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Convergence of Numerical Method For the Solution of Nonlinear Delay Volterra Integral Equations

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Abstract

In this paper, Solvability nonlinear Volterra integral equations with general vanishing delays is stated. So far sinc methods for approximating the solutions of Volterra integral equations have received considerable attention mainly due to their high accuracy. These approximations converge rapidly to the exact solutions as number sinc points increases. Here the numerical solution of nonlinear delay Volterra integral equations is considered by two methods. The methods are developed by means of the sinc approximation with the single exponential (SE) and double exponential (DE) transformations. These numerical methods combine a sinc collocation method with the Newton iterative process that involves solving a nonlinear system of equations. The existence and uniqueness of numerical solutions for these equations are provided. Also an error analysis for the methods is given. So far approximate solutions with polynomial convergence have been reported for this equation. These methods improve conventional results and achieve exponential convergence. Numerical results are included to confirm the efficiency and accuracy of the methods.

Keywords : Nonlinear Volterra integral equations; General delays; Sinc-collocation; Convergence analysis.

1 Introduction

Delay Volterra integral equations arise widely in scientific fields such as physics, biology, ecology, control theory, etc. Due to the practical application of these equations, they must be solved successfully with efficient numerical approaches. In recent years, there have been extensive studies in convergence properties and stability analyses of numerical methods for them, see, for example, [12, 14]. The numerical solutions of integral equations with delays have also been discussed by several authors such as Brunner [2] and Linz Wang [7].

Sinc methods for approximating the solutions of Volterra integral equations have received considerable attention mainly due to their high accuracy. These approximations converge rapidly to the exact solutions as number sinc points increases. Systematic introduction of these methods can be found in [11]. In [13] sinc-collocation method is emplyed to solve Hammerstein Volterra integral equations, but it does not provide existence and uniqueness of the sinc-collocation solution. In this paper we extend the analytical and numerical techniques used in [13] to nonlinear integral equations with general vanishing delay and also show that the sinc-collocation solution exist

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and it is unique. In addition to we give an error analysis.
The main objective of the current study is to implement the sinc-collocation method for nonlinear Volterra integral equation of the form

$$y(t) = g(t) + \int_0^{\theta(t)} K(t, s, y(s))ds, \quad t \in I := [0, T], \qquad (1.1)$$

the delay function θ is subject to the following conditions:

(D1) $\theta(0) = 0$, and θ is strictly increasing on the interval I;

(D2) $\theta(t) \leq t$, $t \in I$;

(D3) $\theta \in C^d(I)$ for some $d \geq 0$.

We will refer to a θ that satisfies (D1) as a vanishing delay function (or, in short, a *vanishing delay*). The linear case, $\theta(t) = qt = t - (1 - q)t =: t - \tau(t)$ $(0 < q < 1)$ (proportional delay) is also known as the pantograph delay function [6]. In this paper we consider vanishing delay but our methods can be use with nonvanishing delay too.
Several methods have been already presented to the numerical solution of (1.1) for the special case pantograph i.e. $\theta(t) = qt$, for example you can see [2, 3]. We propose two numerical algorithms in order to solve the delay integral equation (1.1) where $\theta(t)$ is more general than pantograph.
Our motivation comes from the fact that equation (1.1) may be viewed as a generalization of the equation encountered in the mathematical modeling of electric pantograph dynamics and a partitioning problem in number theory [8]

$$y(t) = g(t) + \int_0^{qt} k(t, s)y(s)ds.$$

The layout of this paper is as follows. In Section 2, the solvability of the Eq. (1.1) is stated. Section 3 outlines some of the main properties of sinc function that is necessary for the formulation of the delay integral equation. Sinc-collocation method is considered in Section 4. In section 5, we analyze the existence and uniqueness of numerical solutions. In Section 6, the orders of scheme convergence using the new approache are described. Finally, Section 7 contains the numerical experiments.

2 Existence and uniqueness of solutions

In the present section, we state the solvability of nonlinear integral equations with vanishing delay. To state the following theorem we will adopt the notation

$$D := \{(t, s) : 0 \leqslant s \leqslant t \leqslant T\},$$
$$\Omega_B := \{(t, s, y) : (t, s) \in D, \quad y \in \mathbb{R} \text{ and } |y - g(t)| \leqslant B\},$$

and we set $M_B := \max\{|K(t, s, y)| : (t, s, y) \in \Omega_B\}$.

