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Research Article

Chaotic convection in couple stress liquid saturated porous layer

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Abstract

In this paper, we have investigated the chaotic behavior of thermal convection in couple stress liquid saturated porous layer subject to gravity, heated from below and cooled from above, based on theory of dynamical system. A low dimensional Lorenz- like model is obtained by using Galerkin-truncation approximation. We found that there is proportional relation between scaled couple stress parameter and rescaled Rayleigh number. We analyzed that increase in level of couple stress parameter increases the level of chaos.

Keywords : Chaotic behavior; Couple stress liquid; Porous media; Lorenz equations.

1 Introduction

Chaotic convection in fluid saturated porous medium has great interest due to its wide range of applications in laboratory and nature. In the laboratory, chaos is used to design electric circuits and mechanical devices. In the nature, chaos theory can be used in the dynamics of satellites in the solar system, thermal insulation and geothermal energy utilization.

The problem of couple stress liquid in porous medium has not much attention till now. The investigations of non-Newtonian fluids with suspended particles in the field of modern technology and industries are of great importance. Stokes [24] has proposed couple stress theory in the simplest polar fluid theory. The main feature of this model is that momentum equation is similar to the Navier-Stokes equation, thereby it gives us facility to compare with the result for the classical non-polar case. Applications of such fluids are occur in industry such as extraction of crude oil from petroleum industry, cooling of metallic plate in a bath, solidification of liquid crystals and exotic lubrication. In the category of non-Newtonian fluids, couple stress fluids have different features such as polar effects, having large viscosity.

There are many researchers who have investigated the effect of couple stress liquid in porous medium. Vadasz and Olek [27] found that for a low Prandtl number, solitary limit cycle obtained by subcritical hopf bifurcation may be associated with a homoclinic explosion. Also, Vadasz [29] suggested an explanation by analytical method for the appearance of this solitary limit cycle. Jawdat and Hasim [9] investigated the chaotic convection in a porous medium for low Prandtl number with internal heat generation. They found that the amount of internal heat genera-

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tion is inversely proportional to scaled Rayleigh number. Mahmud and Hasim [16] investigated the effect of magnetic field on chaotic convection in fluid layer. They found that transition from chaotic convection to steady convection occurs by a subcritical hopf bifurcation producing a homoclinic explosion may be limit cycle as Hartmann number increases. The generalized Lorenz models and their routes to chaos by energy-conserving horizontal mode truncations are investigated by Roy and Musielek [26]. They observed that in horizontal modes, 5D system is the lowest order generalized Lorenz model. Vadasz and Olek [28] investigated the route to chaos occurs by a period doubling sequence of bifurcations when the Prandtl number is moderate. Sheu[21] reported that the route to chaos and its applications of thermal non-equilibrium model tends to stabilize steady convection. Sheu et. al. [22] investigated the onset of chaos through the use of an Oldrydian-fluid. The effect of feedback control on chaos in porous media has been studied by Mahmud and Hasim[15]. They observed that amount of feedback control is proportional to scaled Rayleigh number. Magyari [13] demonstrated that the structure of feedback control system proposed by Mahmud and Hasim[15] does not alter the original uncontrolled system but its effect is in changing the initial condition of the system. Gupta and Singh [5] reported the effect of anisotropic parameters on chaotic convection. They founded a proportional relation between scaled Rayleigh number and scaled anisotropic parameters. Gupta and Bhadauria [6] investigated the double diffusive convection in a couple stress liquid saturated porous layer with soret effect using thermal non-equilibrium model.
From the above paragraph, we observed that a huge amount of analysis on chaotic behavior has been discussed on the onset of convection for various flow models. However, not much work has been done for couple stress liquid to analyze its chaotic behavior. Therefore, in this paper, we have intend to study, the effect of couple stress parameter on Darcy convection by dynamical system approach. for $C = 0.1$, $R_{c2} = 26.4$ for $C = 0.2$, $R_{c2} = 31.25$ for $C = 0.5$,$R_{c2} = 42.85714286$ for $C = 1.0$ and $R = 57.56578947$ for $C = 1.5$

