Available online at http://ijim.srbiau.ac.ir

Int. J. Industrial Mathematics Vol. 1, No. 2 (2009) 105-120

Solving Two-Dimensional Fuzzy Partial Differential Equation by the Alternating Direction Implicit Method

M. Barkhordari Ahmadi [a]*, N. A. Kiani[b]

(a) *Department of Mathematics, Bandar Abbas Branch, Islamic Azad University, Bandar Abbas, Iran.*

(b) *Viroquant Research Group Modeling, University of Heidelberg, Germany.*

Abstract

In this paper, the fuzzy partial differential equation is investigated by using the strongly generalized differentiability concept. The alternating direction implicit(ADI) method is proposed for approximating the solution of the two-dimensional heat equation where the initial and boundary conditions are fuzzy numbers. The algorithm is illustrated by solving several examples.

Keywords: Fuzzy-number, Fuzzy-valued function, Generalized differentiability, Fuzzy partial differential equation, ADI method.

1 Introduction

Proper design for engineering applications requires detailed information of the system-property distributions such as temperature, velocity, density, etc., in the space and time domain. This information can be obtained by either experimental measurement or computational simulation. Although experimental measurement is reliable, it needs a lot of effort and time. Therefore, the computational simulation has become a more and more popular method as a design tool since it needs only a fast computer with a large memory. Frequently, the engineering design problems deal with a set of partial differential equations(PDEs), which are to be numerically solved, such as heat transfer and solid and fluid mechanics. Numerical methods are widely applied to pre-assigned grid points to solve partial differential equations [12]. When a physical problem is transformed into a deterministic parabolic partial differential equation, we cannot usually be sure that this modeling is perfect. Also, the initial and boundary value may not be known exactly. If

*Corresponding author. Email address: mahnaz_barkhordari@yahoo.com.

the nature of errors is random, then instead of a deterministic problem, we get a random partial differential equation with random initial and boundary values. But if the underlying structure is not probabilistic, e.g., because of subjective choice, then it may be appropriate to use fuzzy numbers instead of real random variables. The concept of fuzzy derivative was first introduced by Chang and Zadeh [9], and it was followed up by Dobois and Prade [13], who used the extension principle in their approach. Other methods have been discussed by Puri and Ralescu [23] and by Goetschel and Voxman [16]. Also, strongly generalized differentiability was introduced by Bede in [5, 7] and studied in [6].The notion of fuzzy differential equation was initially introduced by Kandel and Byatt and later applied in fuzzy processes and fuzzy dynamical systems. A thorough theoretical research of fuzzy Cauchy problems was given by Kaleva [19], Seikkala [24], Ouyang and Wu [17], and Kloeden and Wu [21]. A generalization of a fuzzy differential equation was given by Aubin, Baidosov, Leland and Colombo and Krivan. The numerical methods for solving fuzzy differential equations are introduced in [1, 2, 20]. Fuzzy partial differential equations were formulated by Buckly [8]; and Allahviranloo [3] used a numerical method to solve the fuzzy partial differential equation (FPDE).

In this paper, we are going to solve FPDEs by the ADI method. The rest of this paper is organized as follows:

Section 2 contains the basic material to be used in the paper. In section 3, the fuzzy partial differential equations is introduced by using the strongly generalized differentiability concept [6] and we propose the ADI method for approximating the solution of two-dimensional fuzzy partial differential equations. The proposed algorithm is illustrated by solving some examples in section 4, and the conclusion is drawn in section 5.

2 Preliminaries

We now recall some definitions needed throughout the paper. The basic definition of fuzzy numbers is given in [13, 15].

By R we denote the set of all real numbers. A fuzzy number is a mapping $u : R \to [0,1]$ with the following properties:

(a) u is upper semi-continuous,

(b) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq min\{u(x), u(y)\}$ for all $x, y \in R, \lambda \in [0,1]$,

(c) u is normal, i.e.,$\exists x_0 \in R$ for which $u(x_0) = 1$,

(d) supp $u = \{x \in R \mid u(x) > 0\}$ is the support of the u, and its closure cl(supp u) is compact.

Let E be the set of all fuzzy numbers on R. The r-level set of a fuzzy number $u \in E$, $o \leq r \leq 1$, denoted by $[u]_r$, is defined as

$$[u]_r = \begin{cases} \{x \in R \mid u(x) \geq r\} & if \quad 0 \leq r \leq 1 \\ cl(supp\ u) & if \quad r = 0 \end{cases}$$

It is clear that the r-level set of a fuzzy number is a closed and bounded interval $[\underline{u}(r), \overline{u}(r)]$, where $\underline{u}(r)$ denotes the left-hand endpoint of $[u]_r$ and $\overline{u}(r)$ denotes the right-hand endpoint of $[u]_r$. Since each $y \in R$ can be regarded as a fuzzy number $\widetilde{y}$ defined by

$$\widetilde{y}(t) = \begin{cases} 1 & if \quad t = y \\ o & if \quad t \neq y \end{cases}$$

R can be embedded in E.

Remark 2.1. *(See [26]) Let X be the Cartesian product of universes $X = X_1 \times ... \times X_n$, and $A_1, \ldots, A_n$ be n fuzzy numbers in $X_1, \ldots, X_n$, respectively. f is a mapping from X to a universe Y, $y = f(x_1, ..., x_n)$. Then the extension principle allows us to define a fuzzy set B in Y by*

$$B = \{(y, u(y)) \mid y = f(x_1, ..., x_n), (x_1, ..., x_n) \in X\}$$

where

$$u_B(y) = \begin{cases} \sup_{(x_1,\ldots,x_n)\in f^{-1}(y)} \min\{u_{A_1}(x_1), ..., u_{A_n}(x_n))\}, & if \quad f^{-1}(y) \neq 0, \\ 0 & otherwise. \end{cases}$$

where f^{-1} is the inverse of f.
For $n = 1$, the extension principle reduces to

$$B = \{(y, u_B(y)) \mid y = f(x), x \in X\}$$

where

$$u_B(y) = \begin{cases} \sup_{x\in f^{-1}(y)} u_A(x), & if \quad f^{-1}(y) \neq 0, \\ 0 & otherwise. \end{cases}$$

According to Zadeh's extension principle, the addition operation on E is defined by

$$(u \oplus v)(x) = sup_{y\in R} min\{u(y), v(x - y)\}, \quad x \in R$$

and scalar multiplication of a fuzzy number is given by

$$(k \odot u)(x) = \begin{cases} u(x/k), & k > 0, \\ \tilde{0}, & k = 0, \end{cases}$$

where $\tilde{0} \in E$.
It is well known that the following properties are true for all levels

$$[u \oplus v]_r = [u]_r + [v]_r, \qquad [k \odot u]_r = k[u]_r$$

From this characteristic of fuzzy numbers, we see that a fuzzy number is determined by the endpoints of the intervals $[u]_r$. This leads to the following characteristic representation of a fuzzy number in terms of the two "endpoint" functions $\underline{u}(r)$ and $\overline{u}(r)$. An equivalent parametric definition is also given in ([14, 20]) as:

Definition 2.1. *A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$, $0 \le r \le 1$, which satisfy the following requirements:*

1. *$\underline{u}(r)$ is a bounded non-decreasing left continuous function in $(0, 1]$, and right continuous at 0,*
2. *$\overline{u}(r)$ is a bounded non-increasing left continuous function in $(0, 1]$, and right continuous at 0,*
3. *$\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.*

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha,\ 0 \leq r \leq 1$. We recall that for $a < b < c$, where $a, b, c \in R$, the triangular fuzzy number $u = (a, b, c)$ determined by a, b, c is given such that $\underline{u}(r) = a + (b - c)r$ and $\overline{u}(r) = c - (c - b)r$ are the endpoints of the r-level sets, for all $r \in [0, 1]$.
For arbitrary $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and $k > 0$ we define addition $u \oplus v$, subtraction $u \ominus v$ and scalar multiplication by k as (See [14, 20])

(a) Addition:

$$u \oplus v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$$

(b) Subtraction:

$$u \ominus v = (\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r))$$

(c) Scalar multiplication:

$$k \odot u = \begin{cases} (k\underline{u}, k\overline{u}), & k \geq 0, \\ (k\overline{u}, k\underline{u}), & k < 0. \end{cases}$$

The Hausdorff distance between fuzzy numbers given by $D : E \times E \longrightarrow R_+ \bigcup 0$, is

$$D(u, v) = \sup_{r \in [0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\},$$

where $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r)) \subset R$ are utilized (See [6]). Then, it is easy to see that D is a metric in E and has the following properties (See [22])
$(i) D(u \oplus w, v \oplus w) = D(u, v), \quad \forall u, v, w \in E,$
$(ii) D(k \odot u, k \odot v) = |k| D(u, v), \quad \forall k \in R, u, v \in E,$
$(iii) D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e), \quad \forall u, v, w, e \in E,$
$(iV)(D, E)$ is a complete metric space.

Theorem 2.1. *(See [4]) (i) If we define $\widetilde{0} = \chi_0$, then $\widetilde{0} \in E$ is a neutral element with respect to addition, i.e., $u + \widetilde{0} = \widetilde{0} + u = u$, for all $u \in E$.*
(ii) With respect to $\widetilde{0}$, none of $u \in E \setminus R$ has an opposite in E.
(iii) For any $a, b \in R$ with $a, b \geq 0$ or $a, b \leq 0$ and any $u \in E$, we have $(a+b).u = a.u+b.u$; for the general $a, b \in R$, the above property does not necessarily hold.
(iv) For any $\lambda \in R$ and any $u, v \in E$, we have $\lambda.(u + v) = \lambda.u + \lambda.v$;
(v) For any $\lambda, \mu \in R$ and any $u \in E$, we have $\lambda.(\mu.u) = (\lambda.\mu).u$;

Definition 2.2. *Let E be a set of all fuzzy numbers, we say that f is a fuzzy- valued-function if $f : R \rightarrow E$*

Definition 2.3. *(See [23]). Let $x, y \in E$. If there exists $z \in E$ such that $x = y + z$, then z is called the H-difference of x and y, and it is denoted by $x \ominus y$.*

In this paper, the sign "$\ominus$" always stands for H-difference, and also note that $x \ominus y \neq x + (-y)$.

Definition 2.4. *(See [6, 7]) Let $f:(a,b)\to E$ and $x_0\in(a,b)$. f is said to be a strongly generalized differential at x_0 (Bede differential) if there exists an element $f'(x_0)\in E$, such that*
(i) for all $h>0$ sufficiently small, $\exists f(x_0+h)\ominus f(x_0)$, $\exists f(x_0)\ominus f(x_0-h)$ and the limits(in the metric D)
$\lim_{h\searrow 0}\frac{f(x_0+h)\ominus f(x_0)}{h}=\lim_{h\searrow 0}\frac{f(x_0)\ominus f(x_0-h)}{h}=f'(x_0)$,
or
(ii) for all $h>0$ sufficiently small, $\exists f(x_0)\ominus f(x_0+h)$, $\exists f(x_0-h)\ominus f(x_0)$ and the limits (in the metric D)
$\lim_{h\searrow 0}\frac{f(x_0)\ominus f(x_0+h)}{-h}=\lim_{h\searrow 0}\frac{f(x_0-h)\ominus f(x_0)}{-h}=f'(x_0)$,
or
(iii) for all $h>0$ sufficiently small, $\exists f(x_0+h)\ominus f(x_0)$, $\exists f(x_0-h)\ominus f(x_0)$ and the limits (in the metric D)
$\lim_{h\searrow 0}\frac{f(x_0+h)\ominus f(x_0)}{h}=\lim_{h\searrow 0}\frac{f(x_0-h)\ominus f(x_0)}{-h}=f'(x_0)$,
or
(iv) for all $h>0$ sufficiently small, $\exists f(x_0)\ominus f(x_0+h)$, $\exists f(x_0)\ominus f(x_0-h)$ and the limits (in the metric D)
$\lim_{h\searrow 0}\frac{f(x_0)\ominus f(x_0)+h}{-h}=\lim_{h\searrow 0}\frac{f(x_0)\ominus f(x_0-h)}{h}=f'(x_0)$,
(h and $-h$ at the denominators mean $\frac{1}{h}$ and $\frac{-1}{h}$, respectively).

In the special case when$f$ is a fuzzy-valued function, we have the following result.

Theorem 2.2. *(See [10]). Let $f:R\to E$ be a function and denote $f(t)=(\underline{f}(t,r),\overline{f}(t,r))$, for each $r\in[0,1]$. Then*
(1) if f is differentiable in the first form (i), then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are differentiable functions and
$f'(t)=(\underline{f}'(t,r),\overline{f}'(t,r))$.
(2) if f is differentiable in the second form (ii), then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are differentiable functions and
$f'(t)=(\overline{f}'(t,r),\underline{f}'(t,r))$.

Definition 2.5. *We Define the n-th order differential of f as follows: Let $f:(a,b)\to E$ and $x_0\in(a,b)$. We say that f is strongly generalized differentiable of the n-th order at x_0 if there exists an element $f^{(s)}(x_0)\in E$, $\quad\forall s=1\ldots n$, such that*
(i) for all $h>0$ sufficiently small, $\exists f^{(s-1)}(x_0+h)\ominus f^{(s-1)}(x_0)$,
$\exists f^{(s-1)}(x_0)\ominus f^{(s-1)}(x_0-h)$ and the limits(in the metric d_∞)
$\lim_{h\searrow 0}\frac{f^{(s-1)}(x_0+h)\ominus f^{(s-1)}(x_0)}{h}=\lim_{h\searrow 0}\frac{f^{(s-1)}(x_0)\ominus f^{(s-1)}(x_0-h)}{h}=f^{(s)}(x_0)$
or
(ii) for all $h>0$ sufficiently small, $\exists f^{(s-1)}(x_0)\ominus f^{(s-1)}(x_0+h)$,
$\exists f^{(s-1)}(x_0-h)\ominus f^{(s-1)}(x_0)$ and the limits(in the metric d_∞)
$\lim_{h\searrow 0}\frac{f^{(s-1)}(x_0)\ominus f^{(s-1)}(x_0+h)}{-h}=\lim_{h\searrow 0}\frac{f^{(s-1)}(x_0-h)\ominus f(x_0)}{-h}=f^{(s)}(x_0)$
or
(iii) for all $h>0$ sufficiently small, $\exists f^{(s-1)}(x_0+h)\ominus f^{(s-1)}(x_0)$,
$\exists f^{(s-1)}(x_0-h)\ominus f^{(s-1)}(x_0)$ and the limits(in the metric d_∞)
$\lim_{h\searrow 0}\frac{f^{(s-1)}(x_0+h)\ominus f^{(s-1)}(x_0)}{h}=\lim_{h\searrow 0}\frac{f^{(s-1)}(x_0-h)\ominus f^{(s-1)}(x_0)}{-h}=f^{(s)}(x_0)$
or
(iv) for all $h>0$ sufficiently small, $\exists f^{(s-1)}(x_0)\ominus f^{(s-1)}(x_0+h)$,