Theorem 2.1 *Assume:*

(a) *$g \in C(I)$ and $K \in C(\Omega_B)$;*

(b) *$\theta(t)$ is subject to the assumptions (D1)-(D3);*

(c) *K satisfied the Lipschitz condition*

$$|K(t, s, y) - K(t, s, z)| \leqslant L_B|y - z|, \quad \text{for all} \quad (t, s, y), (t, s, z) \in \Omega_B.$$

Then:

(i) *The Picard iteration $y_n(t)$ exist for all $n \geqslant 1$. They are continuous on the interval $I_0 := [0, \delta_0]$, where*

$$\delta_0 := \min\{T, B/M_B\},$$

and they converge uniformly on I_0 to a solution $y \in C(I_0)$ of the nonlinear Volterra integral equation (1.1).

(ii) *This solution y is the unique continuous solution on I_0.*

Proof. The above result is also related to an existence and uniqueness result in [3]. His proof techniques are readily extended to the general delay function in which $\theta(t)$ is subject to the conditions (D1)-(D3). We omit the details. □

3 Review of the sinc approximation

In this section, we will review sinc function properties, sinc quadrature rule, and the sinc method.

These are discussed thoroughly in [11]. The sinc basis functions are given by

$$S(j,h)(z) = \text{sinc}(\frac{z-jh}{h}), \quad j = 0, \pm 1, \pm 2, \ldots \tag{3.2}$$

where

$$\text{sinc}(z) = \begin{cases} \frac{\sin(\pi z)}{\pi z}, & z \neq 0; \\ 1, & z = 0, \end{cases}$$

and h is a step size appropriately chosen depending on a given positive integer N, and j is an integer and (3.2) is called the jth sinc function. Originally, sinc approximation for a function u is expressed as

$$u(t) \approx \sum_{j=-N}^{N} u(jh) S(j,h)(t), \quad t \in \mathbb{R}. \tag{3.3}$$

The above approximation is valid on $\mathbb{R}$, whereas the Eq. (1.1) is defined on finite interval $[0,T]$. The Eq. (3.3) can be adapted to approximate on general intervals with the aid of appropriate variable transformations $t = \phi(x)$. As the transformation function $\phi(x)$ appropriate single exponential (SE) and double exponential (DE) transformations are applied. The single exponential transformation and its inverse can be introduced respectively as below

$$\psi_{SE}(x) = \frac{Te^x}{1+e^x}, \qquad \phi_{SE}(t) = \ln(\frac{t}{T-t}).$$

In order to define a convenient function space, the strip domain $D_d = \{z \in \mathcal{C} : |Imz| < d\}$ for some $d > 0$ is introduced. When incorporated with the SE transformation, the conditions should be considered on the translated domain

$$\psi_{SE}(D_d) = \{z \in \mathcal{C} : |\arg(\frac{z}{T-z})| < d\}.$$

The following definitions and theorems are considered for further details of the procedure.

Definition 3.1 *Let D be a simply connected domain which satisfies $(a,b) \subset D$ and α and c_1 be a positive constant. Then $\mathcal{L}_\alpha(D)$ denotes the family of all functions $u \in \boldsymbol{Hol}(D)$ which satisfy*

$$|u(z)| \leqslant c_1 |Q(z)|^\alpha \tag{3.4}$$

for all z in D where $Q(z) = (z-a)(b-z)$.

The next theorem shows the exponential convergence of the SE-sinc approximation.

Theorem 3.1 *Let $u \in \mathcal{L}_\alpha(D)$, let N be a positive integer, and let h be selected by the formula $h = \sqrt{\frac{\pi d}{\alpha N}}$, then there exists positive constant c_2, independent of N, such that*

$$\sup_{t\in(a,b)} |u(t) - \sum_{j=-N}^{N} u(\psi_{SE}(jh)) S(j,h)(\phi_{SE}(t))| \leqslant c_2 \sqrt{N} e^{-\sqrt{\pi d \alpha N}}.$$

The error analysis of the SE-sinc indefinite integration has been given in [9].