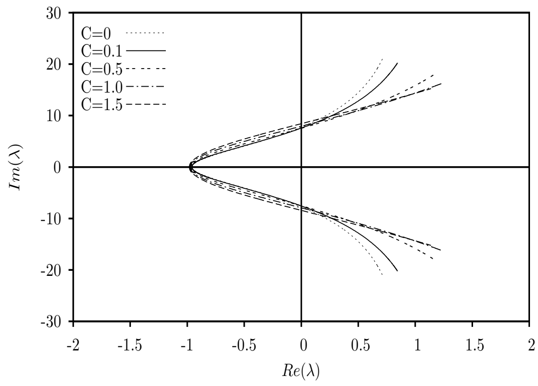

Figure 1: Evolution of the complex eigenvalues with increasing Rayleigh number, for $\sigma = 5$, $\gamma = 0.5$,$C = 0.1, 0.5, 1.0$

2 Mathematical formulation of Problem

We consider a couple stress fluid saturated in horizontal porous layer of depth H_* and width L_* with stress free boundaries, which is heated from below and cooled from above. The x-axis is taken along the lower boundary, and the $Z-$axis vertically upward. The lower surface is held at temperature $T_0+\triangle T$, while the upper surface is at T_0. A uniform adverse temperature gradient $\triangle T/H_*$ is maintained between the lower and upper surfaces. The extended Darcy model that includes the time derivative term and couple stress term is employed for the momentum equation. The continuity and momentum equations governing the motion of an incompressible couple stress fluid in the absence of body couple are given by

$$\nabla.\mathbf{q} = 0 \tag{2.1}$$

$$\frac{1}{\varepsilon}\frac{\partial \mathbf{q}}{\partial t} = -\frac{1}{\rho_0}\nabla p + \frac{\rho}{\rho_0}\mathbf{g} - \frac{1}{K}\frac{1}{\rho_0}(\mu - \mu_c\nabla^2)\mathbf{q} \tag{2.2}$$

$$\varepsilon(\rho_0 c)\frac{\partial T}{\partial t} + (\rho_0 c)(\mathbf{q}.\nabla)T = \varepsilon\kappa\nabla^2 T \tag{2.3}$$

$$\rho = \rho_0[1 - \alpha_T(T - T_0)] \tag{2.4}$$

The boundary conditions are given by

$$\begin{aligned} T = T_0 + \triangle T \quad &\text{at} \quad z = 0 \\ \text{and} \quad T = T_0 \quad &\text{at} \quad z = H_* \end{aligned} \tag{2.5}$$

where $\mathbf{q}$ is the velocity of couple stress fluid in porous medium, ε the porosity, K the permeability of the medium, p the fluid pressure, $\mathbf{g}$ the gravitational acceleration, μ the dynamic viscosity, μ_c is the material constant responsible for

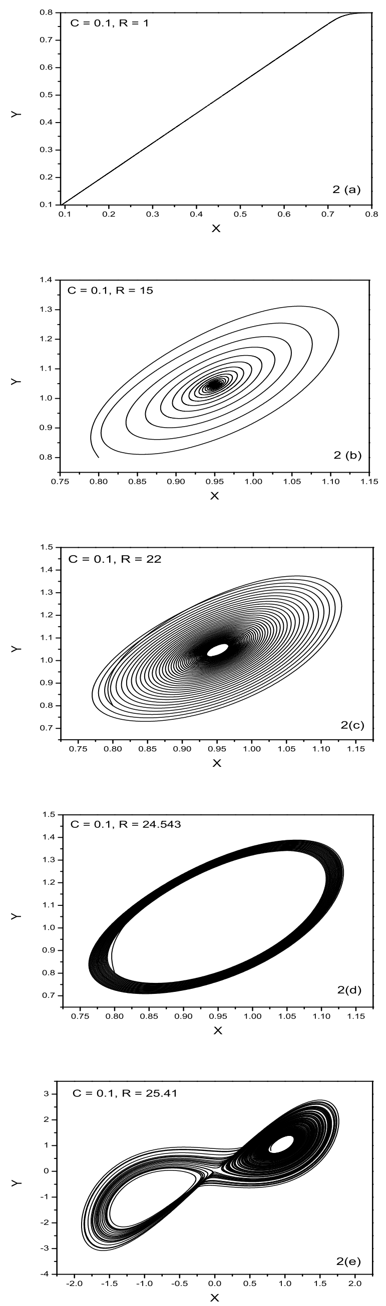

Figure 2: Phase portraits for evolution of trajectories over time in the state space for increasing value of rescaled Rayleigh number (R). The graphs represent the projection of the solution data points onto $Y - X$ plane for $\sigma = 5, \gamma = 0.5$ and $C = 0.1$.