$\exists f^{(s-1)}(x_0) \ominus f^{(s-1)}(x_0-h)$ *and the limits(in the metric* d_∞*)*
$\lim_{h\searrow 0} \frac{f^{(k-1)}(x_0)\ominus f^{(s-1)}(x_0+h)}{-h} = \lim_{h\searrow 0} \frac{f^{(s-1)}(x_0)\ominus f^{(s-1)}(x_0-h)}{h} = f^{(s)}(x_0)$
(h *and* $-h$ *at denominators mean* $\frac{1}{h}$ *and* $\frac{-1}{h}$*, respectively* $\forall i = 1 \ldots n$*)*

3 Two-dimensional fuzzy partial differential equation

The purpose of this section is to present the following 2D fuzzy partial differential equation by using the Bede derivative :

$$\frac{du}{dt} = k(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}) \qquad (k \quad is \quad constant)$$

with the fuzzy initial condition

$$u(0,x,y) = \widetilde{\ell}_1 \in E$$

and the fuzzy boundary conditions

$$u(t,0,y) = \widetilde{\ell}_2 \in E$$
$$u(t,h,y) = \widetilde{\ell}_3 \in E$$
$$u(t,x,0) = \widetilde{\ell}_4 \in E$$
$$u(t,x,b) = \widetilde{\ell}_5 \in E$$

For solving a 2D fuzzy partial differential equation by using the Bede derivative, we have four different cases:

Case(1): If we consider $\frac{du}{dt}$,$\frac{d^2u}{dx^2}$ and $\frac{d^2u}{dy^2}$ by using (i)-differentiability, or $\frac{du}{dt}$,$\frac{d^2u}{dx^2}$ and $\frac{d^2u}{dy^2}$ by using (ii)-differentiability, then we have:

$$\frac{d\underline{u}}{dt}(t,x,y,r) = k(\frac{d^2\underline{u}}{dx^2}(t,x,y,r) + \frac{d^2\underline{u}}{dy^2}(t,x,y,r))$$

and

$$\frac{d\overline{u}}{dt}(t,x,y,r) = k(\frac{d^2\overline{u}}{dx^2}(t,x,y,r) + \frac{d^2\overline{u}}{dy^2}(t,x,y,r)) \tag{3.1}$$

with the initial condition

$$\underline{u}(0,x,y,r) = \underline{\ell}_1(r) \quad and \quad \overline{u}(0,x,y,r) = \overline{\ell}_1(r)$$

and the boundary conditions

$$\underline{u}(t,0,y,r) = \underline{\ell}_2(r) \quad and \quad \overline{u}(t,0,y,r) = \overline{\ell}_2(r)$$
$$\underline{u}(t,h,y,r) = \underline{\ell}_3(r) \quad and \quad \overline{u}(t,h,y,r) = \overline{\ell}_3(r)$$
$$\underline{u}(t,x,0,r) = \underline{\ell}_4(r) \quad and \quad \overline{u}(t,x,0,r) = \overline{\ell}_4(r)$$
$$\underline{u}(t,x,b,r) = \underline{\ell}_5(r) \quad and \quad \overline{u}(t,x,b,r) = \overline{\ell}_5(r).$$

By using the ADI numerical method, we have:

$$\begin{cases} -d_1\underline{u}_{i-1,j}^{n+0.5}(r)+\underline{u}_{i,j}^{n+0.5}(r)+2d_1\overline{u}_{i+1,j}^{n+0.5}(r)-d_1\underline{u}_{i+1,j}^{n+0.5}(r) & =d_2\underline{u}_{i,j+1}^{n}(r)+(1-2d_2)\overline{u}_{i,j}^{n}(r) \\ & +d_2\underline{u}_{i,j-1}^{n}(r) \\ -d_1\overline{u}_{i-1,j}^{n+0.5}(r)+\overline{u}_{i,j}^{n+0.5}(r)+2d_1\underline{u}_{i+1,j}^{n+0.5}(r)-d_1\overline{u}_{i+1,j}^{n+0.5}(r) & =d_2\overline{u}_{i,j+1}^{n}(r)+(1-2d_2)\underline{u}_{i,j}^{n}(r) \\ & +d_2\overline{u}_{i,j-1}^{n}(r) \end{cases} \tag{3.2}$$

and

$$\begin{cases} -d_2\underline{u}_{i,j-1}^{n+1}(r)+(\underline{u}_{i,j}^{n+1}(r)+2d_2\overline{u}_{i,j}^{n+1}(r))-d_2\underline{u}_{i,j+1}^{n+1}(r) & =d_1\underline{u}_{i+1,j}^{n+0.5}(r)+(1-2d_1)\overline{u}_{i,j}^{n+0.5}(r) \\ & +d_1\underline{u}_{i-1,j}^{n+0.5}(r) \\ -d_2\overline{u}_{i,j-1}^{n+1}(r)+(\overline{u}_{i,j}^{n+1}(r)+2d_2\underline{u}_{i,j}^{n+1}(r))-d_2\overline{u}_{i,j+1}^{n+1}(r) & =d_1\overline{u}_{i+1,j}^{n+0.5}(r)+(1-2d_1)\underline{u}_{i,j}^{n+0.5}(r) \\ & +d_1\overline{u}_{i-1,j}^{n+0.5}(r) \end{cases} \tag{3.3}$$

where $d_1=\frac{1}{2}\alpha(\frac{\Delta t}{(\Delta x)^2})$ and $d_2=\frac{1}{2}\alpha(\frac{\Delta t}{(\Delta y)^2})$. Also $t=1,...,n_t$,$i=1,...,n_x$ and $j=1,...,n_y$. From (3.2) and (3.3) we have two crisp linear systems for all i and j which can be displayed as follows:

$$\begin{bmatrix} A_1 & B_1 \\ B_1 & A_1 \end{bmatrix}\begin{bmatrix} \underline{u}_j^{n+0.5} \\ \overline{u}_j^{n+0.5} \end{bmatrix}=\begin{bmatrix} d_2\underline{u}_{j+1}^{n}(r)+(1-2d_2)\overline{u}_j^{n}(r)+d_2\underline{u}_{j-1}^{n}(r) \\ d_2\overline{u}_{j+1}^{n}(r)+(1-2d_2)\underline{u}_j^{n}(r)+d_2\overline{u}_{j-1}^{n}(r) \end{bmatrix} \tag{3.4}$$

$$\begin{bmatrix} A_2 & B_2 \\ B_2 & A_2 \end{bmatrix}\begin{bmatrix} \underline{u}_i^{n+1} \\ \overline{u}_i^{n+1} \end{bmatrix}=\begin{bmatrix} d_1\underline{u}_{i+1}^{n+0.5}(r)+(1-2d_1)\overline{u}_i^{n+0.5}(r)+d_1\underline{u}_{i-1}^{n+0.5}(r) \\ d_1\overline{u}_{i+1}^{n}(r)+(1-2d_1)\underline{u}_i^{n+0.5}(r)+d_1\overline{u}_{i-1}^{n+0.5}(r) \end{bmatrix} \tag{3.5}$$

where $B_1=2d_1I_{n_x\times n_x}$, $B_2=2d_2I_{n_y\times n_y}$,

$$A_1=\begin{bmatrix} 1 & -d_1 & 0 & ... & 0 \\ -d_1 & 1 & -d_1 & ... & 0 \\ \vdots & \vdots & \vdots & ... & \vdots \\ 0 & ... & -d_1 & 1 & -d_1 \\ 0 & 0 & ... & -d_1 & 1 \end{bmatrix}_{n_x\times n_x} \quad \text{and } A_2=\begin{bmatrix} 1 & -d_2 & 0 & ... & 0 \\ -d_2 & 1 & -d_2 & ... & 0 \\ \vdots & \vdots & \vdots & ... & \vdots \\ 0 & ... & -d_2 & 1 & -d_2 \\ 0 & 0 & ... & -d_2 & 1 \end{bmatrix}_{n_y\times n_y}$$

We solve system (3.4), then the solution of system (3.4) is set in system (3.5) to obtain its solutions.

Case(2): If we consider $\frac{du}{dt}$,$\frac{d^2u}{dx^2}$ by using (i)-differentiability and $\frac{d^2u}{dy^2}$ by using (ii)-differentiability, or $\frac{du}{dt}$,$\frac{d^2u}{dx^2}$ by using (ii)-differentiability and $\frac{d^2u}{dy^2}$ by using (i)-differentiability, then we solve the PDE system:

$$\frac{d\underline{u}}{dt}(t,x,y,r)=k(\frac{d^2\underline{u}}{dx^2}(t,x,y,r)+\frac{d^2\overline{u}}{dy^2}(t,x,y,r))$$

and

$$\frac{d\overline{u}}{dt}(t,x,y,r) = k(\frac{d^2\overline{u}}{dx^2}(t,x,y,r) + \frac{d^2\underline{u}}{dy^2}(t,x,y,r)) \tag{3.6}$$

with the initial condition

$$\underline{u}(0,x,y,r) = \underline{\ell}_1(r) \quad and \quad \overline{u}(0,x,y,r) = \overline{\ell}_1(r)$$

and the boundary conditions

$$\begin{aligned}
\underline{u}(t,0,y,r) &= \underline{\ell}_2(r) \quad and \quad \overline{u}(t,0,y,r) = \overline{\ell}_2(r)\\
\underline{u}(t,h,y,r) &= \underline{\ell}_3(r) \quad and \quad \overline{u}(t,h,y,r) = \overline{\ell}_3(r)\\
\underline{u}(t,x,0,r) &= \underline{\ell}_4(r) \quad and \quad \overline{u}(t,x,0,r) = \overline{\ell}_4(r)\\
\underline{u}(t,x,b,r) &= \underline{\ell}_5(r) \quad and \quad \overline{u}(t,x,b,r) = \overline{\ell}_5(r).
\end{aligned}$$

By using the ADI numerical method, we have:

$$\begin{cases}
-d_1\underline{u}^{n+0.5}_{i-1,j}(r) + \underline{u}^{n+0.5}_{i,j}(r) + 2d_1\overline{u}^{n+0.5}_{i,j}(r) - d_1\underline{u}^{n+0.5}_{i+1,j}(r) & = d_2\overline{u}^{n}_{i,j+1}(r) + \overline{u}^{n}_{i,j}(r) \\
& \quad -2d_2\underline{u}^{n}_{i,j}(r) + d_2\overline{u}^{n}_{i,j-1}(r) \\
-d_1\overline{u}^{n+0.5}_{i-1,j}(r) + \overline{u}^{n+0.5}_{i,j}(r) + 2d_1\underline{u}^{n+0.5}_{i,j}(r) - d_1\overline{u}^{n+0.5}_{i+1,j}(r) & = d_2\underline{u}^{n}_{i,j+1}(r) + \underline{u}^{n}_{i,j}(r) \\
& \quad -2d_2\overline{u}^{n}_{i,j}(r) + d_2\underline{u}^{n}_{i,j-1}(r)
\end{cases} \tag{3.7}$$

and

$$\begin{cases}
-d_2\overline{u}^{n+1}_{i,j-1}(r) + (1+2d_2)\underline{u}^{n+1}_{i,j}(r) - d_2\overline{u}^{n+1}_{i,j+1}(r) & = d_1\underline{u}^{n+0.5}_{i+1,j}(r) + (1-2d_1)\overline{u}^{n+0.5}_{i,j}(r) \\
& \quad +d_1\underline{u}^{n+0.5}_{i-1,j}(r) \\
-d_2\underline{u}^{n+1}_{i,j-1}(r) + (1+2d_2)\overline{u}^{n+1}_{i,j}(r) - d_2\underline{u}^{n+1}_{i,j+1}(r) & = d_1\overline{u}^{n+0.5}_{i+1,j}(r) + (1-2d_1)\underline{u}^{n+0.5}_{i,j}(r) \\
& \quad +d_1\overline{u}^{n+0.5}_{i-1,j}(r)
\end{cases} \tag{3.8}$$

where $d_1 = \frac{1}{2}\alpha(\frac{\Delta t}{(\Delta x)^2})$ and $d_2 = \frac{1}{2}\alpha(\frac{\Delta t}{(\Delta y)^2})$. Also $t = 1, ..., n_t$,$i = 1, ..., n_x$ and $j = 1, ..., n_y$.

