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Numerical Solution of Hybrid Fuzzy Differential Equation (IVP) by Improved Predictor-Corrector Method

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Abstract

In this paper, we study the numerical solution of hybrid fuzzy differential equations by using the improved predictor-corrector (IPC) three step method. We state the convergence and stability of this method. Numerical examples will be presented to illustrate this method.

Keywords : Fuzzy differential equations, Fuzzy numbers, Hybrid fuzzy differential system, Explicit three-step method, Implicit two-step method.

1 Introduction

Fuzzy set theory is a powerful tool for modeling uncertainty and for processing vague or subjective information in mathematical models. Its main directions of development and its applications to the very varied real-world problems have been diverse. Fuzzy differential equations (FDEs) are recently gaining more and more attention in the literature. The first and the most popular approach in dealng with FDEs is using the Hukuhara differentiability, or the Seikkala derivative for fuzzy-number-valued functions. Hybrid systems are devoted to modeling, design and validation of interactive systems of computer programs and continuous systems. That is, control systems that are capable of controlling complex systems which have discrete event dynamics as well as continuous time dynamics can be modeled by hybrid systems. The differential systems containing fuzzy-valued functions and interaction with a discrete time controller are named as hybrid fuzzy differential systems. Pederson and Sambandham [8, 9] have investigated the numerical solution of hybrid fuzzy differential equations by using the Runge-kutta method and Euler method, and also they have considered the numerical solution of hybrid fuzzy differential equations by using the characterization theorem for the improved Euler's method [7]. Recently, the

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numerical solution of fuzzy differential equations by the predictor-corrector method has been studied in [1].
In this paper, we develop a numerical solution of hybrid fuzzy differential equation initial value problems by using the improved predictor-corrector method, which is more accurate than the one in [8]. In Section 2, we list some basic definitions for fuzzy-valued functions. Section 3 reviews hybrid fuzzy differential systems. Sections 4 and 5 contain the explicit three-step method and the implicit three-step method for approaching hybrid fuzzy differential equations, respectively. In Section 6, the IPC three -step algorithm is discussed and in Section 7, convergence and stability theorem are provided. Section 8, contains numerical examples to illustrate this method.

2 Preliminaries

Definition 2.1. *[10] Let $\Gamma_k(R^n)$ denote the family of all non-empty, compact, convex subsets of R^n. Denote by E^n the set of $\widetilde{u} : R^n \rightarrow [0,1]$ such that $\widetilde{u}$ satisfies $(i) - (iv)$ mentioned below:*

i. $\widetilde{u}$ *is normal that is, there exists an* $y_0 \in R^n$ *such that* $\widetilde{u}(y_0) = 1$,

ii. $\widetilde{u}$ *is fuzzy convex,*

iii. $\widetilde{u}$ *is upper semi continuous,*

iv. $[\widetilde{u}]^0 = \{\overline{y \in R^n : \widetilde{u}(y) > 0}\}$ *is compact .*

We denote the $\alpha-$ level set $[\widetilde{u}]^\alpha = \{y \in R^n : \widetilde{u}(y) \geq \alpha\}$ for $0 < \alpha \leq 1$. Clearly, the α-level sets $[\widetilde{u}]^\alpha \in \Gamma_k(R^n)$.

Definition 2.2. *[6] Let I be a real interval. A mapping $\widetilde{y} : I \longrightarrow E^1$ is called a fuzzy process and its $\alpha-$ level set is denoted by*

$$[\widetilde{y}]^\alpha = [\underline{y}^\alpha, \overline{y}^\alpha] \quad t \in I, \quad 0 < \alpha \leq 1.$$

Let $\widetilde{x},\ \widetilde{y} \in E^1$. If there exists $\widetilde{z} \in E^1$ such that $\widetilde{x} = \widetilde{y} + \widetilde{z}$, then $\widetilde{z}$ is called the Hukuhara difference of $\widetilde{x}$ and $\widetilde{y}$ and is denoted by $\widetilde{x} \ominus \widetilde{y}$. Note that in this paper, the " $\ominus$ " sign stands always for the Hukuhara difference and that $\widetilde{x} \ominus \widetilde{y} \neq \widetilde{x} + (-1)\widetilde{y}$.

Definition 2.3. *[1] The α-level set of a triangular fuzzy number $\widetilde{T} = (x^l, x^c, x^r)$ in E^1 is given by*

$$[\widetilde{T}]^\alpha = [x^c - (1-\alpha)(x^c - x^l),\ x^c + (1-\alpha)(x^r - x^c)]$$

where $x^l \leq x^c \leq x^r$.

Let us recall the definition of Hukuhara differentiability.

Definition 2.4. *Let $d_H(A, B)$ be the Hausdorff distance between sets $A, B \in \Gamma_k(R^n)$. The supremum metric d_∞ on E^1 is defined by*

$$d_\infty(\widetilde{U}, \widetilde{V}) = \sup\{d_H([\widetilde{U}]^\alpha, [\widetilde{V}]^\alpha) : \alpha \in [0,1]\}$$

and(E^1, d_∞) is a complete metric space; for more details see [11] .

Definition 2.5. *[1] Let $\widetilde{f}: T \to E^1$ and $y_0 \in T \subseteq R$. We say that $\widetilde{f}$ is Hukuhara differentiable at y_0 if there exists an element $\widetilde{f}' \in E^1$ such that for all $h > 0$ sufficiently small, there are $\widetilde{f}(y_0 + h) \ominus \widetilde{f}(y_0)$, $\widetilde{f}(y_0) \ominus \widetilde{f}(y_0 - h))$ and the limits (in the metric d_∞)*

$$\lim_{h\to 0}\left(\frac{\widetilde{f}(y_0+h)\ominus\widetilde{f}(y_0)}{h}\right) = \lim_{h\to 0}\left(\frac{\widetilde{f}(y_0)\ominus\widetilde{f}(y_0-h)}{h}\right) = \widetilde{f}'(y_0).$$

The fuzzy set $\widetilde{f}'(y_0)$ is called the Hukuhara derivative of $\widetilde{f}$ at y_0.