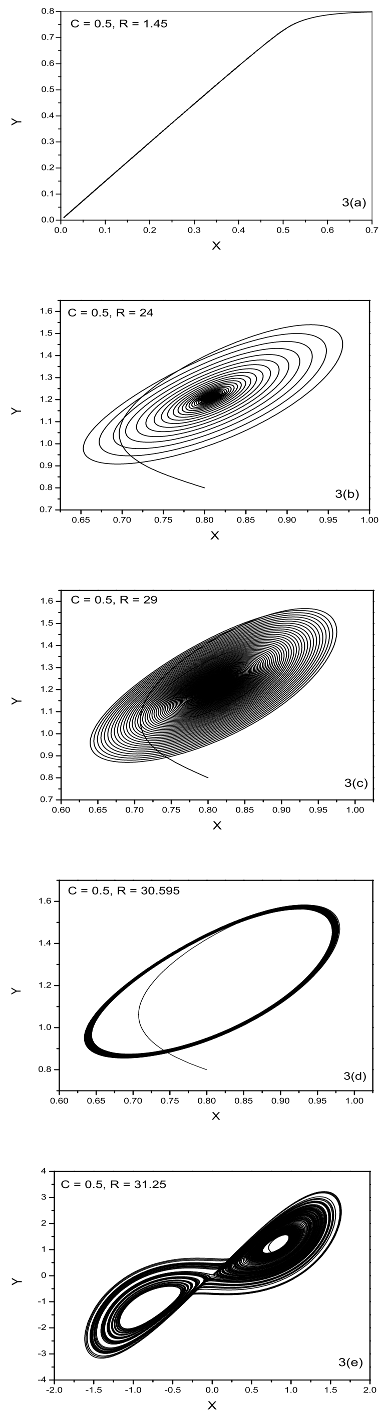

Figure 3: Phase portraits for evolution of trajectories over time in the state space for increasing value of rescaled Rayleigh number (R). The graphs represent the projection of the solution data points onto $Y - X$ plane for $\sigma = 5, \gamma = 0.5$ and $C = 0.5$.

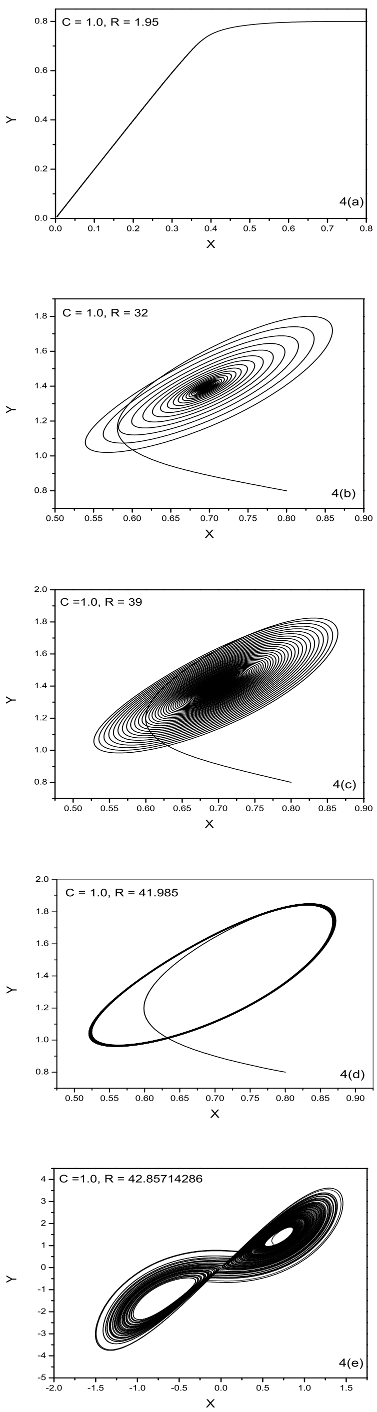

Figure 4: Phase portraits for evolution of trajectories over time in the state space for increasing value of rescaled Rayleigh number (R). The graphs represent the projection of the solution data points onto $Y-X$ plane for $\sigma = 5, \gamma = 0.5$ and $C = 1.0$.

the couple stress property known as the couple stress viscosity and has the dimension of momentum (MLT^{-1}).

