Available online at http://ijim.srbiau.ac.ir/

Int. J. Industrial Mathematics (ISSN 2008-5621)

Vol. 5, No. 1, 2013 Article ID IJIM-00221, 5 pages

Research Article

Science and Research Branch (IAU)

Stability results for set solution of fuzzy integro-differential systems

Ho Vu * †, Ngo Van Hoa ‡, V. I. Slyn'ko §

Abstract

In this paper, we will investigate existence, comparison and some stability results of set solutions of fuzzy intergo-differential systems under the form

$$D_H x(t) = f(t, x(t)) + \int_{t_0}^{t} g(t, \eta, x(\eta))d\eta, \;\; x(t_0) = x_0 \in E^{nN}$$

with some suitable conditions.

Keywords : Fuzzy differential equations; Fuzzy integro-differential equations; Fuzzy integro-differential equations; Stability theory.

1 Introduction

THe fuzzy set theory introduced by Zadeh [24] has emerged as an interesting and fascinating branch of pure and applied sciences. The applications of fuzzy set theory can be found in many branches of regional, physical, mathematical, differential equations and engineering sciences. Recently, the authors have made important research results in the theory of fuzzy differential equations, integro-differential equations, fuzzy integro-differential equations, . . .

On the other hand, in [10] V.Lakshmikantham and Tolstonogov showed the connection between the solutions of fuzzy differential equation and the set differential equation that is generated from it. In [11] V.Lakshmikantham et al studied interconnection between set and fuzzy differential equations and in [12] V. Lakshmikantham, S.Leela studied of fuzzy differential systems is initiated and sufficient condition, in terms of Lyapunov - like functions, are provided for the new concept of stability which unifies Lyapunov and orbital stabilities as well as includes new nontions in between.

In [9], Bashir Ahmad et al studied of stability criteria for set solution of set integro-differential equations. In [1], T. Allahviranloo et al studied of existence and uniqueness of solutions of fuzzy Volterra integro-differential equations of the second kind with fuzzy kernel under strongly generalized differentiability. In [19], Phu N.D *et al* studied of existence, uniqueness and comparisons of solution to fuzzy control integro-differential systems by using some kinds of controls. In [22], Ho Vu *et al* studied of existence, comparison and some stability results of set solutions of fuzzy control intergo-differential systems.

In this paper, we discuss some stability results in terms of Lyapunov-like functions of set solutions of fuzzy intergo-differential systems with some suitable conditions.

2 Preliminaries

We recall some notations and concepts presented in detail in recent series works of Professor Lakshmikantham V. et al . . . ([10]-[16]). Let $K_C(\mathbb{R}^n)$ denote the collection of all nonempty, compact and convex subsets of $\mathbb{R}^n$. Given A, B in $K_C(\mathbb{R}^n)$, the Hausdorff distance between A and B defined as

$$d_H[A, B] = \max\{\sup_{a\in A} \inf_{b\in B} \|a - b\|_{R^n}, \sup_{b\in B} \inf_{a\in A} \|a - b\|_{R^n}\}$$

where $\|.\|_{R^n}$ denotes the Euclidean norm in $\mathbb{R}^n$. It is known that $(K_C(\mathbb{R}^n), d_H)$ is a complete metric space

*Corresponding author. hovumath@gmail.com

†Division of Applied Mathematics, University of Ton Duc Thang, Vietnam.

‡Division of Applied Mathematics, University of Ton Duc Thang, Vietnam.

§S.P. Timoshenko Institute of mechanics National Academy of Sciences of Ukraine.

and if the space $K_C(\mathbb{R}^n)$ is equipped with the natural algebraic operations of addition and nonegative scalar multiplication, then $K_C(\mathbb{R}^n)$ becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space. The set $[\omega]^\alpha = \{z \in \mathbb{R}^n : \omega(z) \geq \alpha, 0 < \alpha \leq 1\}$ is called the α-level set. For all $0 \leq \alpha \leq \beta \leq 1$ then we have $[\omega]^\beta \subset [\omega]^\alpha \subset [\omega]^0$. Set $E^n = \{\omega : \mathbb{R}^n \to [0,1]$ such that $\omega(z)$ satisfies (i)-(iv) stated below$\}$

(i) ω is normal, that is, there exists an $z_0 \in \mathbb{R}^n$ such that $\omega(z_0) = 1$;

(ii) ω is fuzzy convex, that is, for $0 \leq \lambda \leq 1$
$$\omega(\lambda z_1 + (1-\lambda)z_2) \geq \min\{\omega(z_1), \omega(z_2)\};$$

(iii) ω is upper semicontinuous;

(iv) $[\omega]^0 = cl\{z \in \mathbb{R}^n : \omega(z) > 0\}$ is compact. The element $\omega \in E^n$ is called a fuzzy number or fuzzy set.

