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A New Method for Solving Fuzzy Linear System

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Abstract

In this paper, we propose a new form of the fuzzy number of right hand side vector and then solve fuzzy linear system. We show that the solution of fuzzy linear system is always fuzzy vector. We compare the result of proposed idea with other methods by some examples.

Keywords : Fuzzy linear system, Inverse matrix, Fuzzy number, Fuzzy solution

1 Introduction

Systems of simulations linear equations play a major role in various areas such as mathematics, physics, statistics, engineering and social sciences. Since in many applications at least some of the system parameters and measurements are represented by fuzzy rather than crisp numbers, it is important to develop mathematical models and numerical procedures that would appropriately treat general fuzzy linear systems and solve them. The concept of fuzzy numbers and arithmetic operations with these numbers were first introduced and investigated by Zadeh [7] and, etc. One of the major applications of fuzzy number arithmetic is treating linear systems and their parameters that are all partially respected by fuzzy number [1] , etc. In Section 2, the basic concept of fuzzy number operation is brought. In Section 3, the main section of the paper, the new method for solving fuzzy linear system is introduced. The proposed idea is illustrated by some examples and is compared by other methods in Section 4. Finally conclusion is drawn in Section 5.

2 Basic concepts

There are various definitions for the concept of fuzzy numbers ([2, 3])

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Definition 2.1. *An arbitrary fuzzy number u in the parametric form is represented by an ordered pair of functions* (u_α^-, u_α^+) *which satisfy the following requirements:*

1. u_α^- *is a bounded left-continuous non-decreasing function over* $[0, 1]$.
2. u_α^+ *is a bounded left-continuous non-increasing function over* $[0, 1]$.
3. $u_\alpha^- \leq u_\alpha^+, \quad 0 \leq \alpha \leq 1.$

A crisp number r is simply represented by $u_\alpha^- = u_\alpha^+ = r, \quad 0 \leq \alpha \leq 1$. If $u_1^- < u_1^+$, we have a fuzzy interval and if $u_1^- = u_1^+$, we have a fuzzy number. In this paper, we do not distinguish between numbers or intervals and for simplicity we refer to fuzzy numbers as intervals. We also use the notation $u_\alpha = [u_\alpha^-, u_\alpha^+]$ to denote the α-cut of arbitrary fuzzy number u. If $u = (u_\alpha^-, u_\alpha^+)$ and $v = (v_\alpha^-, v_\alpha^+)$ are two arbitrary fuzzy numbers, the arithmetic operations are defined as follows:

Definition 2.2. *(Addition)*

$$u + v = (u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+) \tag{2.1}$$

and in the terms of α*-cuts*

$$(u + v)_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+], \quad \alpha \in [0, 1] \tag{2.2}$$

Definition 2.3. *(Subtraction)*

$$u - v = (u_\alpha^- - v_\alpha^+, u_\alpha^+ - v_\alpha^-) \tag{2.3}$$

and in the terms of α*-cuts*

$$(u - v)_\alpha = [u_\alpha^- - v_\alpha^+, u_\alpha^+ - v_\alpha^-], \quad \alpha \in [0, 1] \tag{2.4}$$

Definition 2.4. *(Scalar multiplication)*
For given $k \in \Re$

$$ku = \begin{cases} (ku_\alpha^-, ku_\alpha^+), & k > 0 \\ (ku_\alpha^+, ku_\alpha^-), & k < 0 \end{cases} \tag{2.5}$$

and

$$(ku)_\alpha = [\min\{ku_\alpha^-, ku_\alpha^+\}, \max\{ku_\alpha^-, ku_\alpha^+\}] \tag{2.6}$$

In particular, if $k = 1$, we have

$$-u = (-u_\alpha^+, -u_\alpha^-)$$

and with α-cuts

$$(-u)_\alpha = [-u_\alpha^+, -u_\alpha^-], \quad \alpha \in [0, 1]$$

Definition 2.5. *(Multiplication)*

$$uv = ((uv)_\alpha^-, (uv)_\alpha^+) \tag{2.7}$$

and

$$\begin{aligned} (uv)_\alpha^- &= \min\{u_\alpha^- v_\alpha^-, u_\alpha^- v_\alpha^+, u_\alpha^+ v_\alpha^-, u_\alpha^+ v_\alpha^+\} \\ (uv)_\alpha^+ &= \max\{u_\alpha^- v_\alpha^-, u_\alpha^- v_\alpha^+, u_\alpha^+ v_\alpha^-, u_\alpha^+ v_\alpha^+\}, \quad \alpha \in [0, 1] \end{aligned} \tag{2.8}$$