The basic state is assumed to be quiescent and quantities in this state are given by

$$\mathbf{q}_b = (0,0,0), \quad p = p_b(z), \quad T = T_b(z) \quad \text{and} \quad \rho = \rho_b(z) \tag{2.6}$$

Substituting Eq. (2.6) in Eqs. (2.2) and (2.3), we get

$$\frac{dp_b}{dz} = -\rho_b g \tag{2.7}$$

$$\frac{d^2T_b}{dz^2} = 0, \tag{2.8}$$

where b refers to the basic state. The solution of Eq. (2.8) subject to the boundary condition (2.5) is given by

$$T_b = T_0 + \triangle T(1 - \frac{z}{H_*}) \tag{2.9}$$

Now, we superimpose the small perturbations at the basic state in the form:

$$\mathbf{q} = \mathbf{q}_b + \mathbf{q}^{'}, \quad T = T_b + T^{'}, \quad p = p_b + p^{'}, \quad \rho = \rho_b + \rho^{'} \tag{2.10}$$

where primes denote the quantities at the perturbations. Using Eqs. (2.9)-(2.10) in Eqs. (2.1)-(2.3) we obtain the following equations,

$$\nabla.\mathbf{q}' = 0 \tag{2.11}$$

$$\frac{1}{\varepsilon}\frac{\partial \mathbf{q}'}{\partial t} = -\frac{1}{\rho_0}\nabla.p' - g\beta_T T' - \frac{1}{K\rho_0}(\mu - \mu_c\nabla^2)\mathbf{q}' \tag{2.12}$$

$$\varepsilon(\rho_0 c)\frac{\partial T^{'}}{\partial t} + (\rho_0 c)(\mathbf{q}'.\nabla)T' + (\rho_0 c)w'\frac{\partial T_b}{\partial z} = \varepsilon\kappa\nabla^2 T^{'} \tag{2.13}$$

Now, non-dimensionalising Eqs.(2.1) − (2.3) by using the following transformations:

$$q_* = (H_*/\kappa)q', \quad p_* = (K/\nu\kappa)p', \quad T_* = \frac{T' - T_0}{\triangle T}, \quad (x_*, y_*, z_*) = H_*(x, y, z), \quad t_* = tH_*^2/\kappa. \tag{2.14}$$

Since, we are considering only two dimensional flow model, therefore introduce the stream function ψ as $u = \partial\psi/\partial z$ and $w = -\partial\psi/\partial x$ obtain the following equations(for simplicity dropping the asterisks)

$$(\frac{1}{Va}\frac{\partial}{\partial t} + 1 - C\nabla^2)\nabla^2\psi + Ra_T\frac{\partial T}{\partial x} = 0 \tag{2.15}$$

$$(\frac{\partial}{\partial t} - \nabla^2)T = -\frac{\partial \psi}{\partial x} + \frac{\partial(\psi, T)}{\partial(x, z)} \qquad (2.16)$$

where $Va = \varepsilon Pr/Da$, the Vadasz number, $Ra_T = \frac{\beta(\triangle T)gH_*K}{\varepsilon\nu\kappa}$, the Darcy- Rayleigh number and $Da = K/L_*^2$, the Darcy number, $C_1 = \mu_c/\mu L_*^2$, couple stress parameter, $Pr = \nu/\kappa$, the Prandtl number.
Assumed, boundaries are stress free and isothermal, therefore the boundary conditions are given by

$$\psi = \frac{\partial^2 \psi}{\partial z^2} = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1 \qquad (2.17)$$

$$T = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1 \qquad (2.18)$$

The set of partial differential Eqs.(2.15) - (2.16) form a non-linear coupled system with the above boundary conditions.

3 Truncated Galerkin expansion

To obtain the solution of non-linear coupled system of partial differential Eqs.(2.15)-(2.16), we represent the stream function ψ and temperature T in the form

$$\psi = A_{11}\sin(\frac{\pi x}{L})\sin(\pi z) \qquad (3.19)$$

$$T = B_{11}\cos(\frac{\pi x}{L})\sin(\pi z) + B_{02}\sin(2\pi z) \qquad (3.20)$$

This representation is equivalent to Galerkin expansion of the solution in both the $X-$ and $Z-$directions. Substituting Eqs. (3.19) and (3.20) in the Eqs. (2.15) - (2.16), multiplying the equations by the orthogonal characteristic functions corresponding to Eqs.(3.19) and (3.20) and integrating them over the domain $[0, L] \times [0, 1]$,yields a set of three ordinary differential equations for the time evolution of the amplitudes:

$$\frac{dA_{11}(\tau)}{d\tau} = -\frac{Va\gamma}{\pi^2}\Big(\frac{Ra_T}{\pi\theta}B_{11}(\tau) + A_{11}(\tau) + C\frac{\pi^2}{\gamma}A_{11}(\tau)\Big) \qquad (3.21)$$

$$\frac{dB_{11}(\tau)}{d\tau} = -\frac{1}{\pi\theta}A_{11}(\tau) - \frac{1}{\theta}A_{11}(\tau)B_{02}(\tau) - B_{11}(\tau) \qquad (3.22)$$

$$\frac{dB_{02}(\tau)}{d\tau} = -4\gamma B_{02}(\tau) + \frac{1}{2\theta}A_{11}(\tau)B_{11}(\tau) \qquad (3.23)$$