Recall that $\widetilde{U} \ominus \widetilde{V} = \widetilde{W} \in E^1$ are defined on $\alpha-$level sets, where $[\widetilde{U}]^\alpha \ominus [\widetilde{V}]^\alpha = [\widetilde{W}]^\alpha$ for all $\alpha \in [0, 1]$. Considering the of definition of the metric d_∞, all the $\alpha-$level sets $[\widetilde{f}(0)]^\alpha$ are Hukuhara differentiable at y_0 with Hukuhara derivatives $[\widetilde{f}'(y_0)]^\alpha$ for each $\alpha \in [0, 1]$, when $\widetilde{f}: T \to E^1$ is Hukuhara differentiable at y_0 with Hukuhara derivative $\widetilde{f}'(y_0)$.

Definition 2.6. *[8] Associated with the difference equation*

$$Y_{i+1} = a_{m-1}y_i + a_{m-2}y_{i-1} + \ldots + a_0 y_{i+1-m} + hF(t_i, h, y_{i+1}, y_i, \ldots, y_{i+1-m}),$$
$$y_0 = \alpha_0, y_1 = \alpha_1, \ldots, y_{m-1} = \alpha_{m-1}, \tag{2.1}$$

the characteristic polynomial of the method is defined by

$$p(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \ldots - a_1\lambda - a_0.$$

If $|\lambda_i| \leq 1$ for each $i = 1, 2, \ldots, m$, and all roots with absolute value 1 are simple roots, then the difference method is said to satisfy the root condition.

Definition 2.7. *A multistep method of the form Eq. (2.1) is stable if and only if it satisfies the root condition.*

Definition 2.8. *An m-step method for solving the initial-value problem is one whose difference equation for finding the approximation $y(t_{i+1})$ at the mesh point t_{i+1} can be represented by the following equation:*

$$y(t_{i+1}) = a_{m-1}y(t_i) + a_{m-2}y(t_{i-1}) + \ldots + a_0 y(t_{i+1-m}) + h(b_m f(t_{i+1}, y_{i-1}) +$$
$$b_{m-1}f(t_i, y_i) + \ldots + b_0 f(t_{i+1-m}, y_{i+1-m})), \tag{2.2}$$

for $i = m-1, m, \ldots, N-1$, such that

$$a = t_0 \leq t_1 \leq \ldots \leq t_N = b, h = (b-a)/N = t_{i+1} - t_i$$

and $a_0, a_1, \ldots, a_{m-1}, b_0, b_1, \ldots . b_m$ are constants with the starting values

$$y_0 = \alpha_0, y_1 = \alpha_1, y_2 = \alpha_2, \ldots, y_{m-1} = a_{m-1}.$$

When $b_m = 0$, the method is known as explicit, since Eq. (2.2) gives y_{i+1} explicit in terms of previously determined values. When $b_m \neq 0$, the method is known as implicit, since y_{i+1} occurs on both sides of Eq. (2.2) and is specified only implicitly.

Definition 2.9. *[6] The fuzzy integral*

$$\int_a^b \widetilde{y}(t)dt, \quad 0 \le a \le b \le 1,$$

is defined by

$$[\int_a^b \widetilde{y}(t)dt]^\alpha = [\int_a^b \underline{y}^\alpha(t)dt, \int_a^b \overline{y}^\alpha(t)dt],$$

provided that the Lebesgue integrals on the right-hand side exist.

Remark 2.1. *[6] If $\widetilde{f} : T \longrightarrow E^1$ is Hukuhara differentiable and its Hukuhara derivative $\widetilde{f}'$ is integrable over $[0,1]$, then*

$$\widetilde{f}(t) = \widetilde{f}(t_0) + \int_{t_0}^t \widetilde{f}'(s)ds,$$

for all $0 \le t_0 \le t \le 1$.
The Seikkala derivative $y'(t)$ of a fuzzy process $\widetilde{y}$ is defined by

$$[\widetilde{y}'(t)]^\alpha = [(\underline{y}^\alpha)'(t), (\overline{y}^\alpha)'(t)], \qquad 0 < \alpha \le 1.$$

provided that its equation defines a fuzzy number $\widetilde{y}'(t) \in E^1$.

Remark 2.2. *[6] If $\widetilde{y} : I \longrightarrow E^1$ is Seikkala differentiable and its Seikkala derivative $\widetilde{y}'$ is integrable over $[0,1]$, then*

$$\widetilde{y}(t) = \widetilde{y}(t_0) + \int_{t_0}^t \widetilde{y}'(s)ds,$$

for all t_0 and $t \in I$.

3 The hybrid fuzzy differential system

Consider the hybrid fuzzy differential system

$$\begin{cases} \widetilde{y}'(t) = \widetilde{f}(t, y, \lambda_k(y_k)), \quad t \in [t_k, t_k + 1] \qquad k = 0, 1, 2, \cdots, \\ \widetilde{y}(t_k) = \widetilde{y}_k, \end{cases} \tag{3.3}$$

where $0 \le t_0 < t_1 < \ldots < t_k < \ldots, t_k \to \infty$, $f \in C[R^+ \times E^1 \times E^1, E^1]$ and $\lambda_k \in C[E^1, E^1]$. Here we assume the existence and uniqueness of solutions of the hybrid system hold on each $[t_k, t_{k+1}]$. To be specific, the system would look like:

$$\begin{cases} \widetilde{y}_0'(t) = \widetilde{f}(t, y_0(t), \lambda_0(y_0)) \quad \widetilde{y}_0(t_0) = \widetilde{y}_0 \quad t_0 \le t \le t_1, \\ \widetilde{y}_1'(t) = \widetilde{f}(t, y_1(t), \lambda_1(y_1)) \quad \widetilde{y}_1(t_1) = \widetilde{y}_1 \quad t_1 \le t \le t_2, \\ \vdots \\ \widetilde{y}_k'(t) = \widetilde{f}(t, y_k(t), \lambda_k(y_k)) \quad \widetilde{y}_k(t_k) = \widetilde{y}_k \quad t_k \le t \le t_{k+1}, \\ \vdots \end{cases}$$

By the solution of Eq. (3.3) we mean the following function:

$$\widetilde{y}(t) = y(t, t_0, \widetilde{y}_0) = \begin{cases} \widetilde{y}_0(t) & , t_0 \leq t \leq t_1 \\ \widetilde{y}_1(t) & , t_1 \leq t \leq t_2 \\ \vdots & \\ \widetilde{y}_k(t) & , t_k \leq t \leq t_k + 1 \\ \vdots & \end{cases} \tag{3.4}$$

We note that the solutions of Eq. (3.3) are piecewise differential in each interval for $t \in [t_k, t_{k+1}]$ for a fixed $\widetilde{y}_k \in E^1$ and $k = 0, 1, 2 \ldots$.