From (3.7) and (3.8) we have two crisp linear systems for all i and j which can be displayed as follows:

$$\begin{bmatrix} A_1 & B_1 \\ B_1 & A_1 \end{bmatrix} \begin{bmatrix} \underline{u}^{n+0.5}_j(r) \\ \overline{u}^{n+0.5}_j(r) \end{bmatrix} = \begin{bmatrix} d_2\overline{u}^{n}_{i,j+1}()r + \overline{u}^{n}_{i,j}(r) - 2d_2\underline{u}^{n}_{i,j}(r) + d_2\overline{u}^{n}_{i,j-1}(r) \\ d_2\underline{u}^{n}_{i,j+1}(r) + \underline{u}^{n}_{i,j}(r) - 2d_2\overline{u}^{n}_{i,j}(r) + d_2\underline{u}^{n}_{i,j-1}(r) \end{bmatrix} \tag{3.9}$$

$$\begin{bmatrix} A_2 & B_2 \\ B_2 & A_2 \end{bmatrix} \begin{bmatrix} \underline{u}^{n+1}_i(r) \\ \overline{u}^{n+1}_i(r) \end{bmatrix} = \begin{bmatrix} d_1\underline{u}^{n+0.5}_{i+1,j}(r) + (1-2d_1)\overline{u}^{n+0.5}_{i,j}(r) + d_1\underline{u}^{n+0.5}_{i-1,j}(r) \\ d_1\overline{u}^{n+0.5}_{i+1,j}(r) + (1-2d_1)\underline{u}^{n+0.5}_{i,j}(r) + d_1\overline{u}^{n+0.5}_{i-1,j}(r) \end{bmatrix} \tag{3.10}$$

where $B_1 = 2d_1 I_{n_x \times n_x}$, $A_2 = (1+2d_2)I_{n_y \times n_y}$,

$$A_1 = \begin{bmatrix} 1 & -d_1 & 0 & ... & 0 \\ -d_1 & 1 & -d_1 & ... & 0 \\ \vdots & \vdots & \vdots & ... & \vdots \\ 0 & ... & -d_1 & 1 & -d_1 \\ 0 & 0 & ... & -d_1 & 1 \end{bmatrix}_{n_x \times n_x} \quad \text{and } B_2 = \begin{bmatrix} 0 & -d_2 & 0 & ... & 0 \\ -d_2 & 0 & -d_2 & ... & 0 \\ \vdots & \vdots & \vdots & ... & \vdots \\ 0 & ... & -d_2 & 0 & -d_2 \\ 0 & 0 & ... & -d_2 & 0 \end{bmatrix}_{n_y \times n_y}$$

We solve system (3.9), then the solution of system (3.9) is set in system (3.10) to obtain its solutions.

Case(3): If we consider $\frac{du}{dt}$,$\frac{d^2u}{dy^2}$ by using (i)-differentiability and $\frac{d^2u}{dx^2}$ by using (ii)-differentiability, or $\frac{du}{dt}$,$\frac{d^2u}{dy^2}$ by using (ii)-differentiability and $\frac{d^2u}{dx^2}$ by using (i)-differentiability, then we solve the PDE system:

$$\frac{d\underline{u}}{dt}(t,x,y,r) = k(\frac{d^2\overline{u}}{dx^2}(t,x,y,r) + \frac{d^2\underline{u}}{dy^2}(t,x,y,r))$$

and

$$\frac{d\overline{u}}{dt}(t,x,y,r) = k(\frac{d^2\underline{u}}{dx^2}(t,x,y,r) + \frac{d^2\overline{u}}{dy^2}(t,x,y,r)) \tag{3.11}$$

with the initial condition

$$\underline{u}(0,x,y,r) = \underline{\ell}_1(r) \quad and \quad \overline{u}(0,x,y,r) = \overline{\ell}_1(r)$$

and the boundary conditions

$$\begin{aligned} \underline{u}(t,0,y,r) = \underline{\ell}_2(r) \quad & and \quad \overline{u}(t,0,y,r) = \overline{\ell}_2(r) \\ \underline{u}(t,h,y,r) = \underline{\ell}_3(r) \quad & and \quad \overline{u}(t,h,y,r) = \overline{\ell}_3(r) \\ \underline{u}(t,x,0,r) = \underline{\ell}_4(r) \quad & and \quad \overline{u}(t,x,0,r) = \overline{\ell}_4(r) \\ \underline{u}(t,x,b,r) = \underline{\ell}_5(r) \quad & and \quad \overline{u}(t,x,b,r) = \overline{\ell}_5(r). \end{aligned}$$

By using the ADI numerical method, we have:

$$\begin{cases} -d_1\overline{u}^{n+0.5}_{i-1,j}(r) + (1+2d_1)\underline{u}^{n+0.5}_{i,j}(r) - d_1\overline{u}^{n+0.5}_{i+1,j}(r) & = d_2\underline{u}^n_{i,j+1}(r) + (1-2d_2)\overline{u}^n_{i,j}(r) \\ & +d_2\underline{u}^n_{i,j-1}(r) \\ -d_1\underline{u}^{n+0.5}_{i-1,j}(r) + (1+2d_1)\overline{u}^{n+0.5}_{i,j}(r) - d_1\underline{u}^{n+0.5}_{i+1,j}(r) & = d_2\overline{u}^n_{i,j+1}(r) + (1-2d_2)\underline{u}^n_{i,j}(r) \\ & +d_2\overline{u}^n_{i,j-1}(r) \end{cases} \tag{3.12}$$

and

$$\begin{cases} -d_2\underline{u}_{i,j-1}^{n+1}(r)+\underline{u}_{i,j}^{n+1}(r)+2d_2\overline{u}_{i,j}^{n+1}(r)-d_2\underline{u}_{i,j+1}^{n+1}(r) & =d_1\overline{u}_{i+1,j}^{n+0.5}(r)+\overline{u}_{i,j}^{n+0.5}(r) \\ & +2d_1\underline{u}_{i,j}^{n+0.5}(r)+d_1\overline{u}_{i-1,j}^{n+0.5}(r) \\ -d_2\overline{u}_{i,j-1}^{n+1}(r)+\overline{u}_{i,j}^{n+1}(r)+2d_2\underline{u}_{i,j}^{n+1}(r)-d_2\overline{u}_{i,j+1}^{n+1}(r) & =d_1\underline{u}_{i+1,j}^{n+0.5}(r)+\underline{u}_{i,j}^{n+0.5}(r) \\ & +2d_1\overline{u}_{i,j}^{n+0.5}(r)+d_1\underline{u}_{i-1,j}^{n+0.5}(r) \end{cases} \tag{3.13}$$

where $d_1=\frac{1}{2}\alpha(\frac{\Delta t}{(\Delta x)^2})$ and $d_2=\frac{1}{2}\alpha(\frac{\Delta t}{(\Delta y)^2})$. Also $t=1,...,n_t$,$i=1,...,n_x$ and $j=1,...,n_y$.