Theorem 3.2 *Let $uQ \in \mathcal{L}_\alpha(D)$ for d with $0 < d < \pi$. Let $h = \sqrt{\frac{\pi d}{\alpha N}}$. Then there exists a constant c_3, which is independent of N, such that*

$$\sup_{t\in(a,b)} |\int_a^t u(s)ds - h \sum_{j=-N}^{N} u(\psi_{SE}(jh)) \psi'_{SE}(jh) J(j,h)(\phi_{SE}(t))| \leqslant c_3 e^{-\sqrt{\pi d \alpha N}} \tag{3.5}$$

where

$$J(j,h)(x) = \frac{1}{2} + \int_0^{\frac{x}{h}-j} \frac{\sin(\pi t)}{\pi t} dt.$$

The double exponential transformation can be used instead of the single exponential transformation. DE-transformation and its inverse are

$$\psi_{DE}(x) = \frac{b-a}{2} \tanh(\frac{\pi}{2} \sinh(x)) + \frac{b+a}{2},$$

$$\phi_{DE}(t) = \ln[\frac{1}{\pi} \ln(\frac{t-a}{b-t}) + \sqrt{1 + \left\{\frac{1}{\pi} \ln\left(\frac{t-a}{b-t}\right)\right\}^2}].$$

This transformation maps D_d onto the domain

$$\psi_{DE}(D_d) = \{z \in \mathcal{C} : |\arg[\frac{1}{\pi} \ln(\frac{t-a}{b-t}) + \sqrt{1 + \{\frac{1}{\pi} \ln(\frac{t-a}{b-t})\}^2}]| < d\}.$$

The following theorem describes the extreme accuracy of DE-sinc approximation when $u \in \mathcal{L}_\alpha(\psi_{DE}(D_d))$.

Theorem 3.3 *Let $u \in \mathcal{L}_\alpha(\psi_{DE}(D_d))$ for d with $0 < d < \frac{\pi}{2}$, N be a positive integer and h is selected by the formula $h = \frac{\ln(2dN/\alpha)}{N}$. Then there exists a constant c_4 which is independent of N, such that*

$$\sup_{t\in(a,b)} |u(t) - \sum_{j=-N}^{N} u(\psi_{DE}(jh)) S(j,h)(\phi_{DE}(t))| \leqslant c_4 e^{-\pi dN/\ln(2dN/\alpha)}.$$

If we use the DE transformation instead of the SE transformation, the DE-sinc quadrature is achieved. The rate of convergence is accelerated as the next theorem states.

Theorem 3.4 *([9]) Let $uQ \in \mathcal{L}_\alpha(\psi_{DE}(D_d))$ for d with $0 < d < \frac{\pi}{2}$. Let $\alpha' = \alpha - \epsilon$ for $0 < \epsilon < \alpha$, N be a positive integer with $N > \alpha'/(2d)$, and h is selected by the formula*

$$h = \frac{\ln(2dN/\alpha')}{N}.$$

Then there exists a constant c_5 which is independent of N, such that

$$\sup_{t\in(a,b)} |\int_a^t u(s)ds - h\sum_{j=-N}^{N} u(\psi_{DE}(jh))\psi'_{DE}(jh) J(j,h)(\phi_{DE}(t))| \leqslant c_5 e^{-\pi dN/\ln(2dN/\alpha')}.$$

4 Sinc-collocation method

In the present section, we apply sinc-collocation method to solve Eq. (1.1) which we state again for the convenience of the reader:

$$y(t) = g(t) + \int_0^{\theta(t)} K(t,s,y(s))ds, \quad t \in I := [0,T],$$

if $t = 0$ we have $y(0) = g(0)$. For ease of calculation, we employ the transformation

$$u(t) = y(t) - \frac{T-t}{T} g(0),$$

in this case $u(0) = 0$. Then the above problem becomes

$$u(t) = f(t) + \int_0^{\theta(t)} K_1(t,s,u(s))ds \qquad (4.6)$$

where

$$f(t) := g(t) - \frac{T-t}{T} g(0),$$

$$K_1(t,s,u(s)) := K(t,s,u(s) + \frac{T-t}{T} g(0)).$$

Now, let $u(t)$ be the exact solution of (4.6).