Definition 2.6. *(Division)*
If $0 \notin [v_0^-, v_0^+]$

$$\frac{u}{v} = ((\frac{u}{v})_\alpha^-, (\frac{u}{v})_\alpha^+) \tag{2.9}$$

and

$$\begin{aligned} (\tfrac{u}{v})_\alpha^- &= \min\{\tfrac{u_\alpha^-}{v_\alpha^-}, \tfrac{u_\alpha^-}{v_\alpha^+}, \tfrac{u_\alpha^+}{v_\alpha^-}, \tfrac{u_\alpha^+}{v_\alpha^+}\} \\ (\tfrac{u}{v})_\alpha^+ &= \max\{\tfrac{u_\alpha^-}{v_\alpha^-}, \tfrac{u_\alpha^-}{v_\alpha^+}, \tfrac{u_\alpha^+}{v_\alpha^-}, \tfrac{u_\alpha^+}{v_\alpha^+}\}, \quad \alpha \in [0,1] \end{aligned} \tag{2.10}$$

Definition 2.7. *, [4]. An arbitrary fuzzy number u is represented by a vector of 8 component of the interval $[0,1]$ without internal points, i.e. $\alpha = 0$ and $\alpha = 1$, as follows:*

$$u = (u_0^-, d_0^-, u_1^-, d_1^-, u_0^+, d_0^+, u_1^+, d_1^+) \tag{2.11}$$

where

$$u_0^- = u^-(0), \quad u_1^- = u^-(1), \quad u_0^+ = u^+(0), \quad u_1^+ = u^+(1)$$

and

$$d_0^- = (u^-)'(0), \quad d_1^- = (u^-)'(1), \quad d_0^+ = (u^+)'(0), \quad d_1^+ = (u^+)'(1)$$

By definition (2.1), it is clear that $d_0^- > 0$, $d_1^- > 0$, $d_0^+ < 0$ and $d_1^+ < 0$. For an arbitrary trapezoidal fuzzy number u, we have

$$d_0^- = d_1^- = (u^-)', \quad d_0^+ = d_1^+ = (u^+)'$$

and if $u_1^- = u_1^+$, then u is an triangular fuzzy number.
A crisp real number a and a crisp interval $[a,b]$ have the forms $(a,0,a,0,a,0,a,0)$ and $(a,0,a,0,b,0,b,0)$, respectively.
Let u and v be two fuzzy numbers in form

$$u = (u_0^-, d_0^-, u_1^-, d_1^-, u_0^+, d_0^+, u_1^+, d_1^+), \quad v = (v_0^-, e_0^-, v_1^-, e_1^-, v_0^+, e_0^+, v_1^+, e_1^+)$$

Definition 2.8. *(Addition) , [4].*

$$u + v = (u_0^- + v_0^-, d_0^- + e_0^-, u_1^- + v_1^-, d_1^- + e_1^-, u_0^+ + v_0^+, d_0^+ + e_0^+, u_1^+ + v_1^+, d_1^+ + e_1^+) \tag{2.12}$$

Definition 2.9. *(Scalar multiplication) , [4]. For given $k \in \Re$*

$$ku = \begin{cases} (ku_0^-, kd_0^-, ku_1^-, kd_1^-, ku_0^+, kd_0^+, ku_1^+, kd_1^+), & k > 0 \\ (ku_0^+, kd_0^+, ku_1^+, kd_1^+, ku_0^-, kd_0^-, ku_1^-, kd_1^-), & k < 0 \end{cases} \tag{2.13}$$

In particular, if $k = -1$ then

$$(-u_0^+, -d_0^+, -u_1^+, -d_1^+, -u_0^-, -d_0^-, -u_1^-, -d_1^-)$$

and subtraction has been defined by

$$u - v = u + (-v)$$

Note that proposed representation in definition (2.7) is exact for the fuzzy numbers having left and right branches described by polynomial up to the third degree; so, the approximated arithmetic operations produce exact results for the low-degree left and right functions and, in particular, the trapezoidal and triangular fuzzy numbers.(see[4])
Guerra et al. [4] used the Hermit approximation to obtain the (approximated) membership function and they suggested how to go from the α-cuts to the membership function and vice versa.

3 Fuzzy Linear System

Definition 3.1. *The $n \times n$ linear system*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_n \end{cases} \tag{3.14}$$

is called a fuzzy linear system(FLS) where the coefficient matrix $A = [a_{ij}]_{i,j=1}^{n}$ is a crisp nonsingular matrix and

$$y_i = (y_{i0}^-, d_{i0}^-, y_{i1}^-, d_{i1}^-, y_{i0}^+, d_{i0}^+, y_{i1}^+, d_{i1}^+), \qquad 1 \leq i \leq n$$

is a fuzzy number.

