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Exact Solutions for the Flow of a Generalized Second Grade Fluid due to a Longitudinal Quadratic Time-dependent Shear Stress

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Abstract

The velocity field and the adequate shear stress, corresponding to the flow of a generalized second grade fluid in an annular region, due to a quadratic time-dependent shear stress, are determined by means of the Laplace and the finite Hankel transforms. The solutions that have been obtained satisfy both the governing equations and all imposed initial and boundary conditions. For $\beta \to 1$ or $\beta \to 1$ and $\alpha_1 \to 0$, the corresponding solutions for a second grade fluid, respectively, for the Newtonian fluid, performing the same motion, are obtained from general solutions. Finally, the influence of the material and fractional parameters on the shear stress as well as a comparison between models is drawn by graphical illustrations.

Keywords : Generalized second grade fluid; Velocity field; Shear stress; Exact solutions.

1 Introduction

The inadequacy of the classical Navier-Stokes theory to describe the characteristics of many rheological complex fluids has led to development of several theories of non-Newtonian fluids. Among the many models that have been used to describe the non-Newtonian behavior exhibited by these fluids, one class that has gained support from both the experimentalists and the theoreticians is that of Rivlin-Erickson fluids of second grade. Although there some criticisms on the applications of this model [1, 14], many papers have been published and a listing of some of them may be found in the literature. Furthermore, it has been shown by Walters [29] that for many types of problems in which the flow is slow enough in the viscoelastic sense, the results given by Oldroyd-B

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constitutive equations will be substantially similar to those of the second or third-order Rivlin-Erickson constitutive equations. Thus, if this is the manner of interpreting the solutions to problems, it would seem reasonable to use the second or third-order constitutive equations in carrying out the calculations. This is particularly so in view of the fact that the calculation is generally simpler. In this paper, the second grade model is used.

As early as Ting [26] provided a set of exact solutions for some flows of second grade fluids. A listing of some problems that have been solved in the next ten years can be found in [6]. During this time a lot of unsteady flows of such fluids have been studied by different authors. However, it is worth pointing out that almost all these studies deal with motion problems in which the velocity is given on the boundary. To the best of our knowledge, the first exact solutions for motions of second grade fluids due to a shear stress on the boundary are those of Bandelli and Rajagopal [2] in cylindrical domains. Recently, similar problems in cylindrical domains have been also studied in [7]-[10]. This is very important as in some problems, what is specified is the force on the boundary. It is also important to bear in mind that the "no slip" boundary condition may not be necessarily applicable to the flows of polymeric fluids that can slip or slide on the boundary. Thus, the shear stress boundary condition is particularly meaningful.

Our purpose here is to extend the results from ([2], Sect. 4) to a motion due to a time-dependent shear stress. However, for completeness, we shall solve the problem for a larger class of non-Newtonian fluids, namely second grade fluids with fractional derivatives or generalized second grade fluids (GSGF). In the last time, the fractional calculus has encountered much success in the description of viscoelasticity. Especially, the rheological constitutive equations with fractional derivative play an important role in description of the behavior of the polymer solutions and melts. In other cases, it has been shown that the constitutive equations employing fractional derivatives are linked to molecular theories [13]. Furthermore, the one-dimensional fractional derivative Maxwell model has been found very useful in modeling the linear viscoelastic response of polymers in the glass transition and the glass state [16]. For a deeper documentation on this subject, we also recommend the books [18, 20] and the recent papers [4]-[15].

2 Problem Formulation

The Cauchy stress tensor $\mathbf{T}$ for second grade fluids is related to the fluid motion in the following manner [2]

$$\mathbf{T} = -p\,\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2\,, \tag{2.1}$$

where $-p$ is the hydrostatic pressure, $\mathbf{I}$ is the unit tensor, μ is the coefficient of viscosity, α_1 and α_2 are the normal stress moduli and $\mathbf{A}_1$, $\mathbf{A}_2$ are the first two Rivlin-Ericksen tensors. Since the fluid is incompressible, it can undergo only isochoric motions, and hence

$$div\,\mathbf{v} = tr\,\mathbf{A}_1 = 0\,. \tag{2.2}$$

2.1 Governing equations

The flows to be here considered have the velocity field of the form

$$\mathbf{v} = \mathbf{v}(r,t) = v(r,t)\,\mathbf{e}_z\,, \tag{2.3}$$

where $\mathbf{e}_z$ is the unit vector along the z-axis of the cylindrical coordinate system r, θ and z. For such flows the constraint of incompressibility (2.2) is automatically satisfied and the governing equations, in the absence of a pressure gradient in the flow direction and neglecting the body forces, are [2, 12, 30]

$$\frac{\partial v(r,t)}{\partial t} = (\nu + \alpha\frac{\partial}{\partial t})\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)v(r,t),\ \ \tau(r,t) = (\mu + \alpha_1\frac{\partial}{\partial t})\frac{\partial v(r,t)}{\partial r}\,, \tag{2.4}$$

where $\nu = \mu/\rho$ is the kinematic viscosity (ρ being the constant density of the fluid), $\alpha = \alpha_1/\rho$ and $\tau(r,t) = S_{rz}(r,t)$ is the non-trivial shear-stress.