4.1 SE-sinc scheme

The approximate solution $\mathcal{U}_N^{SE}$ is considered that has the form

$$\mathcal{U}_N^{SE}(t) = \sum_{j=-N}^{N} u(\psi_{SE}(jh))S(j,h)(\phi_{SE}(t)) + u(T)w_{SE}(t), \quad t \in [0,T] \qquad (4.7)$$

we choose $w_{SE}(t)$ so that above formula interpolate function u at the points

$$t_j^{SE} = \begin{cases} \psi_{SE}(jh), & j = -N,\dots,N; \\ T, & j = N+1, \end{cases}$$

then

$$w_{SE}(t) = \frac{1}{T}(t - \sum_{j=-N}^{N} t_j^{SE} S(j,h)(\phi_{SE}(t))).$$

We replace approximate solution (4.7) in (4.6). Substituting $t = t_k^{SE}$, $k = -N,\dots,N+1$

$$u_k^{SE} = f(t_k^{SE}) + \int_0^{\theta(t_k^{SE})} K_1(t_k^{SE}, s, \sum_{j=-N}^{N} u_j^{SE} S(j,h)(\phi_{SE}(s)) + u_{N+1}^{SE} w_{SE}(s))ds, \qquad (4.8)$$

we approximate the integral in above equation by the quadrature formula presented in (3.5)

$$\int_0^{\theta(t_k^{SE})} K_1(t_k^{SE}, s, \sum_{j=-N}^{N} u_j^{SE} S(j,h)(\phi_{SE}(s)) + u_{N+1}^{SE} w_{SE}(s))ds = h\sum_{l=-N}^{N} \psi'_{SE}(lh) J(l,h)(\phi_k^{SE}) K_1(t_k^{SE}, t_l^{SE}, u_l^{SE}),$$

where

$$\phi_k^{SE} := \phi_{SE}(\theta(t_k^{SE})).$$

Thus Eq. (4.8) is written as

$$u_k^{SE} = f(t_k^{SE}) + h\sum_{l=-N}^{N} \psi'_{SE}(lh) J(l,h)(\phi_k^{SE}) K_1(t_k^{SE}, t_l^{SE}, u_l^{SE}), \qquad (4.9)$$

where $u_k^{SE} = u(t_k^{SE})$.

This linear system of equations is equivalent to (4.6). By solving this system, the unknown coefficients u_k^{SE} are determined. We rewrite the nonlinear system (4.9) in matrix form

$$\mathcal{A}^{SE}(\mathbf{u}^{SE}) = \mathbf{u}^{SE}, \tag{4.10}$$

where

$$[\mathcal{A}^{SE}(\mathbf{u}^{SE})]_{k,l} := f(t_k^{SE}) + h\psi'_{SE}(lh)J(l,h)(\phi_k^{SE}) \\ K_1(t_k^{SE}, t_l^{SE}, u_l^{SE}), \quad k,l = -N,\dots,N+1,$$

$$\mathbf{u}^{SE} := [u_{-N}^{SE},\dots,u_{N+1}^{SE}]^t.$$

4.2 DE-sinc scheme

The DE-sinc case is focused on in this part. Similar to the SE-sinc method, the approximate solution $\mathcal{U}_N^{DE}$ can be defined as follow

$$\mathcal{U}_N^{DE}(t) = \sum_{j=-N}^{N} u_j^{DE} S(j,h)(\phi_{DE}(t)) \\ + u_{N+1}^{SE} w_{DE}(t), \qquad t \in [0,T]. \tag{4.11}$$

By applying (4.11) and setting its collocation on $2N+2$ sampling points at $t = t_k^{DE}$, for $k = N,\dots,N+1$, in Eq. (4.6), the following nonlinear system

$$\mathcal{A}^{DE}(\mathbf{u}^{DE}) = \mathbf{u}^{DE}, \tag{4.12}$$

is achieved. By solving this system, the unknown coefficients in $\mathbf{u}^{DE}$ have been found.

5 Existence and uniqueness of the sinc-collocation solution

In this section, we study the existence and uniqueness of the solution to (4.10) and (4.12). It is necessary to bound the basis function $J(j,h)$. The next lemma gives the bound.

Lemma 5.1 *([11]) For $x \in \mathbb{R}$, the function $J(j,h)(x)$ is bounded by*

$$|J(j,h)(x)| \leqslant 1.1.$$

Theorem 5.1 *Assume that K_1, and f in the nonlinear Volterra equation (4.6) are continuous and*

$$|K_1(t,s,u(t)) - K_1(t,s,v(t))| \\ < L|u(t) - v(t)|.$$

Then the nonlinear algebraic systems (4.10) and (4.12) have a unique solution.