In the Eqs. (3.21)- (3.23), time is rescaled and some additional notations has been used:

$$\tau = \frac{(L^2+1)\pi^2 t}{L^2}, \quad \theta = \frac{L^2+1}{L}$$
$$\gamma = \frac{L}{\theta} = \frac{L^2}{L^2+1} \qquad (3.24)$$

Although we cannot establish the relationship between the solutions of the governing partial differential and the corresponding truncated ordinary differential system, these lower-order spectral models may qualitatively reproduce the convective phenomena observed in the full system. The result can also be used as starting values when discussing the fully non-linear problem.

We introduce the following notations

$$R = \frac{Ra}{\pi^2\theta^2}, \qquad C = \frac{C_1\pi^2}{\gamma}, \quad S = 1 + C, \quad \sigma = \frac{Pr}{\pi^2\gamma} \qquad (3.25)$$

and rescale the amplitudes in the form of

$$X(\tau) = \frac{A_{11}(\tau)}{2\theta\sqrt{2\gamma(R-1)}}, Y(\tau) = \frac{\pi R B_{11}(\tau)}{\sqrt{2\gamma(R-1)}}$$
$$\text{and} \quad Z(\tau) = -\frac{\pi R B_{02}(\tau)}{R-1} \qquad (3.26)$$

to obtain the following set of equations,

$$\dot{X} = \sigma[Y - SX] \qquad (3.27)$$

$$\dot{Y} = RX - Y - (R-1)XZ \qquad (3.28)$$

$$\dot{Z} = 4\gamma(XY - Z) \qquad (3.29)$$

where the dots(.) denote the time derivative $d()/d\tau$. Eqs. (3.27)-(3.29) are like the Lorenz equations (Lorenz [12], Sparrrow [23], with the different coefficients.

4 Stability Analysis

In this section, We consider the thermal instability of buoyancy-driven flow in couple stress liquid confined between stress-free boundaries. The fluid layer is subjected to a constant vertical temperature gradient. Stability analysis of the stationary solutions will be perform in order to determine the nature of dynamics about the fixed points. The non-linear dynamics of Lorenz-like system (3.27) − (3.29) has been analyzed and solved for $\sigma = 5$ and $\gamma = 0.5$ corresponding to convection.

4.1 Dissipation

Let the general form of nonlinear dynamical system of Eqs.(3.27) − (3.29) be $\dot{X}_s = f(X_s)$ and $V(\tau)$ be the volume in phase space of closed surface $s(\tau)$ at instantaneous time τ. So, $f(X_s)$ will be instantaneous velocity. Hence,volume can be obtained by expression (Strogatz[25])

$$\dot{V} = \int_s f(X_s).\hat{n}dA \tag{4.30}$$

where $\hat{n}$ denotes the outward normal on surface $s(\tau)$ and dA denotes the area in time τ. By divergence theorem, above integral can be written as

$$\dot{V} = \int_V \nabla.f(X_s)dV \tag{4.31}$$

Now,

$$\begin{aligned}\nabla.f(X_s) &= \frac{\partial \dot{X}}{\partial X} + \frac{\partial \dot{Y}}{\partial Y} + \frac{\partial \dot{Z}}{\partial Z} \\ &= -[\sigma S + 4\gamma + 1] < 0\end{aligned} \tag{4.32}$$

Since the divergence is constant, Eq. (4.31) reduces to

$$\dot{V} = -(\sigma S + 4\gamma + 1)V \tag{4.33}$$

Hence, if set of initial points in the phase space occupies volume $V(0)$ at time $t = 0$, then volume in the phase space is

$$V(\tau) = V(0)\exp\Big[-(\sigma S + 1 + 4\gamma)\tau\Big] \tag{4.34}$$

The above expression shows that the volume in phase space shrink exponentially. Which indicates that the solution is bounded as time →