Generally, the governing equations corresponding to such a motion of a GSGF, namely (cf. [31], Eqs. (4) and (7)) or ([26], Eq. (8)) with $\lambda = 0$)

$$\frac{\partial v(r,t)}{\partial t} = (\nu + \alpha D_t^\beta)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)v(r,t),\quad \tau(r,t) = (\mu + \alpha_1 D_t^\beta)\frac{\partial v(r,t)}{\partial r}\,, \tag{2.5}$$

are obtained from Eq. (2.4) by replacing the inner time derivatives by the Riemann-Liouville fractional operator [24, 25]

$$D_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^\beta}\,d\tau,\quad 0 \le \beta < 1\,, \tag{2.6}$$

where $\Gamma(.)$ is the Gamma function. When $\beta \to 1$, Eqs. (2.5) reduce to Eq. (2.4) because $D_t^\beta f \to \frac{df}{dt}$. Of course, the new material constants α and α_1 into Eqs. (2.5) (for simplicity, we kept the same notations) are also going to those from Eq. (2.4) if $\beta \to 1$.

In this note, we are interested into the motion of a GSGF whose governing equations are given by Eqs. (2.5). More exactly, we shall determine the velocity field $v(r,t)$ and the shear stress $\tau(r,t)$ corresponding to the motion between two infinite coaxial circular cylinders, one of them applying a shear stress of the form ft^2 to the fluid. Similar solutions for the motion of generalized Oldroyd-B fluids due to an infinite cylinder that applies a constant longitudinal/rotational shear stress to the fluid have been established in [27, 28].

2.2 Axial Couette flow between two cylinders

Suppose that an incompressible generalized second grade fluid at rest is situated in the annular region between two infinite coaxial circular cylinders of radii R_1 and $R_2 (> R_1)$. At time $t = 0^+$ the inner cylinder is pulled with a quadratic time-dependent shear stress ft^2 along its axis, while the outer one is held fixed. Due to the shear, the fluid between cylinders is gradually moved, its velocity being of the form (2.3). The governing equations are given by Eqs. (2.5) and the appropriate initial and boundary conditions are (cf. [2], Eqs. (4.2)-(4.4)))

$$v(r,0) = 0\,;\quad r \in (R_1, R_2], \tag{2.7}$$

$$\tau(R_1,t) = (\mu + \alpha_1 D_t^\beta)\frac{\partial v(r,t)}{\partial r}|_{r=R_1} = ft^2,\ \ v(R_2,t) = 0;\ t > 0\,, \tag{2.8}$$

where f is a negative constant [9, 5].

The partial differential equation (2.5), with the initial and boundary conditions (2.7) and (2.8), can be solved in principle by several methods. The integral transforms technique represents a systematic, efficient and powerful tool. In the following we shall use the

Laplace transform to eliminate the time variable and the finite Hankel transform for the spatial variable. However, in order to avoid the burdensome calculations of residues and contour integrals, as well as in [7, 24, 27, 28, 31], we shall apply the discrete inverse Laplace transform method.

3 Analytical solutions

3.1 Velocity solution

By applying the Laplace transform to the first equation of (2.5) and Eq. (2.8), we find that

$$q\overline{v}(r,q) = (\nu + \alpha q^{\beta})\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\overline{v}(r,q), \tag{3.9}$$

$$\overline{\tau}(R_1,q) = (\mu + \alpha_1 q^{\beta})\frac{\partial \overline{v}(r,q)}{\partial r}|_{r=R_1} = \frac{2f}{q^3}, \qquad \overline{v}(R_2,q) = 0\,, \tag{3.10}$$

where $\overline{v}(r,q)$ and $\overline{\tau}(R_1,q)$ are the Laplace transforms of the functions $v(r,t)$ and $\tau(R_1,t)$, respectively. We denote by [8]

$$\overline{v}_H(r_n,q) = \int_{R_1}^{R_2} r\,\overline{v}(r,q)B(rr_n)\,dr, \tag{3.11}$$

the finite Hankel transform of the function $\overline{v}(r,q)$, where

$$B(rr_n) = J_0(rr_n)Y_1(R_1r_n) - J_1(R_1r_n)Y_0(rr_n)\,, \tag{3.12}$$

r_n being the positive roots of the equation $B(R_2r) = 0$ and $J_p(.)$, $Y_p(.)$ are the Bessel functions of the first and second kind of order p. Using the first equation of (3.10) and Eq. (3.12) and the known relation

$$J_0(z)Y_1(z) - J_1(z)Y_0(z) = -\frac{2}{\pi z}\,, \tag{3.13}$$

we can prove that

$$\int_{R_1}^{R_2} r\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\overline{v}(r,q)B(rr_n)\,dr = -r_n^2\,\overline{v}_H(r_n,q) + \frac{2}{\pi r_n}\,\frac{\partial \overline{v}(r,q)}{\partial r}|_{r=R_1}\,. \tag{3.14}$$