4 The Explicit Three-Step Method

In this section, we develop the numerical solution of Eq. (3.3) by using the explicit three-step method given in [1]. Hence, we present the explicit three-step method for solving the hybrid fuzzy differential system (3.3), when f and λ_k in Eq. (3.3) can be obtained via the Zadeh extension principle from $f \in C[R^+ \times R \times R, R]$ and $\lambda_k \in C[R, R]$. We assume that the existence and uniqueness of the solutions of Eq. (3.3) hold for each $[t_k, t_{k+1}]$. For fixed $k \in \mathbb{Z}^+$, we replace each interval $[t_k, t_{k+1}]$ by a set of N_{K+1} discrete equally spaced grid points, $t_k = t_{k,0} < t_{k,1} < \ldots < t_{k,N} = t_{k+1}$ (including the endpoints), at which the exact solution $\widetilde{Y}(t)$ is approximated by some $\widetilde{y}_k(t)$.
Fixing $k \in \mathbb{Z}^+$, we can solve the hybrid fuzzy initial value problem

$$\begin{cases} \widetilde{y}'(t) = \widetilde{f}(t, y, \lambda_k(y_k)), & t_k \leq t \leq t_{k+1} \\ \widetilde{y}(t_k) = \widetilde{y}_k \end{cases} \tag{4.5}$$

by the explicit three-step method as follows. Let the fuzzy initial values be

$$\widetilde{y}(t_{k,i-1}), \widetilde{y}(t_{k,i}), \widetilde{y}(t_{k,i+1}),$$

i.e.,

$$\widetilde{f}(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k)), \widetilde{f}(t_{k,i}, y(t_{k,i}), \lambda_k(y_k))\ \widetilde{f}(t_{i+1}, y(t_{i+1}), \lambda_k(y_k)),$$

which are triangular fuzzy numbers and are shown as

$$\{\widetilde{f}^l(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k)), \widetilde{f}^c(t_{k,i-1}, y(t_{k,i-1})), \widetilde{f}^r(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k))\},$$

$$\{\widetilde{f}^l(t_{k,i}, y(t_{k,i}), \lambda_k(y_k)), \widetilde{f}^c(t_{k,i}, y(t_{k,i}), \lambda_k(y_k)), \widetilde{f}^r(t_{k,i}, y(t_{k,i})\lambda_k(y_k))\},$$

$$\{\widetilde{f}^l(t_{k,i+1}, y(t_{k,i+1}), \lambda_k(y_k)), \widetilde{f}^c(t_{k,i+1}, y(t_{k,i+1}), \lambda_k(y_k)), \widetilde{f}^r(t_{k,i+1}, y(t_{k,i+1}), \lambda_k(y_k))\}.$$

Consider the following fuzzy equation

$$\widetilde{y}(t_{k,i+2}) = \widetilde{y}(t_{k,i-1}) + \int_{t_{k,i-1}}^{t_{k,i+2}} \widetilde{f}(t, y_k(t), \lambda_k(y_k))\ dt. \tag{4.6}$$

Similar to [1], if we apply fuzzy linear spline interpolation to

$$\widetilde{f}(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k)), \widetilde{f}(t_{k,i}, y(t_{k,i}), \lambda_k(y_k)), \widetilde{f}(t_{k,i+1}, y(t_{k,i+1}), \lambda_k(y_k)),$$

then we have:

$$\underline{f}(t, y_k(t), \lambda_k(y_k)) = \frac{t_{k,i} - t}{t_{k,i} - t_{k,i-1}} \widetilde{f}(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k)) + \frac{t - t_{k,i-1}}{t_{k,i} - t_{k,i-1}}$$

$$\widetilde{f}(t_{k,i}, y(t_{k,i}), \lambda_k(y_k)) \quad t \in [t_{k,i-1}, t_{k,i}]$$

$$\overline{f}(t, y_k(t), \lambda_k(y_k)) = \frac{t_{k,i+1} - t}{t_{k,i+1} - t_{k,i}} \widetilde{f}(t_{k,i}, y(t_{k,i}), \lambda_k(y_k))$$

$$+ \frac{t - t_{k,i}}{t_{k,i+1} - t_{k,i}} \widetilde{f}(t_{k,i+1}, y(t_{k,i+1}), \lambda_k(y_k)) \quad t \in [t_{k,i}, t_{k,i+1}],$$

therefore, the following results will be obtained:

$$\underline{f}^l(t, y_k(t), \lambda_k(y_k)) = \frac{t_{k,i} - t}{t_{k,i} - t_{k,i-1}} \underline{f}^l(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k))$$

$$+ \frac{t - t_{k,i-1}}{t_{k,i} - t_{k,i-1}} \underline{f}^l(t_{k,i}, y(t_{k,i}), \lambda_k(y_k)), \quad t \in [t_{k,i-1}, t_{k,i}], \tag{4.7}$$

$$\underline{f}^c(t, y_k(t), \lambda_k(y_k)) = \frac{t_{k,i} - t}{t_{k,i} - t_{k,i-1}} \underline{f}^c(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k))$$

$$+ \frac{t - t_{k,i-1}}{t_{k,i} - t_{k,i-1}} \underline{f}^c(t_{k,i}, y(t_{k,i}), \lambda_k(y_k)), \quad t \in [t_{k,i-1}, t_{k,i}], \tag{4.8}$$

$$\underline{f}^r(t, y_k(t), \lambda_k(y_k)) = \frac{t_{k,i} - t}{t_{k,i} - t_{k,i-1}} \underline{f}^r(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k))$$

$$+ \frac{t - t_{k,i-1}}{t_{k,i} - t_{k,i-1}} f_1^r(t_{k,i}, y(t_{k,i}), \lambda_k(y_k)), \quad t \in [t_{k.i-1}, t_{k,i}], \tag{4.9}$$

and

$$\overline{f}^l(t, y_k(t), \lambda_k(y_k)) = \frac{t_{k,i+1} - t}{t_{k,i+1} - t_{k,i}} \overline{f}^l(t_{k,i}, y(t_{k,i}), \lambda_k(y_k))$$