From (3.12) and (3.13) we have two crisp linear systems for all i and j which can be displayed as follows:

$$\begin{bmatrix} A_1 & B_1 \\ B_1 & A_1 \end{bmatrix}\begin{bmatrix} \underline{u}_j^{n+0.5}(r) \\ \overline{u}_j^{n+0.5}(r) \end{bmatrix}=\begin{bmatrix} d_2\underline{u}_{i,j+1}^{n}(r)+(1-2d_2)\overline{u}_{i,j}^{n}(r)+d_2\underline{u}_{i,j-1}^{n}(r) \\ d_2\overline{u}_{i,j+1}^{n}(r)+(1-2d_2)\underline{u}_{i,j}^{n}(r)+d_2\overline{u}_{i,j-1}^{n}(r) \end{bmatrix} \tag{3.14}$$

$$\begin{bmatrix} A_2 & B_2 \\ B_2 & A_2 \end{bmatrix}\begin{bmatrix} \underline{u}_i^{n+1} \\ \overline{u}_i^{n+1} \end{bmatrix}=\begin{bmatrix} d_1\overline{u}_{i+1,j}^{n+0.5}(r)+\overline{u}_{i,j}^{n+0.5}(r)+2d_1\underline{u}_{i,j}^{n+0.5}(r)+d_1\overline{u}_{i-1,j}^{n+0.5}(r) \\ d_1\underline{u}_{i+1,j}^{n+0.5}(r)+\underline{u}_{i,j}^{n+0.5}(r)+2d_1\overline{u}_{i,j}^{n+0.5}(r)+d_1\underline{u}_{i-1,j}^{n+0.5}(r) \end{bmatrix} \tag{3.15}$$

where $B_1=2d_2I_{n_y\times n_y}$, $A_1=(1+2d_1)I_{n_x\times n_x}$,

$$B_1=\begin{bmatrix} 0 & -d_1 & 0 & ... & 0 \\ -d_1 & 0 & -d_1 & ... & 0 \\ \vdots & \vdots & \vdots & ... & \vdots \\ 0 & ... & -d_1 & 0 & -d_1 \\ 0 & 0 & ... & -d_1 & 0 \end{bmatrix}_{n_x\times n_x} \quad \text{and } A_2=\begin{bmatrix} 1 & -d_2 & 0 & ... & 0 \\ -d_2 & 1 & -d_2 & ... & 0 \\ \vdots & \vdots & \vdots & ... & \vdots \\ 0 & ... & -d_2 & 1 & -d_2 \\ 0 & 0 & ... & -d_2 & 1 \end{bmatrix}_{n_y\times n_y}$$

We solve system (3.14), then the solution of system (3.14) is set in system (3.15)to obtain its solutions.

Case(4): If we consider $\frac{du}{dt}$ by using (i)-differentiability and ,$\frac{d^2u}{dy^2}$, $\frac{d^2u}{dx^2}$ by using (ii)-differentiability, or $\frac{du}{dt}$ by using (ii)-differentiability and $\frac{d^2u}{dx^2}$,$\frac{d^2u}{dy^2}$ by using (i)-differentiability, then we solve the PDE system:

$$\frac{d\underline{u}}{dt}(t,x,y,r)=k(\frac{d^2\overline{u}}{dx^2}(t,x,y,r)+\frac{d^2\overline{u}}{dy^2}(t,x,y,r))$$

and

$$\frac{d\overline{u}}{dt}(t,x,y,r)=k(\frac{d^2\underline{u}}{dx^2}(t,x,y,r)+\frac{d^2\underline{u}}{dy^2}(t,x,y,r)) \tag{3.16}$$

with the initial condition

$$\underline{u}(0,x,y,r)=\underline{\ell}_1(r) \quad and \quad \overline{u}(0,x,y,r)=\overline{\ell}_1(r)$$

and the boundary conditions

$$\begin{aligned}
\underline{u}(t,0,y,r) &= \underline{\ell}_2(r) \quad and \quad \overline{u}(t,0,y,r) = \overline{\ell}_2(r)\\
\underline{u}(t,h,y,r) &= \underline{\ell}_3(r) \quad and \quad \overline{u}(t,h,y,r) = \overline{\ell}_3(r)\\
\underline{u}(t,x,0,r) &= \underline{\ell}_4(r) \quad and \quad \overline{u}(t,x,0,r) = \overline{\ell}_4(r)\\
\underline{u}(t,x,b,r) &= \underline{\ell}_5(r) \quad and \quad \overline{u}(t,x,b,r) = \overline{\ell}_5(r).
\end{aligned}$$

By using the ADI numerical method, we have:

$$\begin{cases}
-d_1\overline{u}_{i-1,j}^{n+0.5}(r) + (1+2d_1)\underline{u}_{i,j}^{n+0.5}(r) - d_1\overline{u}_{i+1,j}^{n+0.5}(r) & = d_2\overline{u}_{i,j+1}^{n}(r) + (1-2d_2)\underline{u}_{i,j}^{n}(r) \\
& \quad + d_2\overline{u}_{i,j-1}^{n}(r) \\
-d_1\underline{u}_{i-1,j}^{n+0.5}(r) + (1+2d_1)\overline{u}_{i,j}^{n+0.5}(r) - d_1\underline{u}_{i+1,j}^{n+0.5}(r) & = d_2\underline{u}_{i,j+1}^{n}(r) + (1-2d_2)\overline{u}_{i,j}^{n}(r) \\
& \quad + d_2\underline{u}_{i,j-1}^{n}(r)
\end{cases} \tag{3.17}$$

and

$$\begin{cases}
-d_2\overline{u}_{i,j-1}^{n+1}(r) + (1-2d_2)\underline{u}_{i,j}^{n+1}(r) - d_2\overline{u}_{i,j+1}^{n+1}(r) & = d_1\overline{u}_{i+1,j}^{n+0.5}(r) + \overline{u}_{i,j}^{n+0.5}(r) \\
& \quad - 2d_1\underline{u}_{i,j}^{n+0.5}(r) + \overline{u}_{i-1,j}^{n+0.5}(r) \\
-d_2\underline{u}_{i,j-1}^{n+1}(r) + (1-2d_2)\overline{u}_{i,j}^{n+1}(r) - d_2\underline{u}_{i,j+1}^{n+1}(r) & = d_1\underline{u}_{i+1,j}^{n+0.5}(r) + \underline{u}_{i,j}^{n+0.5}(r) \\
& \quad - 2d_1\overline{u}_{i,j}^{n+0.5}(r) + \underline{u}_{i-1,j}^{n+0.5}(r)
\end{cases} \tag{3.18}$$

where $d_1 = \frac{1}{2}\alpha(\frac{\Delta t}{(\Delta x)^2})$ and $d_2 = \frac{1}{2}\alpha(\frac{\Delta t}{(\Delta y)^2})$.
Also $t = 1, ..., n_t$,$i = 1, ..., n_x$ and $j = 1, ..., n_y$.
From (3.17) and (3.18) we have two crisp linear systems for all i and j which can be displayed as follows:

$$\begin{bmatrix} A_1 & B_1 \\ B_1 & A_1 \end{bmatrix} \begin{bmatrix} \underline{u}_j^{n+0.5}(r) \\ \overline{u}_j^{n+0.5}(r) \end{bmatrix} = \begin{bmatrix} d_2\overline{u}_{i,j+1}^{n}(r) + (1-2d_2)\underline{u}_{i,j}^{n}(r) + d_2\overline{u}_{i,j-1}^{n}(r) \\ d_2\underline{u}_{i,j+1}^{n}(r) + (1-2d_2)\overline{u}_{i,j}^{n}(r) + d_2\underline{u}_{i,j-1}^{n}(r) \end{bmatrix} \tag{3.19}$$