From Eqs. (3.9), (3.10) and (3.14), we find that

$$\begin{aligned}\overline{v}_H(r_n,q) &= \tfrac{4f}{\pi r_n}\,\tfrac{1}{q^3}\,\tfrac{1}{\rho q + \alpha_1 q^{\beta} r_n^2 + \mu\, r_n^2} \\ &= \overline{v}_{1H}(r_n,q) + \overline{v}_{2H}(r_n,q),\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}\overline{v}_{1H}(r_n,q) &= \tfrac{4f}{\pi r_n^3}\tfrac{1}{q^3(\mu+\alpha_1 q^{\beta})}, \\ \overline{v}_{2H}(r_n,q) &= -\tfrac{4f}{\pi r_n^3}\tfrac{1}{(\mu+\alpha_1 q^{\beta})}\tfrac{1}{q^2(q+\alpha q^{\beta} r_n^2+\nu r_n^2)}.\end{aligned} \tag{3.16}$$

Applying the inverse Hankel transform to Eqs. (3.16) and using Eqs. (A1) from the Appendix A, we get

$$\begin{aligned}\overline{v}_1(r,q) &= 2R_1 f \ln\left(\tfrac{r}{R_2}\right)\tfrac{1}{q^3(\mu+\alpha_1 q^\beta)},\\ \overline{v}_2(r,q) &= \tfrac{\pi^2}{2}\sum_{n=1}^{\infty}\tfrac{r_n^2 J_0^2(R_2 r_n) B(r r_n)}{J_1^2(R_1 r_n)-J_0^2(R_2 r_n)}\,\overline{v}_{2H}(r_n,q).\end{aligned} \tag{3.17}$$

If we denote by

$$H(q) = \frac{1}{q^2(\mu+\alpha_1 q^\beta)} = \frac{1}{\alpha_1}\frac{q^{-2}}{q^\beta+\frac{\mu}{\alpha_1}}, \tag{3.18}$$

then its inverse Laplace transform is

$$\begin{aligned} h(t) &= L^{-1}[H(q)]\\ &= \tfrac{1}{\alpha_1} G_{\beta,-2,1}\left(-\tfrac{\mu}{\alpha_1}, t\right)\\ &= \tfrac{1}{\alpha_1}\sum_{j=0}^{\infty}\left(-\tfrac{\mu}{\alpha_1}\right)^j \tfrac{t^{(j+1)\beta+1}}{\Gamma[(j+1)\beta+2]},\end{aligned} \tag{3.19}$$

where ([19], Eqs. (97) and (101))

$$\begin{aligned} &G_{a,b,c}(d\,,t) = L^{-1}\left\{\tfrac{q^b}{(q^a-d)^c}\right\} = \sum_{j=0}^{\infty}\tfrac{d^j\,\Gamma(c+j)}{\Gamma(c)\Gamma(j+1)}\tfrac{t^{(c+j)a-b-1}}{\Gamma[(c+j)a-b]},\\ &\mathrm{Re}(ac-b)>0,\\ &|\tfrac{d}{q^a}|<1.\end{aligned} \tag{3.20}$$

By taking the inverse Laplace transform of the first equation of (3.17) and using Eq. (3.19) as well as the convolution theorem, we find that

$$\begin{aligned} v_1(r,t) &= 2R_1 f \ln\left(\tfrac{r}{R_2}\right) L^{-1}\left[\tfrac{1}{q}\,H(q)\right]\\ &= 2R_1 f \ln\left(\tfrac{r}{R_2}\right)\int_0^t h(s)ds\\ &= \tfrac{2R_1 f}{\alpha_1}\ln\left(\tfrac{r}{R_2}\right)\sum_{k=0}^{\infty}\left(-\tfrac{\mu}{\alpha_1}\right)^k\tfrac{t^{(k+1)\beta+2}}{\Gamma[(k+1)\beta+3]}\\ &= \tfrac{2R_1 f}{\alpha_1}\ln\left(\tfrac{r}{R_2}\right) G_{\beta,-3,1}\left(-\tfrac{\mu}{\alpha_1}\,,t\right).\end{aligned} \tag{3.21}$$

In order to determine the inverse Laplace transform of the function $\overline{v}_2(r,q)$, we rewrite the function $\overline{v}_{2H}(r_n,q)$ in the form

$$\begin{aligned}\overline{v}_{2H}(r_n,q) &= -\tfrac{4f}{\pi r_n^3}\,H(q).H_1(r_n,q),\\ H_1(r_n,q) &= \tfrac{1}{q+\alpha q^\beta r_n^2+\nu r_n^2}\,.\end{aligned} \tag{3.22}$$