$$+ \frac{t - t_{k,i}}{t_{k,i+1} - t_{k,i}} \overline{f}^l(t_{k,i+1}, y(t_{k,i+1}), \lambda_k(y_k)), \quad t \in [t_{k,i}, t_{k,i+1}], \tag{4.10}$$

$$\overline{f}^c(t, y_k(t), \lambda_k(y_k)) = \frac{t_{k,i+1} - t}{t_{k,i+1} - t_{k,i}} \overline{f}^c(t_{k,i}, y(t_{k,i}), \lambda_k(y_k))$$

$$+ \frac{t - t_{k,i}}{t_{k,i+1} - t_{k,i}} \overline{f}^c(t_{k,i+1}, y(t_{k,i+1}), \lambda_k(y_k)), \quad t \in [t_{k,i}, t_{k,i+1}], \tag{4.11}$$

$$\overline{f}^r(t, y_k(t), \lambda_k(y_k)) = \frac{t_{k,i+1} - t}{t_{k,i+1} - t_{k,i}} \overline{f}^r(t_{k,i}, y(t_{k,i}), \lambda_k(y_k))$$

$$+ \frac{t - t_{k,i}}{t_{k,i+1} - t_{k,i}} \overline{f}^r(t_{k,i+1}, y(t_{k,i+1}), \lambda_k(y_k)), \quad t \in [t_i, t_{k,i+1}]. \tag{4.12}$$

Now, by using Eq. (4.6) we give the following equations:

$$[\widetilde{y}(t_{k,i+2})]^\alpha = [\underline{y}^\alpha(t_{k,i+2}), \overline{y}^\alpha(t_{k,i+2})]$$

where

$$\underline{y}^{\alpha}(t_{k,i+2}) = \underline{y}^{\alpha}(t_{k,i-1}) + \int_{t_{k,i-1}}^{t_{k,i}} \{\alpha \underline{f}^{c}(t, y_k(t), \lambda_k(y_k)) + (1-\alpha)\underline{f}^{l}(t, y_k(t), \lambda_k(y_k))\}dt$$

$$+ \int_{t_{k,i}}^{t_{k,i+2}} \{\alpha \overline{f}^{c}(t, y_k(t), \lambda_k(y_k)) + (1-\alpha)\overline{f}^{l}(t, y_k(t), \lambda_k(y_k))\}dt \tag{4.13}$$

and

$$\overline{y}^{\alpha}(t_{k,i+2}) = \overline{y}^{\alpha}(t_{k,i-1}) + \int_{t_{k,i-1}}^{t_{k,i}} \{\alpha \underline{f}^{c}(t, y_k(t), \lambda_k(y_k)) + (1-\alpha)\underline{f}^{r}(t, y_k(t), \lambda_k(y_k))\}dt$$

$$+ \int_{t_i}^{t_{k,i+2}} \{\alpha \overline{f}^{c}(t, y_k(t), \lambda_k(y_k)) + (1-\alpha)\overline{f}^{r}(t, y_k(t), \lambda_k(y_k))\}dt. \tag{4.14}$$

If Eqs. (4.7)-(4.8) and Eqs. (4.10)-(4.11) are substituted in Eq. (4.13) and also Eqs. (4.8)-(4.9), Eq. (4.11) and Eq. (4.12) in Eq. (4.14), then we get:

$$\underline{y}^{\alpha}(t_{k,i+2}) = \underline{y}^{\alpha}(t_{k,i-1}) + \int_{t_{k,i-1}}^{t_{k,i}} \{\alpha\{\frac{t_{k,i}-t}{t_{k,i}-t_{k,i-1}}\underline{f}^{c}(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k))$$

$$+\tfrac{t-t_{k,i-1}}{t_{k,i}-t_{k,i-1}}\underline{f}^{c}(t_{k,i}, y(t_{k,i}), \lambda_k(y_k))\} + (1-\alpha)\{\tfrac{t_{k,i}-t}{t_{k,i}-t_{k,i-1}}\underline{f}^{l}(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k))$$

$$+\tfrac{t-t_{k,i-1}}{t_{k,i}-t_{k,i-1}}\underline{f}^{l}(t_{k,i}, y(t_{k,i}), \lambda_k(y_k))\}\}dt + \textstyle\int_{t_{k,i}}^{t_{k,i+2}}\{\alpha\{\tfrac{t_{k,i+1}-t}{t_{k,i+1}-t_{k,i}}\overline{f}^{c}(t_{k,i}, y(t_{k,i}), \lambda_k(y_k))$$

$$+\tfrac{t-t_{k,i}}{t_{k,i+1}-t_{k,i}}\overline{f}^{c}(t_{k,i+1}, y(t_{k,i+1}), \lambda_k(y_k))\} + (1-\alpha)\{\tfrac{t_{k,i+1}-t}{t_{k,i+1}-t_{k,i}}\overline{f}^{l}(t_{k,i}, y(t_{k,i}), \lambda_k(y_k))$$

$$+\tfrac{t-t_{k,i}}{t_{k,i+1}-t_{k,i}}\overline{f}^{l}(t_{k,i+1}, y(t_{k,i+1}), \lambda_k(y_k))\}\}dt,$$

and

$$\overline{y}^{\alpha}(t_{k,i+2}) = \overline{y}^{\alpha}(t_{k,i-1}) + \textstyle\int_{t_{k,i-1}}^{t_{k,i}} \{\alpha\{\tfrac{t_{k,i}-t}{t_{k,i}-t_{k,i-1}}\underline{f}^{c}(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k))$$

$$+\tfrac{t-t_{k,i-1}}{t_{k,i}-t_{k,i-1}}\underline{f}^{c}(t_{k,i}, y(t_{k,i}), \lambda_k(y_k))\} + (1-\alpha)\{\tfrac{t_{k,i}-t}{t_{k,i}-t_{k,i-1}}\underline{f}^{r}(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k))$$

$$+\tfrac{t-t_{k,i-1}}{t_{k,i}-t_{k,i-1}}\underline{f}^{r}(t_{k,i}, y(t_{k,i}), \lambda_k(y_k))\}\}dt + \textstyle\int_{t_{k,i}}^{t_{k,i+2}}\{\alpha\tfrac{t_{k,i+1}-t}{t_{k,i+1}-t_{k,i}}\overline{f}^{c}(t_{k,i}, y(t_{k,i}), \lambda_k(y_k))$$