4.2 Equilibrium points

System of Eqs.(3.27)-(3.29) has the general form $\dot{X}_s = f(X_s)$ and the equilibrium (fixed or stationary) points are obtained by $f(X_s) = 0$. The equilibrium points of the rescaled system are

$$(X_1, Y_1, Z_1) = (0, 0, 0) \tag{4.35}$$

and

$$X_{2,3} = \pm\sqrt{\frac{R - S}{(R-1)S}} \tag{4.36}$$

$$Y_{2,3} = \pm\sqrt{\frac{(R - S)S}{R-1}} \tag{4.37}$$

$$Z_{2,3} = \frac{R - S}{R-1} \tag{4.38}$$

corresponding to the motionless and convection solutions respectively. When $C = 0$, the values coincides with Vadasz and Olek [27]

4.3 Stability of equilibrium points

By linearizing system of Eqs.(3.27)-(3.29), we obtain its Jacobian matrix as follows:

$$J = \begin{bmatrix} -\sigma S & \sigma & 0 \\ R - (R-1)Z & -1 & -(R-1)X \\ 4\gamma Y & 4\gamma X & -4\gamma \end{bmatrix} \tag{4.39}$$

The Routh-Hurwitz criteria has been used to determine the stability of fixed points. The stability condition are obtained by solving the zeros of characteristic polynomial of the Jacobian matrix. If the eigen polynomial of the Jacobian matrix of a system of a differential equation at an equilibrium point, it is of the form

$$a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{4.40}$$

where $a_0 > 0$ and $\triangle_1 = a_1$, $\triangle_2 = det \begin{bmatrix} a_1 & a_0 \\ a_3 & a_2 \end{bmatrix}$, the equilibrium points are stable iff $\triangle_1 > 0$, $\triangle_2 > 0$ and $a_3 > 0$. The system of equations (3.27)-(3.29) has a feasible equilibrium point $(0, 0, 0)$, and the associated characteristic equation of Jacobian matrix is,

$$\begin{aligned}&\lambda^3 + (\sigma S + 4\gamma + 1)\lambda^2 + \sigma(4\gamma + S - R)\lambda \\ &+ 4\gamma\sigma(S - R) = 0\end{aligned} \tag{4.41}$$

Here, $\triangle_1 = a_1 = (\sigma S + 4\gamma + 1) > 0$ and $a_3 = 4\gamma\sigma(S - R) > 0$ if $R < S$, the first eigenvalue $\lambda_1 = -4\gamma$ is always negative as $\gamma > 0$, but the other two eigenvalues are given by equation

$$\begin{aligned}\lambda_{2,3} =& \frac{1}{2}\Big[-\{\sigma S + 1\} \\ &\pm \sqrt{(\sigma\ S + 1)^2 + 4\sigma(R - S)}\Big]\end{aligned} \tag{4.42}$$

λ_3 is always negative and λ_2 provides the stability condition for the motionless solution in the form $R < S$. Therefore the critical value of R, where the motionless solution loses stability and the convection solution (expressed by the other two fixed points) takes over, is determined as

$$R_{c1} = R_{cr} = S = 1 + C \tag{4.43}$$

which corresponds to $Ra_{cr} = 4\pi^2 S$.

The stability of the fixed points corresponding to the convective solution $(X_{2,3}, Y_{2,3}, Z_{2,3})$ is controlled by the following equation for the eigenvalues, $\lambda_i (i = 1, 2, 3)$:

$$\begin{aligned} &S\lambda^3 + S(\sigma S + 4\gamma + 1)\lambda^2 + 4\gamma(S^2 + R)\lambda \\ &+ 8S\sigma\gamma(R - S) = 0 \end{aligned} \tag{4.44}$$

Eq.(4.44) yields three eigenvalues, all the roots are real and negative at slightly supercritical value of R, such that the convection fixed points are stable, that is simple nodes. These roots move on the real axis towards the origin as the value of R increases. These roots become equal when $R =$

$$R_{c2} = \frac{\sigma S^2(\sigma S + 4\gamma + 3)}{\sigma S - 4\gamma - 1} \tag{4.45}$$

When $C = 0$, value of R_{c2} coincides with in Vadasz and Olek [9]. The loss of stability of the convective fixed points for $\sigma = 5, \gamma = 0.5$, using Eq.(4.45) is evaluated to be $R_{c2} = 25$, which recover Vadasz and Olek's solution for $C = 0$, $R_{c2} = 25.41$

5 Result and discussion

In this section, we perform some numerical solution of the system of Eqs.(3.27) – (3.29) for the time domain $0 \leq t \leq 80$. Numerical solutions are obtained by using fourth-order Runge- Kutta method on double precision with the step size 0.001, fixing the values $\sigma = 5, \gamma = 0.5$ and taking the initial conditions $X(0) = 0.8, Y(0) = 0.8$ and $Z(0) = 0.9$.