$$\begin{bmatrix} A_2 & B_2 \\ B_2 & A_2 \end{bmatrix} \begin{bmatrix} \underline{u}_i^{n+1} \\ \overline{u}_i^{n+1} \end{bmatrix} = \begin{bmatrix} d_1\overline{u}_{i+1,j}^{n+0.5}(r) + \overline{u}_{i,j}^{n+0.5}(r) - 2d_1\underline{u}_{i,j}^{n+0.5}(r) + \overline{u}_{i-1,j}^{n+0.5}(r) \\ d_1\underline{u}_{i+1,j}^{n+0.5}(r) + \underline{u}_{i,j}^{n+0.5}(r) - 2d_1\overline{u}_{i,j}^{n+0.5}(r) + \underline{u}_{i-1,j}^{n+0.5}(r) \end{bmatrix} \tag{3.20}$$

where $A_1 = (1+2d_1)I_{n_x\times n_x}$, $A_2 = (1-2d_2)I_{n_y\times n_y}$,

$$B_1 = \begin{bmatrix} 0 & -d_1 & 0 & ... & 0 \\ -d_1 & 0 & -d_1 & ... & 0 \\ \vdots & \vdots & \vdots & ... & \vdots \\ 0 & ... & -d_1 & 0 & -d_1 \\ 0 & 0 & ... & -d_1 & 0 \end{bmatrix}_{n_x\times n_x} \quad \text{and } B_2 = \begin{bmatrix} 0 & -d_2 & 0 & ... & 0 \\ -d_2 & 0 & -d_2 & ... & 0 \\ \vdots & \vdots & \vdots & ... & \vdots \\ 0 & ... & -d_2 & 0 & -d_2 \\ 0 & 0 & ... & -d_2 & 0 \end{bmatrix}_{n_y\times n_y}$$

First, we solve system (3.19), then the solution of the system (3.19) is set in system (3.20) to obtain its solutions.

4 Numerical example

Example 4.1. *Consider the one-dimensional heat equation*

$$\begin{cases} \frac{du}{dt} = k(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}) & & & \\ & & & \\ u(0,x,y) = \widetilde{0} & at \quad t = 0 & and & 0 \leq x \leq l \\ u(t,0,y) = \widetilde{200} = (198 + 2r, 204 - 4r) & at \quad x = 0 & and & t > 0 \\ u(t,h,y) = \widetilde{10} = (9 + r, 11 - r) & at \quad h = 3.5ft; & and & t > 0 \\ u(t,x,0) = \widetilde{200} = (198 + 2r, 204 - 4r) & at \quad y = 0 & and & t > 0 \\ u(t,x,b) = \widetilde{10} = (9 + r, 11 - r) & at \quad b = 3.5ft & and & t > 0 \end{cases}$$

Distributions of temperature are compared for $r = 0, 0.1, ..., 1$ in the following Tables 1, 2,...,7.

Table 1
$\underline{T}$

y,r=0	$x = 0$	$x = 0.5$	$x = 1$	$x = 1.5$	$x = 2$	$x = 2.5$	$x = 3$	$x = 3.5$
0	198	198	198	198	198	198	198	198
0.5	198	136.36	108.69	90.1	87.87	85.02	70.92	9
1	198	102.75	49.65	32.78	29.9	27.22	22.28	9
1.5	198	90.22	30.49	11.92	8.99	6.75	6.9	9
2	198	86.42	25.11	6.16	3.24	3.05	3.1	9
2.5	198	83.64	23.86	5.6	2.72	2.91	3.17	9
3	198	70.85	21.32	7.19	4.95	5.2	5.41	9
3.5	198	9	9	9	9	9	9	9

Table 2
$\overline{T}$

y ,r=0	$x = 0$	$x = 0.5$	$x = 1$	$x = 1.5$	$x = 2$	$x = 2.5$	$x = 3$	$x = 3.5$
0	204	204	204	204	204	204	204	204
0.5	204	146.23	108.06	93.62	89	87.04	75.2	11
1	204	117.78	61.75	37.64	30.69	31.22	28.02	11
1.5	204	107.59	44.84	17.54	9.77	11.6	13.24	11
2	204	103.56	39.36	11.74	3.96	6.12	9.21	11
2.5	204	98.08	35.74	10.28	3.35	5.21	8.32	11
3	204	79.78	28.11	10.15	5.53	6.44	8.75	11
3.5	204	11	11	11	11	11	11	11

$\vdots$

Table 3
$\underline{T}$

y,r=0.7	$x=0$	$x=0.5$	$x=1$	$x=1.5$	$x=2$	$x=2.5$	$x=3$	$x=3.5$
0	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.4
0.5	199.4	139.32	104.01	91.01	87.95	85.43	72.18	9.7
1	199.4	107.63	53.69	34.35	30.07	28.42	24.22	9.7
1.5	199.4	95.96	35.39	13.83	9.23	8.42	9.11	9.7
2	199.4	92.09	29.99	8.08	3.48	3.02	5.24	9.7
2.5	199.4	88.38	27.91	7.22	2.93	2.63	4.98	9.7
3	199.4	73.72	23.63	8.21	5.12	5.04	6.58	9.7
3.5	199.4	9.7	9.7	9.7	9.7	9.7	9.7	9.7

Table 4
$\overline{T}$

y,r=0.7	$x=0$	$x=0.5$	$x=1$	$x=1.5$	$x=2$	$x=2.5$	$x=3$	$x=3.5$
0	201	201	201	201	201	201	201	201
0.5	201	142.28	106.01	92.07	88.3	86.03	73.46	10.3
1	201	112.14	57.33	35.82	30.31	29.67	25.94	10.3
1.5	201	101.17	39.69	15.52	9.47	9.88	11.01	10.3
2	201	97.23	34.27	9.76	3.7	4.45	7.07	10.3
2.5	201	92.72	31.5	8.6	3.12	3.82	6.52	10.3
3	201	76.4	25.66	9.1	5.33	5.69	7.58	10.3
3.5	201	10.3	10.3	10.3	10.3	10.3	10.3	10.3

Table 5
$\underline{T}$

y,r=0.8	$x=0$	$x=0.5$	$x=1$	$x=1.5$	$x=2$	$x=2.5$	$x=3$	$x=3.5$
$0, r=0.8$	199.6	199.6	199.6	199.6	199.6	199.6	199.6	199.6
0.5	199.6	139.75	104.29	91.14	87.97	85.49	72.36	9.8
1	199.6	108.32	54.28	34.58	30.1	28.61	24.5	9.8
1.5	199.6	96.78	36.09	14.11	9.27	8.66	9.42	9.8
2	199.6	92.9	30.69	8.36	3.52	3.26	5.55	9.8
2.5	199.6	89.06	28.5	7.45	2.97	2.83	5.23	9.8
3	199.6	74.13	23.96	8.36	5.18	5.15	6.75	9.8
3.5	199.6	9.8	9.8	9.8	9.8	9.8	9.8	9.8
$0, r=0.9$	199.8	199.8	199.8	199.8	199.8	199.8	199.8	199.8
0.5	199.8	140.17	104.58	91.28	87.98	85.55	72.54	9.9
1	199.8	109.02	54.85	34.81	30.12	28.81	24.77	9.9
1.5	199.8	97.6	36.79	14.38	9.3	8.89	9.73	9.9
2	199.8	93.71	31.38	8.63	3.55	3.49	5.85	9.9
2.5	199.8	89.74	29.08	7.68	3	3.03	5.49	9.9
3	199.8	74.54	24.29	8.5	5.2	5.26	6.91	9.9
3.5	199.8	9.9	9.9	9.9	9.9	9.9	9.9	9.9