Using the following expansion of the function $H_1(r_n,q)$

$$\begin{aligned} H_1(r_n,q) &= \tfrac{q^{-\beta}}{(q^{1-\beta}+\alpha r_n^2)+\nu r_n^2 q^{-\beta}}\\ &= \sum_{k=0}^{\infty}\tfrac{(-\nu r_n^2)^k\,q^{-\beta(k+1)}}{(q^{1-\beta}+\alpha r_n^2)^{k+1}}\,,\end{aligned} \tag{3.23}$$

as well as the formula (3.20), we get

$$\begin{aligned} h_1(r_n,t) &= L^{-1}[H_1(r_n,q)] \\ &= \sum_{k=o}^{\infty}(-\nu r_n^2)^k\, G_{1-\beta,\,-\beta(k+1),\,k+1}(-\alpha r_n^2,t)\,. \end{aligned} \tag{3.24}$$

Applying the inverse Laplace transform to the second equation of (3.17) and using Eqs. (3.19), (3.22), (3.24) and the property

$$\begin{aligned} L^{-1}[H(q)H_1(r_n,q)] &= h(t) * h_1(r_n,t) \\ &= \int_0^t h(t-s)h_1(r_n,s)ds, \end{aligned}$$

we find that

$$\begin{aligned} v_2(r,t) &= L^{-1}[\overline{v}_2(r,q)] \\ &= -\tfrac{2\pi f}{\alpha_1}\sum_{n=1}^{\infty}\tfrac{J_0^2(R_2r_n)B(rr_n)}{r_n[J_1^2(R_1r_n)-J_0^2(R_2r_n)]} \\ &\times \sum_{k=o}^{\infty}(-\nu r_n^2)^k\ \int_0^t G_{\beta,\,-2,\,1}\big(-\tfrac{\mu}{\alpha_1},t-s\big)G_{1-\beta,\,-\beta(k+1),\,k+1}(-\alpha r_n^2,s)ds\,. \end{aligned} \tag{3.25}$$

Consequently, the velocity field $v(r,t)$ is given by the relation

$$\begin{aligned} v(r,t) &= \tfrac{2R_1f}{\alpha_1}\ln(\tfrac{r}{R_2})G_{\beta,\,-3,\,1}\big(-\tfrac{\mu}{\alpha_1},t\big) - \tfrac{2\pi f}{\alpha_1}\sum_{n=1}^{\infty}\tfrac{J_0^2(R_2r_n)B(rr_n)}{r_n[J_1^2(R_1r_n)-J_0^2(R_2r_n)]} \\ &\times \sum_{k=o}^{\infty}(-\nu r_n^2)^k\ \int_0^t G_{\beta,\,-2,\,1}\big(-\tfrac{\mu}{\alpha_1},t-s\big)G_{1-\beta,\,-\beta(k+1),\,k+1}(-\alpha r_n^2,s)ds\,. \end{aligned} \tag{3.26}$$

Of course, in view of the known relation

$$D_t^{\beta}(t^a) = \frac{\Gamma(a+1)}{\Gamma(a-\beta+1)}t^{a-\beta}, \qquad 0 \le \beta < 1,$$

it is easy to show that $v(r,t)$ satisfies the second boundary condition (2.8). Indeed, using (3.26), we have

$$\begin{aligned} (\mu+\alpha_1 D_t^{\beta})\tfrac{\partial v(r,t)}{\partial r}|_{r=R_1} &= -2f\sum_{j=0}^{\infty}\left(-\tfrac{\mu}{\alpha_1}\right)^{j+1}\tfrac{t^{(j+1)\beta+2}}{\Gamma[(j+1)\beta+3]} \\ &+2f\sum_{j=0}^{\infty}\left(-\tfrac{\mu}{\alpha_1}\right)^{j}\tfrac{t^{j\beta+2}}{\Gamma[j\beta+3]} \\ &= -2f\sum_{k=1}^{\infty}\left(-\tfrac{\mu}{\alpha_1}\right)^{k}\tfrac{t^{k\beta+2}}{\Gamma[k\beta+3]} \\ &+2f\sum_{j=0}^{\infty}\left(-\tfrac{\mu}{\alpha_1}\right)^{j}\tfrac{t^{j\beta+2}}{\Gamma[j\beta+3]} \\ &= ft^2. \end{aligned}$$

A simpler but equivalent expression for the velocity field $v(r,t)$, can be also obtained by rewriting Eq. (3.15) under the form

$$\overline{v}_H(r_n,q) = \frac{4f}{\mu\pi r_n^3}\,\frac{1}{q^3} - \frac{4f}{\mu\pi r_n^3}\,\frac{q^{-2}+\alpha q^{\beta-3}r_n^2}{q+\alpha q^{\beta}r_n^2+\nu\, r_n^2}. \tag{3.27}$$