$$+\tfrac{t-t_{k,i}}{t_{k,i+1}-t_{k,i}}\overline{f}^{c}(t_{k,i+1}, y(t_{k,i+1}), \lambda_k(y_k))\} + (1-\alpha)\{\tfrac{t_{k,i+1}-t}{t_{k,i+1}-t_{k,i}}\overline{f}^{r}(t_{k,i}, y(t_{k,i}), \lambda_k(y_k))$$

$$+\tfrac{t-t_{k,i}}{t_{k,i+1}-t_{k,i}}\overline{f}^{r}(t_{k,i+1}, y(t_{k,i+1}), \lambda_k(y_k))\}\}dt,$$

and hence we have:

$$\underline{y}^{\alpha}(t_{k,i+2}) = \underline{y}^{\alpha}(t_{k,i-1}) + \frac{h}{2}[\alpha f^{c}(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k))$$

$$+(1-\alpha)f^{l}(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k))] + \frac{h}{2}[\alpha f^{c}(t_{k,i}, y(t_{k,i}), \lambda_k(y_k))$$

$$+(1-\alpha)f^{l}(t_{k,i},y(t_{k,i}),\lambda_k(y_k))]+2h[\alpha f^{c}(t_{k,i+1},y(t_{k,i+1}),\lambda_k(y_k))$$
$$+(1-\alpha)f^{l}(t_{k,i+1},y(t_{k,i+1}),\lambda_k(y_k))],$$

and

$$\overline{y}^{\alpha}(t_{k,i+2})=\overline{y}^{\alpha}(t_{k,i-1})+\frac{h}{2}[\alpha f^{c}(t_{k,i-1},y(t_{k,i-1}),\lambda_k(y_k))$$
$$+(1-\alpha)f^{r}(t_{k,i-1},y(t_{k,i-1}),\lambda_k(y_k))]+\frac{h}{2}[\alpha f^{c}(t_{k,i},y(t_{k,i}),\lambda_k(y_k))$$
$$+(1-\alpha)f^{r}(t_{k,i},y(t_{k,i}),\lambda_k(y_k))]+2h[\alpha f^{c}(t_{k,i+1},y(t_{k,i+1}),\lambda_k(y_k))$$
$$+(1-\alpha)f^{r}(t_{k,i+1},y(t_{k,i+1}),\lambda_k(y_k))],$$

thus

$$\underline{y}^{\alpha}(t_{k,i+2})=\underline{y}^{\alpha}(t_{k,i-1})+\frac{h}{2}[\underline{f}^{\alpha}(t_{k,i-1},y(t_{k,i-1}),\lambda_k(y_k)))+\underline{f}^{\alpha}(t_{k,i},y(t_{k,i}),\lambda_k(y_k)$$
$$+4\underline{f}^{\alpha}(t_{k,i+1},y(t_{k,i+1}),\lambda_k(y_k))] \tag{4.15}$$
$$\overline{y}^{\alpha}(t_{k,i+2})=\overline{y}^{\alpha}(t_{k,i-1})+\tfrac{h}{2}[\overline{f}^{\alpha}(t_{k,i-1},y(t_{k,i-1}),\lambda_k(y_k))+\overline{f}^{\alpha}(t_{k,i},y(t_{k,i}),\lambda_k(y_k))$$
$$+4\overline{f}^{\alpha}(t_{k,i+1},y(t_{k,i+1}),\lambda_k(y_k))]. \tag{4.16}$$

Therefore, the explicit three-step method for solving the hybrid fuzzy initial value problem Eq. (3.3) is obtained as follows:

$$\underline{y}^{\alpha}(t_{k,i+2})=\underline{y}^{\alpha}(t_{k,i-1})+\tfrac{h}{2}[\underline{f}^{\alpha}(t_{k,i-1},y(t_{k,i-1}),\lambda_k(y_k))+\underline{f}^{\alpha}(t_{k,i},y(t_{k,i}),\lambda_k(y_k))$$
$$+4\underline{f}^{\alpha}(t_{k,i+1},y(t_{k,i+1}),\lambda_k(y_k))],$$

$$\overline{y}^{\alpha}(t_{k,i+2})=\overline{y}^{\alpha}(t_{k,i-1})+\tfrac{h}{2}[\overline{f}^{\alpha}(t_{k,i-1},y(t_{k,i-1}),\lambda_k(y_k))+\overline{f}^{\alpha}(t_{k,i},y(t_{k,i}),\lambda_k(y_k))$$
$$+4\overline{f}^{\alpha}(t_{k,i+1},y(t_{k,i+1}),\lambda_k(y_k))],$$

$$\underline{y}^{\alpha}(t_{k,i-1})=\alpha_0,\underline{y}^{\alpha}(t_{k,i})=\alpha_1,\underline{y}^{\alpha}(t_{k,i+1})=\alpha_2,$$
$$\overline{y}^{\alpha}(t_{k,i-1})=\alpha_3,\overline{y}^{\alpha}(t_{k,i})=\alpha_4,\overline{y}^{\alpha}(t_{k,i+1})=\alpha_5. \tag{4.17}$$

5 Implicit two-step method

In this section, the main aim is solving the fuzzy initial value problem $\widetilde{y}'_k(t) = \widetilde{f}(t, y_k(t), \lambda_k(y_k))$ by the implicit two-step method as given in [1]. Let the fuzzy initial values be $\widetilde{y}(t_{k,i-1}), \widetilde{y}(t_{k,i})$, i.e.,

$$\widetilde{f}(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k)), \widetilde{f}(t_{k,i}, y(t_{k,i}), \lambda_k(y_k)),$$

which are tTriangular fuzzy numbers and are shown as

$$\{\widetilde{f}^l(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k)), \widetilde{f}^c(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k)), \widetilde{f}^r(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k))\},$$
$$\{\widetilde{f}^l(t_{k,i}, y(t_{k,i}), \lambda_k(y_k)), \widetilde{f}^c(t_{k,i}, y(t_{k,i}), \lambda_k(y_k)), \widetilde{f}^r(t_{k,i}, y(t_{k,i}), \lambda_k(y_k))\}.$$

Consider the following fuzzy equation,

$$\widetilde{y}(t_{k,i+1}) = \widetilde{y}(t_{k,i-1}) + \int_{t_{k,i-1}}^{t_{k,i+2}} \widetilde{f}(t, y(t), \lambda_k(y_k))\ dt. \tag{5.18}$$

