac{s_n^2}{2}) \\ \qquad \times \frac{f(y_n)}{f[y_n, w_n] + q_n f(w_n) + t_n(y_n - x_n)(y_n - w_n)}, \end{cases} \tag{3.20}$$

Method 2

$$\begin{cases} x_0, \gamma_0, q_0, t_0, \quad \text{are given suitably.} \\ \gamma_n = -\frac{1}{N_3'(x_n)}, q_n = -\frac{N_4''(w_n)}{2N_4'(w_n)}, \\ t_n = \frac{1}{6}N_4'''(w_n), n = 1, 2, 3, \ldots, \\ w_n = x_n + \gamma_n f(x_n), n = 0, 1, 2, \ldots, \\ y_n = x_n - \frac{f(x_n)}{f[x_n, w_n] + q_n f(w_n)}, \\ s_n = \frac{f(y_n)}{f(x_n)} \\ x_{n+1} = y_n - (\frac{2+s_n}{2-s_n}) \\ \qquad \times \frac{f(y_n)}{f[y_n, w_n] + q_n f(w_n) + t_n(y_n - x_n)(y_n - w_n)}, \end{cases} \tag{3.21}$$

Method 3

$$\begin{cases} x_0, \gamma_0, q_0, t_0, \quad \text{are given suitably.} \\ \gamma_n = -\frac{1}{N_3'(x_n)}, q_n = -\frac{N_4''(w_n)}{2N_4'(w_n)}, \\ t_n = \frac{1}{6}N_4'''(w_n), n = 1, 2, 3, \ldots, \\ w_n = x_n + \gamma_n f(x_n), n = 0, 1, 2, \ldots, \\ y_n = x_n - \frac{f(x_n)}{f[x_n, w_n] + q_n f(w_n)}, \\ s_n = \frac{f(y_n)}{f(x_n)} \\ x_{n+1} = y_n - (\frac{2}{2-2s_n+s_n^2}) \\ \qquad \times \frac{f(y_n)}{f[y_n, w_n] + q_n f(w_n) + t_n(y_n - x_n)(y_n - w_n)}, \end{cases} \tag{3.22}$$

4 Numerical performances, comparisons, and concluding remarks

To implement our proposed methods in action, we test them using some examples. By $|x_k - \alpha|$ we denote the error to the sought zeros. $A(-h)$ stands for $A \times 10^{-h}$.The software Mathematica 10, with multi-precision arithmetic has been used in our computations. Moreover, r_c indicates computational order of convergence, and is computed [4],

$$r_c = \frac{\log(|f(x_n)/f(x_{n-1})|)}{\log(|f(x_{n-1})/f(x_{n-2})|)}.$$

We compare our Methods 1-3 (see (3.20)-(3.22)) with the Method (7) in [5] and the Method (1.1) in [7] with the similar features. The following test functions are used:

$$f_1(x) = \exp(x^2 - 3x)\sin(x) + \log(x^2 + 1),$$
$$x_0 = 0.35, \ \alpha = 0,$$
$$f_2(x) = \exp(2 + x - x^2) + \sin(\pi x) \times \exp(x^2 + c\cos(x) - 1) + 1,$$
$$x_0 = 1.3, \ \alpha = 1.5503...$$

Methods	$\|f(x_1)\|$	$\|f(x_2)\|$	$\|f(x_3)\|$	COC
Method [7]	.495(-4)	.811(-21)	.213(-133)	7.23
Method [5]	0.002	.227(21)	.276(142)	7.23
Method 1	.184(-3)	.672(-22)	.620(-182)	7.24
Method 2	0.662(-2)	.200(23)	.271(154)	7.23
Method 3	0.729(-3)	.300(-9)	.116(-59)	7.23

Table 1: Results of $f_1(x) = 0$ for different methods with $\gamma_0 = q_0 = t_0 = -0.1$

heightMethods	$\|f(x_1)\|$	$\|f(x_2)\|$	$\|f(x_3)\|$	COC
Method [7]	.177(-3)	.179(-26)	.283(-182)	7.23
Method [5]	0.006	.303(25)	.115(175)	7.21
Method 1	.837(-2)	.908(-10)	.620(-74)	7.24
Method 2	.471(-2)	.739(-19)	.863(-137)	7.26
Method 3	.593(-4)	.920(-26)	.992(-184)	7.27

Table 2: Results of $f_2(x) = 0$ for different methods with $\gamma_0 = q_0 = t_0 = -0.1$

As can be observed form the Tables 1 and 2, our proposed methods have good performances. Moreover, they behave similar to the compared methods asymptotically. To sum up, we have developed a new class of with memory methods in which it uses only three functional evaluations per full cycle. This class uses three accelerators in such a way that it reaches convergence order 7.23. This class has higher efficiency index than the two accelerators introduced in [1] and [6]. Numerical examples showed that some concrete methods derived form the mentioned class can compete the existing methods with the similar properties suggested in [4, 7]. Finally, it is recommended that further research should be undertaken to carrying out higher convergence rate applying the idea of adaptation. In this case, it seems that the rate of convergence could increase to 7.77200.

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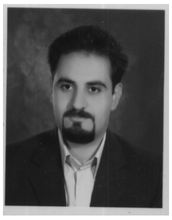

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