The matrix form of the system (3.14) is

$$AX = Y$$

Then let $B = A^{-1}$ be inverse of matrix A, so we have

$$X = A^{-1}Y = BY \tag{3.15}$$

therefore,

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} (y_{10}^-, d_{10}^-, y_{11}^-, d_{11}^-, y_{10}^+, d_{10}^+, y_{11}^+, d_{11}^+) \\ (y_{20}^-, d_{20}^-, y_{21}^-, d_{21}^-, y_{20}^+, d_{20}^+, y_{21}^+, d_{21}^+) \\ \vdots \\ (y_{n0}^-, d_{n0}^-, y_{n1}^-, d_{n1}^-, y_{n0}^+, d_{n0}^+, y_{n1}^+, d_{n1}^+) \end{bmatrix} \tag{3.16}$$

so,

$$\begin{cases} b_{11}(y_{10}^-, d_{10}^-, y_{11}^-, d_{11}^-, y_{10}^+, d_{10}^+, y_{11}^+, d_{11}^+) + b_{12}(y_{20}^-, d_{20}^-, y_{21}^-, d_{21}^-, y_{20}^+, d_{20}^+, y_{21}^+, d_{21}^+) \\ \quad + \cdots + b_{1n}(y_{n0}^-, d_{n0}^-, y_{n1}^-, d_{n1}^-, y_{n0}^+, d_{n0}^+, y_{n1}^+, d_{n1}^+) = x_1 \\ \vdots \\ b_{n1}(y_{10}^-, d_{10}^-, y_{11}^-, d_{11}^-, y_{10}^+, d_{10}^+, y_{11}^+, d_{11}^+) + b_{n2}(y_{20}^-, d_{20}^-, y_{21}^-, d_{21}^-, y_{20}^+, d_{20}^+, y_{21}^+, d_{21}^+) \\ \quad + \cdots + b_{nn}(y_{n0}^-, d_{n0}^-, y_{n1}^-, d_{n1}^-, y_{n0}^+, d_{n0}^+, y_{n1}^+, d_{n1}^+) = x_n \end{cases} \tag{3.17}$$

Using definition (2.9), it is clear that

$$x_i = (x_{i0}^-, e_{i0}^-, x_{i1}^-, e_{i1}^-, x_{i0}^+, e_{i0}^+, x_{i1}^+, e_{i1}^+), \qquad 1 \leq i \leq n \tag{3.18}$$

is a fuzzy number, therefore fuzzy linear system (3.14) always has a fuzzy solution.

4 Numerical Examples

Example 4.1. *,(Example 4.) [5]. Consider* 3×3 *fuzzy system*

$$\begin{aligned} x_1 + \ x_2 - \ x_3 &= (0, 1, 1, 1, 2, -1, 1, -1) \\ x_1 - 2x_2 + \ x_3 &= (2, 1, 3, 0, 3, 0, 3, 0) \\ 2x_1 + \ x_2 + 3x_3 &= (-2, 0, -2, 0, -1, -1, -2, -1) \end{aligned} \tag{4.19}$$

then

$$\begin{aligned} x_1 &= (6/13, 11/13, 17/13, 11/13, 25/13, -8/13, 17/13, -8/13) \\ &\simeq (0.46, 0.85, 1.32, 0.85, 1.9, -0.62, 1.32, -0.62) \\ x_2 &= (-19/13, 1/13, -18/13, 1/13, -10/13, -8/13, -18/13, -8/13) \\ &\simeq (-1.46, 0.07, -1.38, 0.07, 0.77, -0.62, -1.38, -0.62) \\ x_3 &= (-19/13, 5/13, -14/13, 5/13, -5/13, -9/13, -14/13, -9/13) \\ &\simeq (-1.46, 0.38, -1.08, 0.38, -0.38, -0.69, -1.08, -0.69) \end{aligned}$$

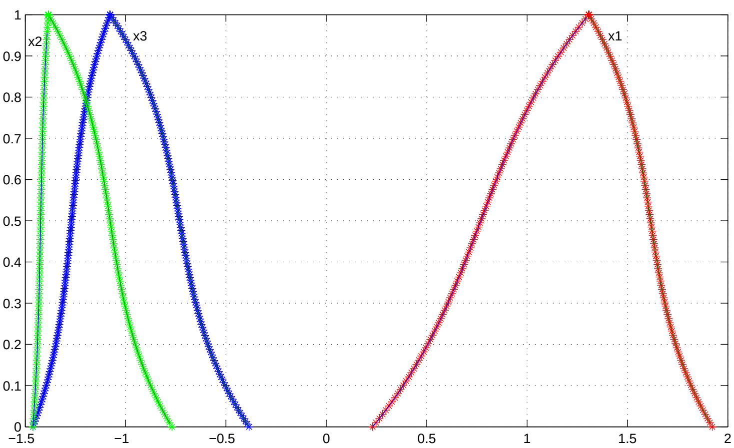

Fig. 1.