In a similar manner to [2], by using fuzzy linear spline interpolation for

$$\widetilde{f}(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k)), \widetilde{f}(t_{k,i}, y(t_{k,i}), \lambda_k(y_k)), \widetilde{f}(t_{k,i+1}, y(t_{k,i+1}), \lambda_k(y_k))$$

and proceeding as above we have:

$$[\widetilde{y}(t_{k,i+2})]^\alpha = [\underline{y}^\alpha(t_{k,i+2}), \overline{y}^\alpha(t_{k,i+2})],$$

Thus, the implicit two-step method is obtained as follows:

$$\begin{aligned}
\underline{y}^\alpha(t_{k,i+1}) &= \underline{y}^\alpha(t_{k,i-1}) + \frac{h}{2}[\underline{f}^\alpha(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k)) + 2\underline{f}^\alpha(t_{k,i}, y(t_{k,i}), \lambda_k(y_k)) \\
&\quad + \underline{f}^\alpha(t_{k,i+1}, y(t_{k,i+1}), \lambda_k(y_k))], \\
\overline{y}^\alpha(t_{k,i+1}) &= \overline{y}^\alpha(t_{k,i-1}) + \frac{h}{2}[\overline{f}^\alpha(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k)) + 2\overline{f}^\alpha(t_{k,i}, y(t_{k,i}), \lambda_k(y_k) \\
&\quad + \overline{f}^\alpha(t_{k,i+1}, y(t_{k,i+1}), \lambda_k(y_k))], \\
\underline{y}^\alpha(t_{k,i-1}) &= \alpha_0, \quad \underline{y}^\alpha(t_{k,i}) = \alpha_1, \quad \overline{y}^\alpha(t_{k,i-1}) = \alpha_2, \quad \overline{y}^\alpha(t_{k,i}) = \alpha_3.
\end{aligned} \tag{5.19}$$

6 Improved predictor-corrector three-step method (IPCTSM)

In this section, we present an algorithm for solving Eq. (3.3) based on the explicit three-step method as a predictor and an iteration of the implicit two-step method as a corrector.

ALGORITHM (*Improved predictor-corrector three-step method (IPCTSM)*)
Fix $k \in \mathbb{Z}^+$. To approximate the solution of the following hybrid fuzzy differential system

$$\begin{cases}
\widetilde{y}'_k(t) = \widetilde{f}(t, y_k(t), \lambda_k(y_k)), & t_k \le t \le t_{k+1}, \\
\underline{y}^\alpha(t_{k,0}) = \alpha_0, \underline{y}^\alpha(t_{k,1}) = \alpha_1, \underline{y}^\alpha(t_{k,2}) = \alpha_2, & \\
\overline{y}^\alpha(t_{k,0}) = \alpha_3, \overline{y}^\alpha(t_{k,1}) = \alpha_4, \overline{y}^\alpha(t_{k,2}) = \alpha_5, &
\end{cases}$$

an arbitrary positive integer N_k is chosen.

Step 1. Let $h = \frac{t_{k+1}-t_k}{N_k}$,

$$\underline{y}^{\alpha}(t_{k,0}) = \alpha_0, \underline{y}^{\alpha}(t_{k,1}) = \alpha_1, \underline{y}^{\alpha}(t_{k,2}) = \alpha_2,$$

$$\overline{y}^{\alpha}(t_{k,0}) = \alpha_3, \overline{y}^{\alpha}(t_{k,1}) = \alpha_4, \overline{y}^{\alpha}(t_{k,2}) = \alpha_5.$$

Step 2. Let $i = 1$.
Step 3. Let

$$\underline{y}^{(0)\alpha}(t_{k,i+2}) = \underline{y}^{\alpha}(t_{k,i-1}) + \frac{h}{2}[\underline{f}^{\alpha}(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k))$$
$$+\underline{f}^{\alpha}(t_{k,i}, y(t_{k,i}), \lambda_k(y_k)) + 4\underline{f}^{\alpha}(t_{k,i+1}, y(t_{k,i+1}), \lambda_k(y_k))],$$
$$\overline{y}^{(0)\alpha}(t_{k,i+2}) = \overline{y}^{\alpha}(t_{k,i-1}) + \frac{h}{2}[\overline{f}^{\alpha}(t_{k,i-1}, y(t_{k,i-1}), \lambda_k(y_k))$$
$$+\overline{f}^{\alpha}(t_{k,i}, y(t_{k,i}), \lambda_k(y_k)) + 4\overline{f}^{\alpha}(t_{k,i+1}, y(t_{k,i+1}), \lambda_k(y_k))].$$

Step 4. Let $t_{k,i+2} = t_{k,0} + (i + 2)h$.
Step 5. Let

$$\underline{y}^{\alpha}(t_{k,i+2}) = \underline{y}^{\alpha}(t_{k,i}) + (\frac{h}{2})\underline{f}^{\alpha}(t_{k,i}, y(t_{k,i}), \lambda_k(y_k))$$
$$+h\underline{f}^{\alpha}(t_{k,i+1}, y(t_{k,i+1}), \lambda_k(y_k)) + (\frac{h}{2})\underline{f}^{\alpha}(t_{k,i+2}, y^{(0)}(t_{k,i+2}), \lambda_k(y_k)),$$

$$\overline{y}^{\alpha}(t_{k,i+2}) = \overline{y}^{\alpha}(t_{k,i}) + (\frac{h}{2})\overline{f}^{\alpha}(t_{k,i}, y(t_{k,i}), \lambda_k(y_k))$$
$$+h\overline{f}^{\alpha}(t_{k,i+1}, y(t_{k,i+1}), \lambda_k(y_k)) + (\frac{h}{2})\overline{f}^{\alpha}(t_{k,i+2}, y^{(0)}(t_{k,i+2}), \lambda_k(y_k)).$$

Step 6. $i = i + 1$.

Step 7. If $i \leq N - 2$, go to step 3.

Step 8. The algorithm will end and $(\underline{y}^{\alpha}(t_{k+1}), \overline{y}^{\alpha}(t_{k+1}))$ approximates the real value of $(\underline{Y}^{\alpha}(t_{k+1}), \overline{Y}^{\alpha}(t_{k+1}))$.