Proof. Using Lemma 5.1 and continuity K_1 we have

$$\|\mathcal{A}^{SE}(\mathbf{u}) - \mathbf{F}^{SE}\|_\infty \\ = \max_k |h \sum_{l=-N}^{N} \psi'_{SE}(lh)J(l,h)(\phi_k^{SE}) \\ K_1(t_k^{SE}, t_l^{SE}, u_l^{SE})| \\ \leqslant 1.1h \sup_x |\psi'(x)| \\ \sum_{l=-N}^{N} |K_1(t_k^{SE}, t_l^{SE}, u_l^{SE})| \\ \leqslant 1.1he^{-Nh} \sum_{l=-N}^{N} |K_1(t_k^{SE}, t_l^{SE}, u_l^{SE})| \\ \leqslant 1.1he^{-Nh}(2N+1)M,$$

where M is upper bound of $|K_1(t,s,u(t))|$ and $\mathbf{F}^{SE} := [f(t_{-N}^{SE}),\dots,f(t_{N+1}^{SE})]^t$.

By using fixed point theorem, this proves that the nonlinear system has a solution in the closed ball with center $\mathbf{F}^{SE}$ and radius $1.1hT(2N+1)M$. It may be shown that, if K_1 is Lipschitz with respect to $u(t)$, the solution is unique. For suppose that $\mathbf{u}$, $\mathbf{v}$ are two possible solutions

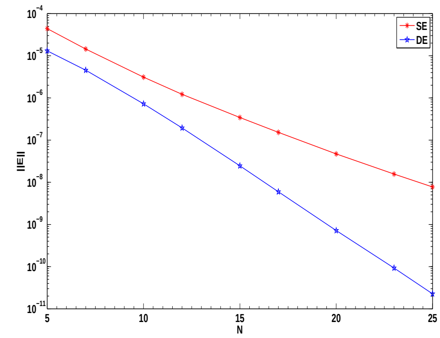

Figure 1: $q = 0.9$

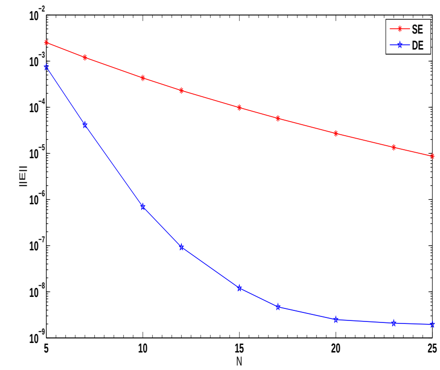

Figure 2: $r = 0.01$

$$\|\mathbf{u}-\mathbf{v}\|_\infty = \|\mathcal{A}^{SE}(\mathbf{u})-\mathcal{A}^{SE}(\mathbf{v})\|_\infty$$
$$= \max_k \Big| h \sum_{l=-N}^{N} \psi'_{SE}(lh) J(l,h)(\phi_k^{SE})$$

K˙1(t˙kˆSE,t˙lˆSE,u˙l)-K˙1(t˙kˆSE,t˙lˆSE,v˙l)

|

$$\leqslant 1.1he^{-Nh} \max_k \sum_{l=-N}^{N} |K_1(t_k^{SE}, t_l^{SE}, u_l)$$
$$-K_1(t_k^{SE}, t_l^{SE}, v_l)|$$
$$\leqslant 1.1he^{-Nh} \max_k \sum_{l=-N}^{N} L\,|u_l - v_l|$$
$$\leqslant 1.1he^{-Nh}(2N+1)L\|\mathbf{u}-\mathbf{v}\|_\infty$$
$$< \|\mathbf{u}-\mathbf{v}\|_\infty,$$

because $\lim_{N\to\infty} he^{-Nh}(2N+1) = 0$, we can write the last inequality for some N. It follows that $\|\mathbf{u}-\mathbf{v}\|_\infty$ vanishes and there is thus uniqueness.
The similar conclusions are achieved for DE case.
□

6 Convergence analysis

The convergence of the two sinc-collocation methods which are introduced in the previous sections is discussed in the present section. We first consider the SE case. It is assumed that $\mathbf{u}^{SE}$ is the exact solution of Eq. (4.10) and $\mathbf{u}_{(m)}^{SE}$ is an approximation of $\mathbf{u}^{SE}$ obtained from Newton's iterative process.
In the following theorem, we will find an upper bound for the error.