Also, we display the stability diagrams in Fig. 1 for the complex eigenvalues versus scaled Rayleigh number R and different values of scaled couple stress parameter$C$. From Fig. 1, We observed that the increase in value of couple-stress parameter C increases the Rayleigh number at which there are exactly two complex conjugate roots and they have still negative real parts, therefore, the convection fixed points are stable.

For $C = 0.1$, we obtained from Eq. (4.43), the motionless solution loses their stability at $R_{c1} = 1.1$ as well as the convective solution present in system. The convective fixed points become unstable at $R = 25.41$ and chaos is obtained. The evolution of trajectories over time in the state space for increasing value of scaled Rayleigh number is shown in Fig.2 in terms of projection of trajectories onto $Y - X$ plane. In Fig. 2a, we see that the trajectory moves to the steady convection points on a straight line for a Rayleigh number slightly above the loss of stability of the motionless solution ($R = 1.1$). At $R = 15$ the trajectories approach the fixed point on a spiral as shown in Fig. 2b. At the critical value of $R = 25.41$ in Fig. 2e, we observe that transition from laminar to chaotic behavior occurs via limit cycle at $R = 24.543$ (Fig. 2d).

From Fig. 3, we reveal that the convective solution will start from $R_{c1} = 1.5$ at $C = 0.5$ while chaotic solution occurs at $R = 31.25$. The detail relation between phase portraits X and Y are shown by Figs. 3(a) -(e). In Fig. 3a, we see that the trajectory moves to the steady convection point on a straight line for a Rayleigh number slightly above the critical value of scaled Rayleigh number for the motionless solution ($R = 1.5$). At $R = 24$ the trajectories approach to the fixed point in a spiral as shown in Fig. 3b. At the critical value of $R = 31.25$ in Fig. 3e, we observe that transition to chaotic behavior occurs via limit cycle at $R = 30.595$ (Fig. 3d).

For $C = 1$, we found that at $R_{c1} = 2$, the motionless solution takes over by the convective solution. We found a chaotic behavior of convective fixed points at $R = 42.85714286$. The evolution of trajectories over time in the state space for different values of scaled Rayleigh number is shown in Fig.4 in terms of projection of trajectories onto $Y - X$ plane. In Fig. 4a, we see that the trajectory moves to the steady convection points on a straight line for a Rayleigh number slightly above the loss of stability of the motionless solution ($R = 2$). At $R = 32$ the trajectories approach the fixed point on a spiral as shown in Fig. 4b. At the critical value of $R = 42.85714286$ in

Fig. 4e, we observe that transition to chaotic behavior occurs via limit cycle at $R = 41.985$ (Fig. 4d).

6 Conclusion

In this paper, we have investigated the chaotic behavior under the effect of different scaled couple stress parameter, in a couple stress liquid saturated in a porous layer, subjected to gravity and heated from below. We found that there is proportional relation between the scaled couple stress parameter and scaled Rayleigh number R. We found that increase in scaled couple stress parameter increases the level of chaos.

References

[1] B. S. Bhadauria, P. Kiran, *Chaotic and oscillatory magneto-convection in a binary viscoelastic fluid under g-jitter*, Int. J. Heat and mass Transfer 84 (2015) 610-624.

[2] Z. M. Chen, W. G. Price, *On relation between Rayleigh-Bénard convection and Lorenz system*, Chaos Solitons Fractals 28 (2006) 571-578.

[3] C. Ferrarioa, A Passerinib, Th. Gudrun, *Generalization of the Lorenz model to the two-dimensional convection of second-grade fluids*, Int. J. of Non-Linear Mech 39 (2004) 581-591.

[4] J. P. Garandet, T. Alboussiere, R. Moreau, *Buoyancydriven convection in a rectangular enclosure with a transverse magnetic field*, Int. J. Heat Mass Transf 35 (1992) 741-748.

[5] K. Gupta Vinod, A. K. Singh, *A Study of Chaos in an anisotropic porous cavity*, Int. J. of Energy and Technology 5 (2013) 1-12.

[6] K. Gupta Vinod, B. S. Bhadauria, *Double diffusive convection in a couple stress fluid saturated sparsely packed porous layer with soret effect using thermal non-equilibrium model*, Int. Conference on Modeling and Simulation of diffusive Process and Applications 5 (2012) 89-95.