It is clear that $X = (x_1, x_2, x_3)$ *is a fuzzy solution but in [5] the system (4.19) has not a fuzzy solution.*

Example 4.2. *,(Example 5.* $(\beta = 1.9)$*) [5] Consider* 2×2 *fuzzy system*

$$\begin{aligned} x_1 - 2x_2 &= (0, 1, 1, 1, 2, -1, 1, -1) \\ x_2 + 3x_2 &= (0, 1, 1, 1, 2.9, -0.53, 1, -0.53) \end{aligned} \tag{4.20}$$

then

$$x_1 \simeq (0, 1, 1, 1, 2.36, -1, 1, -1)$$
$$x_2 \simeq (-0.4, 0.4, 0, 0.4, 0.58, -0.31, 0, -0.31)$$

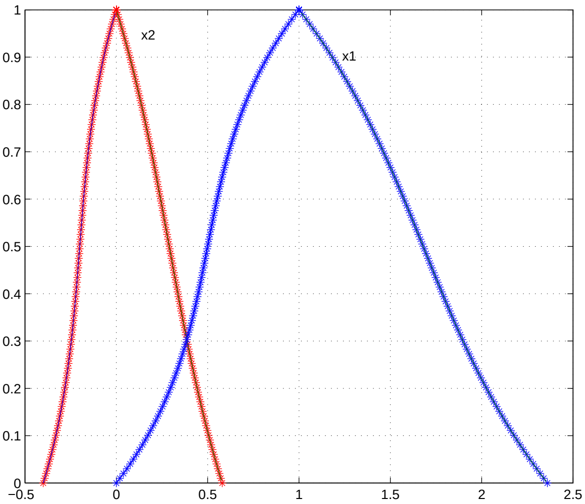

Fig. 2.

It is clear that $X = (x_1, x_2)$ *is a fuzzy solution but in [5] the system (4.20) has not a fuzzy solution.*

Example 4.3. *,(Example 4.2) [6]. Consider* 3×3 *fuzzy system*

$$\begin{aligned} 3x_1 \quad\quad - \; x_3 &= (1, 1, 2, 1, 3, -1, 2, -1) \\ x_1 + 2x_2 \quad\quad &= (0, 1, 1, 1, 2, -1, 1, -1) \\ -\,2x_2 + 5x_3 &= (-3, 0, -3, 0, -2, -1, -3, -1) \end{aligned} \tag{4.21}$$

then

$$\begin{aligned} x_1 &= (4/32, 12/32, 16/32, 12/32, 30/32, -14/32, 16/32, -14/32) \\ &\simeq (0.012, 0.375, 0.5, 0.375, 0.937, -0.437, 0.5, -0.437) \\ x_2 &= (-3/32, 11/32, 8/32, 11/32, 18/32, -10/32, 8/32, -10/32) \\ &\simeq (-0.093, 0.344, 0.25, 0.344, 0.562, -0.312, 0.25, -0.312) \\ x_3 &= (-24/32, 8/32, -16/32, 8/32, -2/32, -14/32, -16/32, -14/32) \\ &\simeq (-0.75, 0.25, -0.5, 0.25, -0.062, -0.437, -0.5, -0.437) \end{aligned}$$

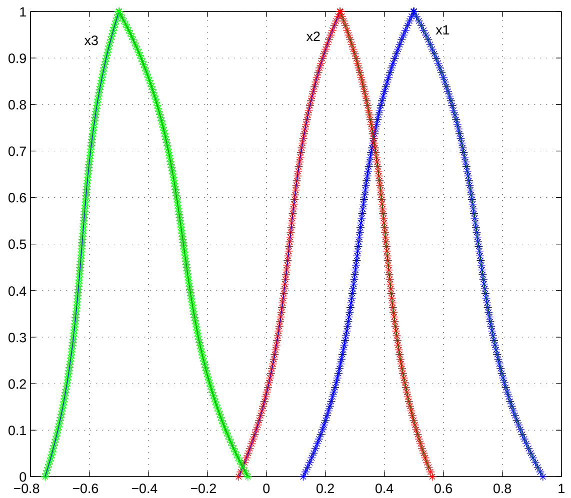

Fig. 3.

It is clear that $X = (x_1, x_2, x_3)$ is a fuzzy solution too but in [6] the system (4.21) has not a fuzzy solution.

5 Conclusion

In this work we proposed a new model for solving a system of n fuzzy linear equations with n variables. If A, the coefficient matrix, is nonsingular then we make $X = A^{-1}Y$ and the new system is then solved. The solution vector of fuzzy linear system is always a strong fuzzy solution.

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