Indeed, applying the inverse Hankel transform to Eq. (3.27) and following the same way as before, we find for velocity the simpler expression

$$\begin{aligned} v(r,t) &= \tfrac{R_1 f}{\mu}\ln\left(\tfrac{r}{R_2}\right)t^2 - \tfrac{2\pi f}{\mu}\sum_{n=1}^{\infty}\tfrac{J_0^2(R_2 r_n)B(rr_n)}{r_n\left[J_1^2(R_1 r_n)-J_0^2(R_2 r_n)\right]}\sum_{k=0}^{\infty}\left(-\nu r_n^2\right)^k \\ &\times\left[G_{1-\beta,-\beta k-\beta-2,k+1}\left(-\alpha r_n^2,t\right)+\alpha r_n^2 G_{1-\beta,-\beta k-3,k+1}\left(-\alpha r_n^2,t\right)\right]. \end{aligned} \tag{3.28}$$

3.2 Shear stress solution

By applying the Laplace transform to the second equation of (2.5), we find that

$$\begin{aligned} \overline{\tau}(r,q) &= (\mu+\alpha_1 q^\beta)\tfrac{\partial\overline{v}(r,q)}{\partial r} \\ &= (\mu+\alpha_1 q^\beta)\left[\tfrac{\partial\overline{v}_1(r,q)}{\partial r}+\tfrac{\partial\overline{v}_2(r,q)}{\partial r}\right]. \end{aligned} \tag{3.29}$$

In view of Eqs. (3.16) and (3.17), it results that

$$\overline{\tau}(r,q)=\frac{2R_1 f}{r}\frac{1}{q^3}+2\pi f\sum_{n=1}^{\infty}\frac{J_0^2(R_2 r_n)B_1(rr_n)}{J_1^2(R_1 r_n)-J_0^2(R_2 r_n)}\frac{1}{q^2(q+\alpha q^\beta r_n^2+\nu r_n^2)}, \tag{3.30}$$

where

$$B_1(rr_n)=J_1(rr_n)Y_1(R_1 r_n)-J_1(R_1 r_n)Y_1(rr_n)$$

Now taking the inverse Laplace transform of both sides of Eq. (3.30) and using (3.23), we find that

$$\begin{aligned} \tau(r,t) &= \tfrac{R_1 f t^2}{r}+2\pi f\sum_{n=1}^{\infty}\tfrac{J_0^2(R_2 r_n)B_1(rr_n)}{J_1^2(R_1 r_n)-J_0^2(R_2 r_n)} \\ &\times\sum_{k=o}^{\infty}(-\nu r_n^2)^k\, G_{1-\beta,-\beta k-\beta-2,k+1}(-\alpha r_n^2,t). \end{aligned} \tag{3.31}$$

4 The special case $\beta\to 1$

By making $\beta\to 1$ into Eqs. (3.26) and (3.31), we obtain the similar solutions

$$\begin{aligned} v_{SG}(r,t) &= \tfrac{2R_1 f}{\alpha_1}\ln\left(\tfrac{r}{R_2}\right)G_{1,-3,1}\left(-\tfrac{\mu}{\alpha_1},t\right)-\tfrac{2\pi f}{\alpha_1}\sum_{n=1}^{\infty}\tfrac{J_0^2(R_2 r_n)B(rr_n)}{r_n\left[J_1^2(R_1 r_n)-J_0^2(R_2 r_n)\right]} \\ &\times\sum_{k=o}^{\infty}(-\nu r_n^2)^k\int_0^t G_{1,-2,1}\left(-\tfrac{\mu}{\alpha_1},t-s\right)G_{0,-k-1,k+1}(-\alpha r_n^2,s)ds \end{aligned} \tag{4.32}$$

and

$$\begin{aligned} \tau_{SG}(r,t) &= \tfrac{R_1 f t^2}{r}+2\pi f\sum_{n=1}^{\infty}\tfrac{J_0^2(R_2 r_n)B_1(rr_n)}{J_1^2(R_1 r_n)-J_0^2(R_2 r_n)} \\ &\sum_{k=o}^{\infty}(-\nu r_n^2)^k\, G_{0,-k-3,k+1}(-\alpha r_n^2,t), \end{aligned} \tag{4.33}$$

corresponding to a second grade fluid performing the same motion. These solutions can be also simplified to give (see also Eqs. (A2) – (A6) from the Appendix *A*)

$$\begin{aligned} v_{SG}(r,t) &= \tfrac{2R_1 f}{\mu}\ln(\tfrac{r}{R_2})\left[\left(\tfrac{\alpha_1}{\mu}\right)^2\left\{1-exp\left(-\tfrac{\mu t}{\alpha_1}\right)\right\}-\tfrac{\alpha_1 t}{\mu}+\tfrac{t^2}{2}\right] \\ &+\tfrac{2\pi f\alpha_1}{\mu^2}\sum_{n=1}^{\infty}\tfrac{J_0^2(R_2 r_n)B(rr_n)}{r_n\left[J_1^2(R_1 r_n)-J_0^2(R_2 r_n)\right]} \\ &\times\tfrac{1}{1+\alpha r_n^2}\int_0^t\left[1-\tfrac{\mu}{\alpha_1}(t-s)-exp\left\{-\tfrac{\mu}{\alpha_1}(t-s)\right\}\right]exp\left(-\tfrac{\nu r_n^2 s}{1+\alpha r_n^2}\right)ds, \end{aligned} \tag{4.34}$$