7 Convergence

By an application of the methods suggested in [1] (Theorem (6.1) and Theorem (6.2)) and [4] (Lemma (3.1) and Theorem (3.2)), the following results can prove the convergence of the methods provided in sections 4, 5 and 6. For a fixed $k \in \mathbb{Z}^+$, to integrate the system (3.3) in $[t_0, t_1], \ldots [t_k, t_{k+1}], \ldots$, we replace each interval by a set of N_{k+1} discrete equally spaced grid point at which the exact solution $(\underline{Y}^{\alpha}(t), \overline{Y}^{\alpha}(t)$ is approximated by some $(\underline{y}^{\alpha}(t), \overline{y}^{\alpha}(t)$.

For the chosen grid point on $[t_k, t_{k+1}]$ at $t_{k,n} = t_k + nh_k$, $h_k = \frac{t_{k+1}-t_k}{N_k}$, the exact and approximate solutions are denoted by $[\widetilde{Y}_{k,n}]^\alpha = [\underline{Y}^\alpha_{k,n}, \overline{Y}^\alpha_{k,n}]$, and $[\widetilde{y}_{k,n}]^\alpha = [\underline{y}^\alpha_{k,n}, \overline{y}^\alpha_{k,n}]$, respectively.

Theorem 7.1. *For any arbitrary fixed α : $0 \le \alpha \le 1$, and $k \in Z^+$, the implicit two-step approximates of Eq. (5.19) converge to the exact solution $\underline{Y}^\alpha(t)$,$\overline{Y}^\alpha(t)$ for $\underline{Y}, \overline{Y} \in C^3[t_0, t_{k+1}]$.*

Proof: It is sufficient to show

$$\lim_{h_0,\cdots,h_k \to 0} \underline{y}^\alpha_{k,N_k} = \underline{Y}^\alpha(t_{k+1}),$$

$$\lim_{h_0,\cdots,h_k \to 0} \overline{y}^\alpha_{k,N_k} = \overline{Y}^\alpha(t_{k+1}).$$

The proof is similar to that of Theorem (6.1) in [1].

Theorem 7.2. *For any arbitrary fixed α : $0 \le \alpha \le 1$, and $k \in \mathbb{Z}^+$, the explicit three-step approximates of Eq.(4.17) converge to the exact solution $\underline{Y}^\alpha(t)$,$\overline{Y}^\alpha(t)$ for $\underline{Y}, \overline{Y} \in C^3[t_0, t_{k+1}]$.*

Proof. The proof is similar to that of Theorem (6.2) in [1].

Theorem 7.3. *The explicit three-step method is stable.*

Proof. For the explicit three-step method, there exists only one characteristic polynomial $p(\lambda) = \lambda^3 - \lambda$, then it satisfies the root condition and, therefore, it is a stable method.

Theorem 7.4. *The implicit two-step method is stable.*

Proof. Similar to Theorem (7.3).

Clearly, according to the above-mentioned theorems, the IPCTSM is convergent and stable.

8 Examples

In this section, we present two examples to illustrate the IPCTSM and also compare the results of this method with Euler's method.

Example 8.1. *[8] Consider the initial value problem*

$$\begin{cases} \widetilde{y}'(t) = \widetilde{y}(t) + m(t)\lambda_k(y(t_k)), & t \in [t_k, t_{k+1}], t_k = k, \\ [\widetilde{y}(0)] = [0.75 + 0.25\alpha, 1.125 - 0.125\alpha] & 0 \le \alpha \le 1, \\ [\widetilde{y}(0.1)] = [(0.75 + 0.25\alpha)e^{0.1}, (1.125 - 0.125\alpha)e^{0.1}], & \\ [\widetilde{y}(0.2)] = [(0.75 + 0.25\alpha)e^{0.2}, (1.125 - 0.125\alpha)e^{0.2}] & \end{cases} \tag{8.20}$$

where

$$m(t) = \begin{cases} 2(t(mod1)) & if \quad t(mod1) \leq (0.5), \\ 2(1 - t(mod1)) & if \quad t(mod1) > (0.5), \end{cases}$$

$$\lambda_k(\mu) = \begin{cases} \widehat{0} & if \quad k = 0, \\ \mu & if \quad k \in \{1, 2, \cdots\}. \end{cases}$$

Now, by using the example presented in [6], it is clear that for each $k = 0, 1, 2, \cdots$, the fuzzy initial value problem

$$\begin{cases} \widetilde{y}'(t) = \widetilde{y}(t) + m(t)\lambda_k(y(t_k)), \quad t \in [t_k, t_{k+1}], \quad t_k = k, \\ y(t_k) = y_{t_k}, \\ y(t_{k-1}) = y_{t_{k-1}}, \\ y(t_{k-2}) = y_{t_{k-2}}, \end{cases} \tag{8.21}$$

has a unique solution on $[t_k, t_{k+1}]$.
For $t \in [0, 1]$, the exact solution of Eq. (8.20) satisfies

$$[y(t)]^\alpha = [(0.75 + 0.25\alpha)e^t, (1.125 - 0.125\alpha)e^t]. \tag{8.22}$$

For $t \in [1, 2]$, the exact solution of Eq. (8.20) satisfies the following equation

$$[y(t)]^\alpha = y^\alpha(1)(2t - 2 + e^{t-1.5}(3\sqrt{e} - 4)). \tag{8.23}$$

By using the IPCTSM, we have presented the numerical solution of this example at $t = 2$ in Fig. 1. Also, by using Euler's method for this example at $t = 2$ with $N = 10$ and the initial value as

$$y(0)]^\alpha = [(0.75 + 0.25\alpha)e, (1.125 - 0.125\alpha)e],$$

the results are shown in Fig. 2.