[7] R. Idris, I. Hasim, *Effects of a magnetic field on chaos for low Prandtl number convection in porous maedia*, Nonlinear Dyn. 62 (2010) 905- 917.

[8] D. B. Ingham, I. Pop, *Transport Phenomena in Porous Media*, Elsevier Oxford (2005).

[9] J. M. Jawdat, I. Hashim, *Low Prandtl number chaotic convection in porous media with uniform internal heat generation*, Int. Comm. in Heat and Mass Transf. 37 (2010) 629-636.

[10] J. M. Jawdat, I. Hasim, S. Momani, *Dynamical dsystem analysis of thermal convection in a horizontal layer of Nanofluids heated from below*, Mathematical Problems in Engg., Hindwai 13 (2012) http://dx.doi.org/10.1155/2012/128943/.

[11] E. R. Lapwood, *Convection of a fluid in a porous medium*, Proc. Camb. Phil. Soc. 44 (1948) 508-521.

[12] E. N. Lorenz,*Deterministic non-periodic flow*, J. Atmos. Sci. 20 (1963) 130-141.

[13] E. Magyari, *The "Butterfly Effect" in a Porous Slab*, Transp Porous Med 84 (2011) 711-715.

[14] M. N. Mahmud, I. Hasim, *Effect of magnetic field on chaotic convection in fluid layer heated from below, Int. Comm. in Heat and Mass Transf. 38 (2011) 481-486.*

[15] M. N. Mahmud, I Hasim, Small and Moderate Prandtl Number Chaotic Convection in Porous Media in the Presence of Feedback Control, Transp. Porous Med. 84 (2010) 421-440.

[16] M. N. Mahmud, I. Hasim, Small and Moderate Prandtl Number Chaotic Convection in Porous Media in the Presence of Feedback Control, Transp. Porous Med. 84 (2010) 421-440.

[17] M. S. Malashetty, I. S. Shivkumara, K. Sridhar, The onset of convection in a couple stress fluid saturated porous layer using a thermal non-equilibrium, Physics Letters A 373 (2009) 781-790.

[18] M. S. Malashetty, D. K. Pal, Premila, Double-diffusive convection in a Darcy

porous medium saturated with a couple-stress fluid, Fluid Dyn. Res. 42 (2010) 035502 (21pp).

[19] D. A. Nield, A. Bejan, Convection in Porous Media, Third ed., Springer-Verlag, New York (2006).

[20] D. Roy, Z. E. Musielak, Generalized Lorenz models and their routes to chaos. II. Energy-conserving horizontal mode truncations, Chaos Solitons Fractals 31 (2007) 747-756.

[21] L. J. Sheu, An autonomus system for chaotic convection in a porous medium using a thermal non-eqilibrium model, Chaos,Solutions and Fractals 30 (2006) 672-689.

[22] L. J. Sheu, L. M. Tam, J. H. Chen, H. K. Chen, K. T. Lin, Y. Kang, Chaotic convection of viscoelastic fluids in porous media, Chaos Solitons and Fractals 37 (2008) 113-124.

[23] C. Sparrow, The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors, Springer-Verlag, New York (1982).

[24] V. K. Stokes,Couple stresses in fluids, Phys. Fluids 9 (1966) 1709-1716.

[25] H. Strogatz,Nonlinear Dynamics and Chaos, Levant Books Publication, Indian edition (2007).

[26] P. Vadasz, S. Olek, Route to chaos for moderate Prandtl number convection in a porous layer heated from below, Transp. Porous Med. 41 (2000) 211-239.

[27] P. Vadasz, S. Olek, Weak turbulence and chaos for low Prandtl number gravity driven convection in porous media, Transp. Porous Med. 37 (1999) 69-91.

[28] P. Vadasz, S. Olek, Transition and chaos for free convection in a rotating porous layer, Int. J. Heat Mass Transf. 41 (1999) 1417-1435.

[29] P. Vadasz, Subcritical transitions to chaos and hysteresis in a fluid layer heated from below, Int. J. Heat Mass Transf. 43 (2000) 705-724.

[30] P. Vadasz, Local and global transitions to chaos and hysteresis in a porous layer heated from below, Transp. Porous Med. 37 (1999) 213-245.

[31] K. Vafai,Handbook of Porous Media, Taylor and Francis (CRC), Boca Raton (2005).

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