and

$$\begin{aligned}\tau_{SG}(r,t) &= \frac{R_1 f t^2}{r} + \frac{2\pi f}{\nu^2}\sum_{n=1}^{\infty}\frac{J_0^2(R_2 r_n)B_1(r r_n)}{r_n^2[J_1^2(R_1 r_n)-J_0^2(R_2 r_n)]} \\ &\times\left[\frac{1+\alpha r_n^2}{r_n^2}\left\{exp\left(\frac{-\nu r_n^2 t}{1+\alpha r_n^2}\right)-1\right\}+\nu t\right].\end{aligned} \tag{4.35}$$

The above expressions for the velocity $v_{SG}(r,t)$ and the shear stress $\tau_{SG}(r,t)$ can be also written in the simpler forms as

$$\begin{aligned}v_{SG}(r,t) &= \frac{R_1 f}{\mu}\ln\left(\frac{r}{R_2}\right)\left[\left(t-\frac{\alpha_1}{\mu}\right)^2+\left(\frac{\alpha_1}{\mu}\right)^2\right] \\ &-\frac{2\pi f}{\mu\nu}\sum_{n=1}^{\infty}\frac{J_0^2(R_2 r_n)B(r r_n)}{r_n^3[J_1^2(R_1 r_n)-J_0^2(R_2 r_n)]} \\ &\times\left[t-\frac{\alpha_1}{\mu}-\frac{1+\alpha r_n^2}{\nu r_n^2}\left\{1-(1+\alpha r_n^2)exp\left(\frac{-\nu r_n^2 t}{1+\alpha r_n^2}\right)\right\}\right],\end{aligned} \tag{4.36}$$

and

$$\begin{aligned}\tau_{SG}(r,t) &= \frac{R_1 f t^2}{r} + \frac{2\pi f}{\nu}\sum_{n=1}^{\infty}\frac{J_0^2(R_2 r_n)B_1(r r_n)}{r_n^2[J_1^2(R_1 r_n)-J_0^2(R_2 r_n)]} \\ &\times\left[t-\frac{1+\alpha r_n^2}{\nu r_n^2}\left\{1-exp\left(\frac{-\nu r_n^2 t}{1+\alpha r_n^2}\right)\right\}\right].\end{aligned} \tag{4.37}$$

Making $\alpha_1 \to 0$ and then $\alpha \to 0$ into Eqs. (4.36) and (4.37), we obtain the velocity field

$$\begin{aligned}v_N(r,t) &= \frac{R_1 f t^2}{\mu}\ln\left(\frac{r}{R_2}\right) - \frac{2\pi f}{\mu\nu}\sum_{n=1}^{\infty}\frac{J_0^2(R_2 r_n)B(r r_n)}{r_n^3[J_1^2(R_1 r_n)-J_0^2(R_2 r_n)]} \\ &\times\left[t-\frac{1}{\nu r_n^2}\left\{1-exp\left(-\nu r_n^2 t\right)\right\}\right]\end{aligned} \tag{4.38}$$

and the associated shear stress

$$\begin{aligned}\tau_N(r,t) &= \frac{R_1 f t^2}{r} + \frac{2\pi f}{\nu}\sum_{n=1}^{\infty}\frac{J_0^2(R_2 r_n)B_1(r r_n)}{r_n^2[J_1^2(R_1 r_n)-J_0^2(R_2 r_n)]} \\ &\times\left[t-\frac{1}{\nu r_n^2}\left\{1-exp\left(-\nu r_n^2 t\right)\right\}\right],\end{aligned} \tag{4.39}$$

corresponding to a Newtonian fluid performing the same motion. Of course, by making $\beta \to 1$ in Eq. (3.28), we attain to the same expression (4.36) for the velocity field corresponding to a second grade fluid. Direct computations show that the expression of $v_{SG}(r,t)$, given by Eq. (4.36), is wholly in accordance with the known result ([2], Eq. (4.34)) corresponding to a constant shear on the boundary.

5 Numerical results and discussion

In this paper the velocity field and the shear stress corresponding to the motion of a generalized second grade fluid due to a longitudinal quadratic time-dependent shear stress have been determined using Laplace and finite Hankel transforms. The solutions that have been obtained, presented under integral and series form in terms of the generalized $G_{a,b,c}(\cdot,\, t)$ functions, satisfy all imposed initial and boundary conditions. They can be easily reduced to give the similar solutions for second grade and Newtonian fluids, performing the same motion. These last solutions, as it results from Eqs. (4.36)-(4.39),

are presented as a sum between the large time and transient solutions. The large time solutions corresponding to second grade fluids, for instance, are