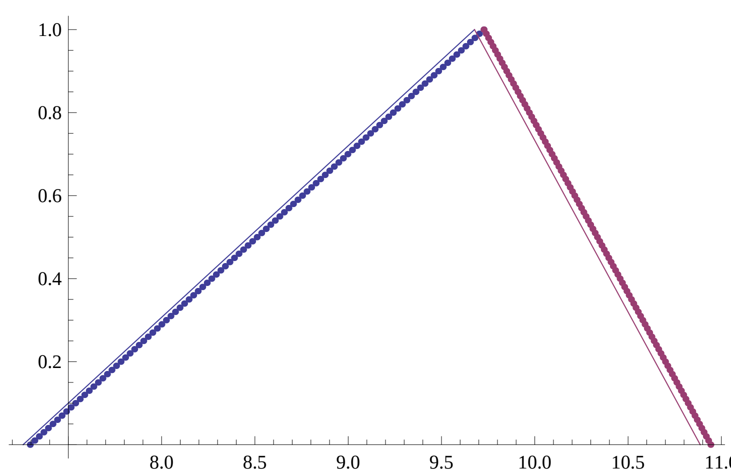

Fig. 1. The solid line and the dotted line show the initial fuzzy number and the numerical solution by using the IPCTSM, respectively.

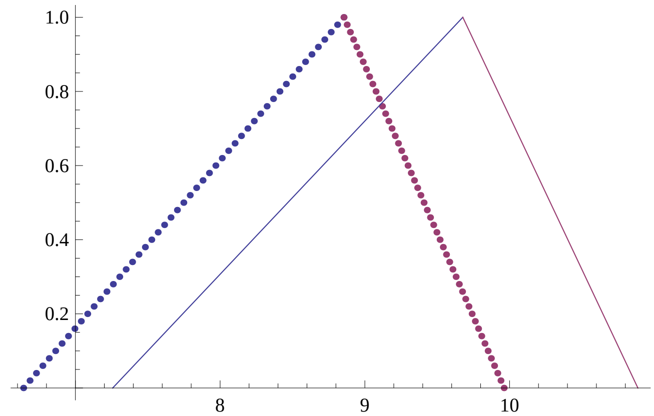

Fig. 2. The solid line and the dotted line show the initial fuzzy number and the numerical solution by using Euler's method, respectively.

Example 8.2. *[8] Consider the initial value problem*

$$\begin{cases} \widetilde{y}'(t) = -\widetilde{y}(t) + m(t)\lambda_k(y(t_k)), & t \in [t_k, t_{k+1}], t_k = k, \\ [\widetilde{y}(0)] = [0.75 + 0.25\alpha, 1.125 - 0.125\alpha] & 0 \le \alpha \le 1, \\ [\widetilde{y}(0.1)] = [-0.1875e^{0.1} + 0.9375e^{-0.1} + (e^{-0.1} + 0.1875e^{0.1} - 0.9375e^{-0.1})\alpha, \\ \qquad 0.1875e^{0.1} + 0.9375e^{-0.1} + (e^{-0.1} - 0.1875e^{0.1} - 0.9375e^{-0.1})\alpha], \\ [\widetilde{y}(0.2)] = [-0.1875e^{0.2} + 0.9375e^{-0.2} + (e^{-0.2} + 0.1875e^{0.2} - 0.9375e^{-0.2})\alpha, \\ \qquad 0.1875e^{0.2} + 0.9375e^{-0.2} + (e^{-0.2} - 0.1875e^{0.2} - 0.9375e^{-0.2})\alpha], \end{cases} \tag{8.24}$$

where

$$m(t) = |\sin(\pi t)|,$$

$$\lambda_k(\mu) = \begin{cases} \widehat{0} & if \quad k = 0 \\ \mu & if \quad k \in \{1, 2, \cdots\}. \end{cases}$$

For $t \in [0, 1]$ *and for each* $k = 0, 1, 2, \cdots$, *the exact solution of Eq.* (8.24) *satisfies in the following equation; see [1],*

$$y^\alpha(t) = [(-0.1875e^t + 0.9375e^{-t}) + (e^{-t} + 0.1875e^t - 0.9375e^{-t})\alpha,$$

$$(0.1875e^t + 0.9375e^{-t}) + (e^{-t} - 0.1875e^t - 0.9375e^{-t})\alpha]. \tag{8.25}$$

For more details, see Fig. 3 and Fig. 4.

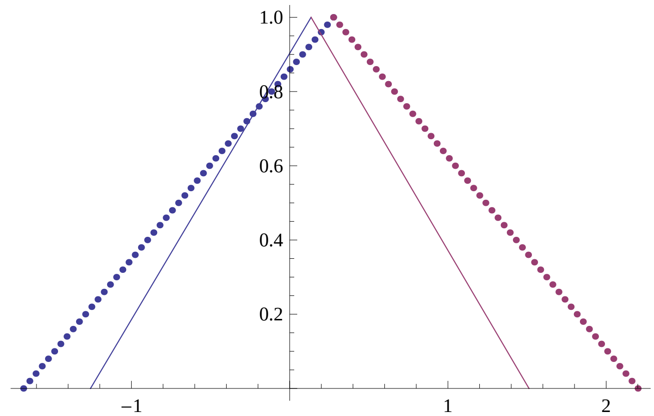

Fig. 3. The solid line and the dotted line show the initial fuzzy number and the numerical solution by using the IPCTSM, respectively .

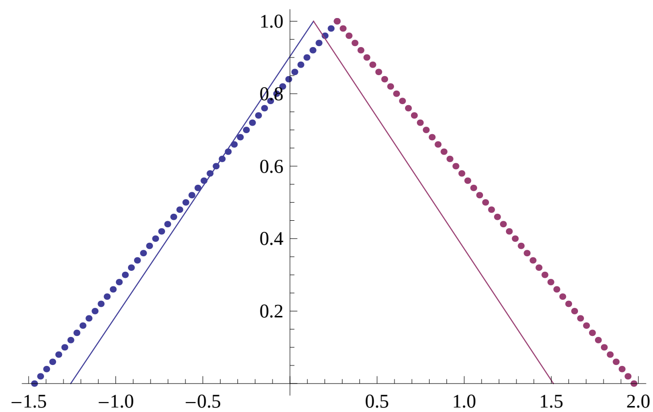

Fig. 4. The solid line and the dotted line show the initial fuzzy number and the numerical solution by using Euler's method, respectively.

9 Conclusion

In this paper, we developed the numerical solution of the hybrid fuzzy differential system by the improved predictor-corrector method, in which the explicit three-step method is used as a predictor and also an iteration of the implicit two-step method as a corrector. By Theorems (7.1), (7.2), (7.3) and (7.4), we proved that the improved predictor corrector three-step method is convergent and stable. We presented the examples based on examples in [1, 7, 8, 9, 10] to illustrate our results using the improved predictor-corrector three-step method and the Euler method.

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