Theorem 6.1 *Let $\mathcal{U}_N^{SE}(t)$ is the approximate solution of integral equation (4.6). Then there exists a constant c_5 independent of N such that*

$$\sup_{t\in(0,T)} |u(t)-\mathcal{U}_N^{SE}(t)| \leqslant c_5\sqrt{N}e^{-\sqrt{\pi d\alpha N}}. \tag{6.13}$$

Proof. The error between u and $\mathcal{U}_N^{SE}$ can be expressed by

$$u(t)-\mathcal{U}_N^{SE}(t)$$
$$= f(t) + \int_0^{\theta(t)} K_1(t,s,u(s))$$
$$- \sum_{j=-N}^{N} u_j^{SE} S(j,h)(\phi_{SE}(t))$$
$$-u_{N+1}^{SE} w_{SE}(t)$$
$$= f(t) + \int_0^{\theta(t)} K_1(t,s,u(s))$$
$$- \sum_{j=-N}^{N} \Big\{ f(t_j^{SE}) + h \sum_{l=-N}^{N} \psi'_{SE}(lh)$$
$$J(l,h)(\phi_j^{SE}) K_1(t_j^{SE}, t_l^{SE}, u_l^{SE}) \Big\}$$
$$S(j,h)(\phi_{SE}(t)) - \Big\{ f(t_{N+1}^{SE})$$
$$+h \sum_{l=-N}^{N} \psi'_{SE}(lh) J(l,h)(\phi_{N+1}^{SE})$$
$$K_1(t_{N+1}^{SE}, t_l^{SE}, u_l^{SE}) \Big\} w_{SE}(t)$$
$$= f(t) - \sum_{j=-N}^{N} f(t_j^{SE}) S(j,h)(\phi_{SE}(t))$$
$$-f(t_{N+1}^{SE}) w_{SE}(t) + \int_0^{\theta(t)} K_1(t,s,u(s))$$
$$-h \sum_{j=-N}^{N} \sum_{l=-N}^{N} \psi'_{SE}(lh) J(l,h)(\phi_j^{SE})$$
$$K_1(t_j^{SE}, t_l^{SE}, u_l^{SE}) S(j,h)(\phi_{SE}(t))$$
$$-h \sum_{l=-N}^{N} \psi'_{SE}(lh) J(l,h)(\phi_{N+1}^{SE})$$
$$K_1(t_{N+1}^{SE}, t_l^{SE}, u_l^{SE}) w_{SE}(t)$$
$$= c_1\sqrt{N}e^{-\sqrt{\pi d\alpha N}} + c_2\sqrt{N}e^{-\sqrt{\pi d\alpha N}}$$
$$+c_3 h\sqrt{N}e^{-\sqrt{\pi d\alpha N}}$$
$$= c_5\sqrt{N}e^{-\sqrt{\pi d\alpha N}}.$$

Because $w_{SE}(t_j^{SE}) = 0$ for $j = -N,\ldots,N$, when we use Theorem 3.1 we obtain

$$\sum_{l=-N}^{N} \psi'_{SE}(lh) J(l,h)(\phi_{N+1}^{SE})$$
$$K_1(t_{N+1}^{SE}, t_l^{SE}, u_l^{SE}) w_{SE}(t)$$
$$= c_3\sqrt{N}e^{-\sqrt{\pi d\alpha N}}.$$

If we replace the SE transformation ϕ_{SE} by DE transformation ϕ_{DE} and assume $0 < d < \frac{\pi}{2}$

in Theorem , then the similar conclusions are achieved for DE case.

Theorem 6.2 *Let* $\mathcal{U}_N^{DE}(t)$ *is the approximate solution of integral equation (4.6). Then there exists a constant* c_6 *independent of* N *such that*

$$\sup_{t\in(0,T)} |u(t) - \mathcal{U}_N^{DE}(t)| \leqslant c_6 e^{-\pi dN/ln(2dN/\alpha)}.$$

In the following we are trying to discuss the conditions under which Newtons method is convergent. For this reason we will state and prove the following theorem.