Table 6
$\overline{T}$

y,r=0.8	$x=0$	$x=0.5$	$x=1$	$x=1.5$	$x=2$	$x=2.5$	$x=3$	$x=3.5$
$0, r=0.8$	200.8	200.8	200.8	200.8	200.8	200.8	200.8	200.8
0.5	200.8	141.72	105.62	91.85	88.2	85.89	73.21	10.2
1	200.8	111.33	56.69	35.55	30.26	29.44	25.64	10.2
1.5	200.8	100.25	38.96	15.23	9.42	9.63	10.69	10.2
2	200.8	96.33	33.54	9.47	3.66	4.21	6.77	10.2
2.5	200.8	91.95	30.87	8.39	3.09	3.63	6.26	10.2
3	200.8	75.92	25.31	8.95	5.3	5.58	7.41	10.2
3.5	200.8	10.2	10.2	10.2	10.2	10.2	10.2	10.2
$0, r=0.9$	204.4	204.4	204.4	204.4	204.4	204.4	204.4	204.4
0.5	204	141.16	105.24	91.62	88.09	85.75	72.97	10.1
1	204.4	110.52	56.06	35.29	30.2	29.22	25.35	10.1
1.5	204.4	99.33	38.22	14.94	9.4	9.4	10.37	10.1
2	204.4	95.42	32.81	9.19	3.62	3.42	6.46	10.1
2.5	204.4	91.18	30.27	8.15	3.06	3.43	6.01	10.1
3	204.4	75.44	24.96	8.8	5.27	5.48	7.25	10.1
3.5	204.4	10.1	10.1	10.1	10.1	10.1	10.1	10.1

Table 7
$\underline{T}=\overline{T}$

y,r=0.8	$x=0$	$x=0.5$	$x=1$	$x=1.5$	$x=2$	$x=2.5$	$x=3$	$x=3.5$
$0, r=1$	200	200	200	200	200	200	200	200
0.5	200	140.59	104.85	91.41	88	85.6	72.72	10
1	200	109.72	55.43	35.03	30.15	29	25.05	10
1.5	200	98.41	37.49	14.66	9.34	9.14	10.05	10
2	200	94.52	32.08	8.91	3.6	3.73	6.15	10
2.5	200	90.42	29.66	7.9	3.03	3.23	5.75	10
3	200	74.96	24.62	8.65	5.24	5.37	7.08	10
3.5	200	10	10	10	10	10	10	10

We see that the solution of a PDE is dependent on the selection of the derivative: whether it is (i)-differentiable or (ii)-differentiable. In this example, the solution of a PDE is of the case(1) type.

5 Conclusion

In this paper, we proposed a numerical method for solving a two-dimensional heat equation. This numerical method is based on the definition of the strongly generalized derivative.

References

[1] S. Abbasbandy, T.Allahviranloo, Numerical solution of fuzzy differential equation by Tailor method, Journal of Computational Methods in Applied Mathematics 2 (2002) 113-124

[2] T. Allahviranloo, N. Ahmady, E. Ahmady, Numerical solution of fuzzy differential equations by predictor-corrector method, Information Sciences 177/7 (2007) 1633-1647.

[3] T. Allahviranloo, Difference methods for fuzzy partial differential equations, Computational methods in appliead mathematics 2 (3) (2002) 1-10.

[4] G. A. Anastassiou, S.G. Gal, On a fuzzy trigonometric approximation theorem of Weierstrass-type, Journal of Fuzzy Mathematics 9 (2004) 701-708.

[5] B. Bede, S. G. Gal, Almost periodic fuzzy -number-valued functions, Fuzzy Sets and Systems 147 (2004) 385-403.

[6] B. Bede, S. G. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, Fuzzy Sets and Systems 151 (2005) 581-599.

[7] B. Bede, I. Rudas, A. Bencsik, First order linear fuzzy differential equations under generalized differentiablity, Information sciences 177 (2006) 3627-3635.

[8] J. J. Buckly, Thomas Feuring, Introduction to fuzzy partial differential equations, Fuzzy Sets and Systems 105 (1999) 241-248.

[9] S. S. L. Chang and L.A. Zadeh, On fuzzy mapping and control, IEEE Trans. Systems Man Cybernet. 2 (1972) 30-34.

[10] Y. Chalco-Cano, H. Roman-Flores, On new solutions of fuzzy differential equations . Chaos, Solitons and Fractals (2006)1016-1043.

[11] C. K. Chen and S. H. Ho, Solving partial differential equations by two-dimensional differential transform method, Applied Mathematics and Computation 106 (1999) 171-179.

[12] M. Dehghan, A finite difference method for a non-local boundary value problem for two-dimensional heat equation, Applied Mathematics and Computation 112 (2000) 133-142.

[13] D. Dubios, H. Prade, Fuzzy numbers: an overview, in: J. Bezdek (Ed.), Analysis of Fuzzy Information, CRC Press, (1987) 112-148.

[14] M. Friedman, M. Ming, A. Kandel, Numerical solution of fuzzy differential and integral equations, Fuzzy Sets and Systems 106 (1999) 35-48.

[15] S. G. Gal, Approximation theory in fuzzy setting, in: G.A. Anastassiou (Ed.), Handbook of Analytic-Computational Methods in Applied Mathematics, Chapman Hall CRC Press, (2000) 617-666.

[16] R. Goetschel, W. Voxman, Elementary calculus, Fuzzy Sets and Systems 18 (1986) 31-43.

[17] O. He, Wu Yi, On fuzzy differential equations, Fuzzy Sets and Systems 32 (1989) 321-325.

[18] M. J. Jang, C. L. Chen and Y. C. Liy, On solving the initial-value problems using the differential transformation method, Applied Mathematics and Computation 115 (2000) 145-160.

[19] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems 24 (1987) 301-317.

[20] M. Ma, M. Friedman, A. Kandel, Numerical solution of fuzzy differential equations, Fuzzy Sets and Systems 105 (1999) 133-138.

[21] W. Menda, Linear fuzzy differential equation systems on R, J. Fuzzy Systems Math. 2 (1988) 51-56 (in Chinese).

[22] M. L. Puri, D. Ralescu, Fuzzy random variables, Journal of Mathematical Analysis and Applications 114 (1986) 409-422

[23] M. L. Puri, D. Ralescu, Differential for fuzzy function, Journal of Mathematical Analysis and Applications 91 (1983) 552-558.

[24] S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems 24 (1987) 319-330.

[25] J. K. Zhou, Differential transformation and its applications for Electrical Circuits, Huarjung University Press, (1986).

[26] H. J. Zimmermann, Fuzzy programming and linear programming with several objective functions, Fuzzy Sets and Systems 1 (1987) 45-55.