$$\begin{aligned} v_{LSG}(r,t) &= \frac{R_1 f}{\mu}\ln\left(\frac{r}{R_2}\right)\left[\left(t-\frac{\alpha_1}{\mu}\right)^2+\left(\frac{\alpha_1}{\mu}\right)^2\right] \\ &\quad -\frac{2\pi f}{\mu\nu}\sum_{n=1}^{\infty}\frac{J_0^2(R_2 r_n)B(r r_n)}{r_n^3[J_1^2(R_1 r_n)-J_0^2(R_2 r_n)]}\left[t-2\frac{\alpha_1}{\mu}-\frac{1}{\nu r_n^2}\right], \end{aligned} \tag{5.40}$$

and

$$\begin{aligned} \tau_{LSG}(r,t) &= \frac{R_1 f t^2}{r}+\frac{2\pi f}{\nu}\sum_{n=1}^{\infty}\frac{J_0^2(R_2 r_n)B_1(r r_n)}{r_n^2[J_1^2(R_1 r_n)-J_0^2(R_2 r_n)]} \\ &\quad \times\left[t-\frac{\alpha_1}{\mu}-\frac{1}{\nu r_n^2}\right]. \end{aligned} \tag{5.41}$$

For $\alpha \to 0$ they tend to the Newtonian large time solutions $v_{LN}(r,t)$ and $\tau_{LN}(r,t)$.

Now, in order to reveal some relevant physical aspects of the obtained results, the diagrams of the shear stress $\tau(r,t)$ are depicted against r for different values of t, α_1 and of the fractional parameter β. In Fig. 1 the diagrams of the shear stress are presented at three different times. The shear stress, in absolute value, is an increasing function of t. Fig. 2 and Fig. 3 show the influence of the material constant α_1 and the fractional parameter β on the shear stress $\tau(r,t)$. Their effect, as it was to be expected, is opposite. On the first part of the flow domain, near the moving cylinder, the shear stress is a decreasing function with respect to α_1 and an increasing one of β. In Fig. 4, for comparison, the diagrams of the shear stress corresponding to the three models (Newtonian, second grade and generalized second grade) are together depicted for the same values of the common parameters and the time t. In the neighborhood of the inner cylinder, the shear stress corresponding to a GSGF is the biggest and that for a Newtonian fluid is the lowest. The units of the material constants into Figs. 1-4 are SI units and the roots r_n have been approximated by $(2n-1)\pi/[2(R_2-R_1)]$.

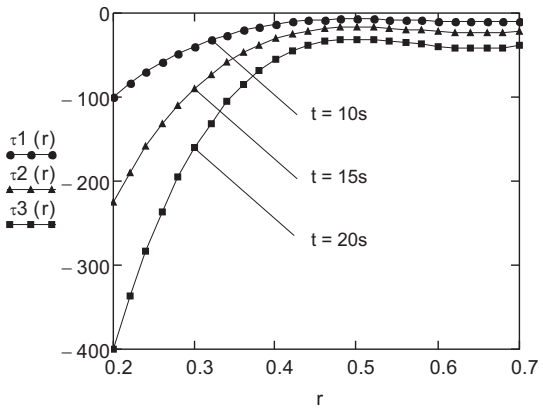

Fig. 1. Profiles of the shear stress $\tau(r,t)$ given by Eq. (3.31) - curves $\tau 1(r)$, $\tau 2(r)$, $\tau 3(r)$ for $f=-1$, $R_1=0.2$, $R_2=0.7$, $\nu=0.0001637$, $\rho=880$, $\alpha_1=5$, $\beta=0.8$ and different values of t.

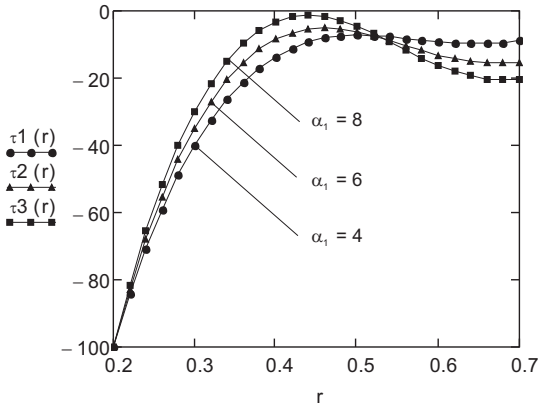

Fig. 2. Profiles of the shear stress $\tau(r,t)$ given by Eq. (3.31) - curves $\tau 1(r)$, $\tau 2(r)$, $\tau 3(r)$ for $f=-1$, $R_1=0.2$, $R_2=0.7$, $\nu=0.0001637$, $\rho=880$, $t=10s$, $\beta=0.8$ and different values of α_1.