Theorem 6.3 *Assume* $\boldsymbol{u}^{SE}$ *is the exact solution of the nonlinear system (4.9), and hypotheses of Theorem 5.1 are satisfied. Also, suppose that* $\frac{\partial K_1}{\partial u}$ *is Lipschitz with respect to* u*. Then there exist* $\delta > 0$ *and* $\bar{h} > 0$ *such that if* $\|\boldsymbol{u}_{(0)}^{SE} - \boldsymbol{u}^{SE}\| \leqslant \delta$*, the Newton's sequence* $\{\boldsymbol{u}_{(m)}^{SE}\}$ *for any* $h \in (0, \bar{h})$ *is well-defined and convergence to* $\boldsymbol{u}^{SE}$*. Furthermore, for some constant* l *with* $l\delta < 1$*, we have the error bounds*

$$\|\boldsymbol{u}_{(m)}^{SE} - \boldsymbol{u}^{SE}\| \leqslant \frac{(l\delta)^{2^m}}{l}.$$

Proof. We must solve the nonlinear system

$$\mathbf{u} - \mathcal{A}(\mathbf{u}) = 0.$$

The Newton method reads as follow. Choose an initial guess u_0; for $m = 0, 1, \ldots$, compute

$$\mathbf{u}_{(m+1)} = \mathbf{u}_{(m)} - [\mathcal{I} - \mathcal{A}'(\mathbf{u}_{(m)})]^{-1} [\mathbf{u}_{(m)} - \mathcal{A}(\mathbf{u}_{(m)})], \tag{6.14}$$

we know that

$$[\mathcal{A}'(\mathbf{u})]_{k,l} = h\psi'(lh)J(l,h)(\phi_k^{SE}) \frac{\partial K_1}{\partial u}(t_k^{SE}, t_l^{SE}, u_k^{SE}),$$
$$k, l = -N \ldots, N,$$
$$[\mathcal{A}'(\mathbf{u})]_{N+1,l} = 0, \quad l = -N \ldots, N,$$

using Lemma 5.1 and differentiable K_1, there exists a $c > 0$ so that $|[\mathcal{A}'(\mathbf{u})]_{k,l}| < ch$ and then $\|\mathcal{A}'(\mathbf{u})\| < 1$ whenever h is sufficiently small. In other words, there is a $\bar{h} > 0$ so that for any $h < \bar{h}$ matrix $(\mathcal{I} - \mathcal{A}'(\mathbf{u}))$ has a uniformaly bounded inverse.
The conclusion is straightforwardly achievable by applying Theorem 5.4.1 in [1] and the above discussion. □

In the following theorem, we summarize the conclusions of theorems proved in this section.

Theorem 6.4 *Assume that* u *is an isolated solution of Eq. (4.6), Furthermore,* $\mathcal{U}_N^{SE}$ *and* $u_{N,(m)}^{SE}$ *are the solutions of Eqs. (4.9) and (6.14), respectively. Suppose that hypotheses of Theorems 6 and 6.3 are satisfied. Then there exists a positive constant* $C(m)$ *independent of* N *and dependant on* m *such that*

$$\|u - u_{N,(m)}^{SE}\| \leqslant C(m)\sqrt{N} \ln N e^{-\sqrt{\pi d\alpha N}}.$$

Proof. The conclusion is obtained by using the triangular inequality and conclusions of Theorems 6 and 6.3. □

The proof of the similar theorem goes almost in the same way as in the SE case.

Theorem 6.5 *Assume that* u *is an isolated solution of Eq. (4.6), Furthermore,* $\mathcal{U}_N^{DE}$ *and* $u_{N,(m)}^{DE}$ *are the solutions of Eqs. (4.12) and (6.14), respectively. Suppose that hypotheses of Theorems 6.2 and 6.3 are satisfied. Then there exists a positive constant* $C(m)$ *independent of* N *and dependant on* m *such that*

$$\|u - u_{N,(m)}^{DE}\| \leqslant C(m) e^{-\pi dN/ln(2dN/\alpha)}.$$

7 Illustrative examples

In this section, the theoretical results of the previous sections are used for two numerical examples. The numerical experiments are implemented in *Matlab*. In these examples, Newton's method is iterated until the accuracy 10^{-8} is obtained.
It is assumed that $\alpha = 1$. The d values are $\frac{\pi}{2}$ and $\frac{\pi}{4}$ for the SE-sinc and DE-sinc methods, respectively. The errors of the two methods for N = 5, 10, 15, 20 and 25 are reported. These tables show that increasing N the error significantly is reduced. As expected, the tables show that the convergence rate of the DE-sinc method is faster than the SE-sinc scheme.

Example 7.1 *We consider the following pantograph Volterra integral equation*

$$y(t) = g(t) + \int_0^{qt} (x + t)[u(t)]^3 dt,$$

$g(t)$ *chosen so that its exact solution is* $y(t) = t^2 - t$*. The results are shown in Table 1. The errors of the method for* $q = 0.9$ *are reported.*

Table 1: Values of $\|E\|_\infty$ for Example 7.1

N	5	10	15	20	25
SE	4.4085E-5	3.1099E-6	3.4367E-7	4.7006E-8	7.7329E-9
DE	1.3091E5	7.2505E-7	2.4544E-8	7.1655E-10	2.2621E-11

Table 2: Values of $\|E\|_\infty$ for Example 7.2

N	5	10	15	20	25
SE	2.5315E-3	4.3114E-4	9.8027E-5	2.7038E-5	8.6518E-6
DE	7.4395E-4	6.9632E-7	1.2007E-8	2.4890E-9	1.9521E-9

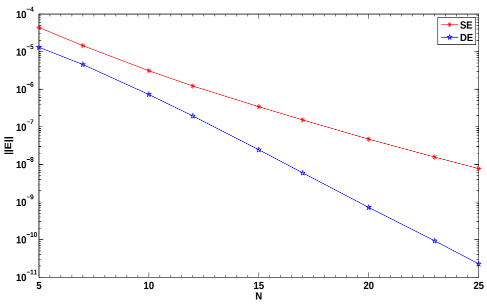

Figure 3: Example 7.1, $q = 0.9$

Example 7.2 *Consider the following equation*

$$y(t) = g(t) + \int_0^{t^r} 2ste^{-y(s)^2} ds,$$

with $g(t) = te^{-t^{2r}}$ *has the solution* $y(t) = t$. *The errors of the method for* $r = 0.01$ *are reported. Table 2 shows the numerical results.*

8 Conclusion

We propose two numerical methods based on the sinc function, the SE-sinc and DE-sinc, in order to solve the nonlinear delay integral equation (Eq. (1.1)) where θ is general function. Our methods have been shown theoretically and numerically that it is extremely accurate and achieve exponential convergence with respect to N. These two methods have some strengths and weaknesses. In comparison with each other, as the theorems show, it is understood that the SE-sinc formulas are applicable to larger classes of functions than the DE-sinc formulas, whereas the DE-sinc formulas are more efficient for well-behaved functions.

References

[1] K. E. Atkinson, W. Han, Theoretical Numerical Analysis: A Functional Analysis Framework, *Springer*, New York, (2001).

[2] H. Brunner, Iterated collocation methods for Volterra integral equations with delay arguments, *Math. Comput.* 62 (1994) 581-599.

[3] H. Brunner, Collocation Methods for Volterra Integral and Related Functional Equations, *Cambridge*, (2004).

[4] A. M. Denisov, A. Lorenzi, Existence results and regularisation techniques for severely ill-posed integrofunctional equations, *Boll. Un. Mat. Ital.* 11 (1997) 713-731.

[5] S. Haber, Two formulas for numerical indefinite integration, *Mathematics of Computation* 60 (1993) 279-296.

[6] A. Iserles, On the generalized pantograph functional differential equation, *European J. Appl. Math.* 4 (1993) 1-38.

[7] P. Linz, R. L. C. Wang, Error bounds for the solution of Volterra and delay equations, *Appl. Numer. Math.* 9 (1992) 201-207.

[8] W. Liu, J. C. Clements, On Solutions of Evolution Equations with Proportional Time Delay, *Int. J. Differ. Equ. Appl.* 4 (2002) 229-254.

[9] T. Okayama, T. Matsuo, M. Sugihara, Error estimates with explicit constants for Sinc approximation, Sinc quadrature

and Sinc indefinite integration, *Mathematical Engineering Technical Reports* 2009-01, The University of Tokyo, (2009).

[10] F. Stenger, Handbook of Sinc Numerical Methods, *Springer*, New York, (2011).

[11] F. Stenger, Numerical Methods Based on Sinc and Analytic Functions, *Springer*, New York, (1993).

[12] K. Yang, R. Zhang, Analysis of continuous collocation solutions for a kind of Volterra functional integral equations with proportional delay, *Comput. Appl. Math.* 236 (2011) 743-752.

[13] M. Zarebnia, J. Rashidinia, Convergence of the Sinc method applied to Volterra integral equations, *Appl. Math.* 5 (2010) 198-216.

[14] K. Zhang, J. Li, H. Song, Collocation methods for nonlinear convolution Volterra integral equations with multiple proportional delays, *Appl. Math. Comput.* 218 (2012) 10848-10860.

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