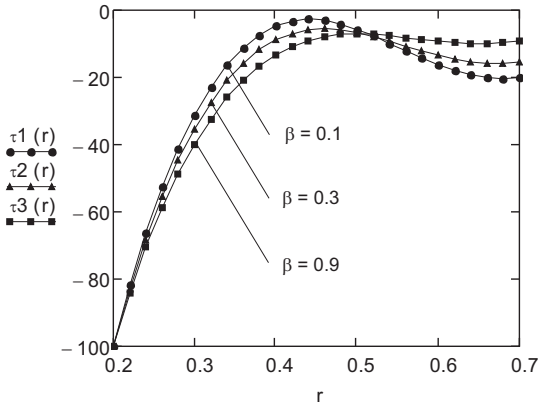

Fig. 3. Profiles of the shear stress $\tau(r,t)$ given by Eq. (3.31) - curves $\tau 1(r)$, $\tau 2(r)$, $\tau 3(r)$ for $f=-1$, $R_1=0.2$, $R_2=0.7$, $\nu=0.0001637$, $\rho=880$, $\alpha_1=5$, $t=10s$ and different values of β.

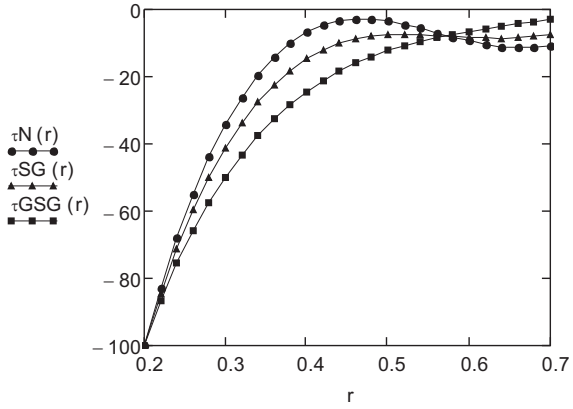

Fig. 4. Profiles of the shear stress $\tau(r,t)$ given by Eq. (3.31) - curves $\tau 1(r)$, $\tau 2(r)$, $\tau 3(r)$ for $f=-1$, $R_1=0.2$, $R_2=0.7$, $\nu=0.0001637$, $\rho=880$, $\alpha_1=5$, $t=10s$ and $\beta=0.8$.

6 Conclusions

Exact solutions for the motion of a generalized second grade fluid between two infinite coaxial circular cylinders are established by means of integral transforms. The motion is produced by the inner cylinder that applies a longitudinal quadratic time-dependent shear to the fluid.

The limiting solutions corresponding to ordinary second grade and Newtonian fluids are presented as a sum of large time and transient solutions. They describe the motion of the fluid some time after its initiation. After that time, when the transients disappear, they tend to large time solutions.

The shear stress, in absolute value, is an increasing function of t and a decreasing one with respect to the spatial variable.

In a large neighbourhood of the inner cylinder, that produce the motion, the shear stress decreases for increasing α_1. The effect of the fractional parameter β on the shear stress is opposite to that of the material parameter α_1.

In the neighbourhood of the inner cylinder the shear stress is the lowest for Newtonian fluids and the highest for the generalized fluid. Consequently, as it was to be expected, the Newtonian fluid is the swiftest and the generalized second grade fluid is the slowest.

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Appendix A

$$v(r,t) = \frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{r_n^2 J_0^2(R_2 r_n) B(r r_n)}{J_1^2(R_1 r_n) - J_0^2(R_2 r_n)}\, v_H(r_n, t),$$

$$R_1 \ln\left(\frac{r}{R_2}\right) = \pi \sum_{n=1}^{\infty} \frac{J_0^2(R_2 r_n) B(r r_n)}{r_n [J_1^2(R_1 r_n) - J_0^2(R_2 r_n)]}, \tag{A1}$$

$$G_{1,-2,1}\left(-\frac{\mu}{\alpha_1}, t\right) = \left(\frac{\alpha_1}{\mu}\right)^2 \left[exp\left(-\frac{\mu\, t}{\alpha_1}\right) + \frac{\mu t}{\alpha_1} - 1\right], \tag{A2}$$

$$G_{1,-3,1}\left(-\frac{\mu}{\alpha_1}, t\right) = \left(-\frac{\alpha_1}{\mu}\right)^3 \left[exp\left(-\frac{\mu\, t}{\alpha_1}\right) - \left(\frac{\mu}{\alpha_1}\right)^2 \frac{t^2}{2} + \frac{\mu t}{\alpha_1} - 1\right], \tag{A3}$$

$$\sum_{k=o}^{\infty} (-\nu r_n^2)^k\, G_{0,-k-1,k+1}(-\alpha r_n^2, t) = \frac{1}{1+\alpha r_n^2}\, exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right), \tag{A4}$$

$$\sum_{k=o}^{\infty} (-\nu r_n^2)^k\, G_{0,-k-2,k+1}(-\alpha r_n^2, t) = \frac{1}{\nu r_n^2}\left[1 - exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)\right], \tag{A5}$$

$$\sum_{k=o}^{\infty} (-\nu r_n^2)^k\, G_{0,-k-3,k+1}(-\alpha r_n^2, t) = \frac{t}{\nu r_n^2} - \frac{1+\alpha r_n^2}{(\nu r_n^2)^2}\left[1 - exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)\right]. \tag{A6}$$

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