For two fuzzy sets $\omega_1, \omega_2 \in \mathbb{E}^n$, we denote $\omega_1 \leq \omega_2$ if and only if $[\omega_1]^\alpha \subset [\omega_2]^\alpha$. Let us denote
$$D_0[\omega_1, \omega_2] = \sup\{d_H\Big[[\omega_1]^\alpha, [\omega_2]^\alpha\Big] : 0 \leq \alpha \leq 1\}$$
the distance between ω_1 and ω_2 in E^n, where $d_H\Big[[\omega]^\alpha, [\omega]^\alpha\Big]$ is Hausdorff distance between two set $[\omega_1]^\alpha, [\omega_2]^\alpha$ of $K_C(\mathbb{R}^n)$. Then (E^n, d_H) is a complete space. Some properties of metric D_0 are as follows.
$$\begin{aligned} D_0[\omega_1+\omega_3, \omega_2+\omega_3] &= D_0[\omega_1, \omega_2], \\ D_0[\lambda\omega_1, \lambda\omega_2] &= |\lambda| D_0[\omega_1, \omega_2], \\ D_0[\omega_1, \omega_2] &\leq D_0[\omega_1, \omega_3] + D_0[\omega_3, \omega_2], \end{aligned}$$

for all $\omega_1, \omega_2, \omega_3 \in E^n$ and $\lambda \in \mathbb{R}$. Given an interval $J = [t_0, T] \subseteq \mathbb{R}_+$.

Let us denote $\theta^n \in E^n$ the zero element of E^n as follows:
$$\theta^n(z) = \begin{cases} 1 \text{ if } z = \widehat{0} \\ 0 \text{ if } z \neq \widehat{0} \end{cases}$$
where $\widehat{0}$ is the zero element of $\mathbb{R}^n$. Let $u, v \in E^n$. The set $w \in E^n$ satisfying $w = u + v$ is known as the geometric difference of the set u and v and is denoted by the symbol $u - v$. The mapping $F : \mathbb{R}_+ \supset J = [t_0, T] \to E^n$ is said to have a Hukuhara derivative $D_H F(\tau)$ at a point $\tau \in J$, if
$$\lim_{h\to 0^+} \frac{F(\tau+h) - F(\tau)}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{F(\tau) - F(\tau-h)}{h}$$
exist and equal to $D_H F(\tau)$. Here limits are taken in the metric space (E^n, D_0). If $F : J \to E^n$ is continuous, then it is integralble and
$$\int_{t_0}^{t_2} F(s)\,ds = \int_{t_0}^{t_1} F(s)\,ds + \int_{t_1}^{t_2} F(s)\,ds \tag{2.1}$$

If $F, G : J \to E^n$ are integralble, $\lambda \in \mathbb{R}$, then some properties below hold
$$\int_{t_0}^{t} (F(s) + G(s))\,ds = \int_{t_0}^{t} F(s)\,ds + \int_{t_0}^{t} G(s)\,ds \tag{2.2}$$
$$\int_{t_0}^{t} \lambda F(s)\,ds = \lambda \int_{t_0}^{t} F(s)\,ds, \lambda \in R, t_0 \leq t \leq T. \tag{2.3}$$
$$D_0\left[\int_{t_0}^{t} F(s)\,ds, \int_{t_0}^{t} G(s)\,ds\right] \leq \int_{t_0}^{t} D_0[F(s), G(s)]\,ds. \tag{2.4}$$

Let $F : J \to E^n$ be continuous. Then integral $\int_{t_0}^{t} F(s)\,ds$ is differentiable and $D_H G(t) = F(t)$.

In [12] the authors have some definitions on the fuzzy mapping set:$x_i : I \to E^n, x_i(t) \in E^n$ where $[x_i(t)]^\alpha \in K_C(\mathbb{R}^n)$, and $x(t) = x_1(t) \times x_2(t) \times \ldots \times x_N(t) \in E^{nN} = E^n \times E^n \times \ldots \times E^n$, where every $x_i(t) \in E^n, i = 1, 2, \ldots, N$. The fuzzy set must be $x(t) = (x_1(t), x_2(t), \ldots, x_N(t))$.

Let $\bar{x}, x \in E^{nN}$. If there exists a set $z \in E^{nN}$ satisfying $\bar{x} = x + y$, then y is called the Hukuhara difference of the set $\bar{x}$ and x and is denoted by $\bar{x} - x$.

We have some possibilities to measure the new fuzzy variables $x, \bar{x}, f$ that are
$$d_0[x, \bar{x}] = \sum_{i=1}^{N} d[x_i, \bar{x}_i]$$
or
$$d_0[x, \bar{x}] = \frac{1}{N}\sqrt{\sum_{i=1}^{N} d^2[x_i, \bar{x}_i]}$$
or
$$d_0[x, \bar{x}] = max(d[x_1, \bar{x}_1], d[x_2, \bar{x}_2], \ldots, d[x_N, \bar{x}_N])$$
and employ the metric space (E^{nN}, d_0) as a fuzzy Hausdorff metric space, where if $x \in E^{nN}$, then $\|x\| = d_0[x, \theta^{nN}]$.

We say that fuzzy mapping set $x(t) \in E^{nN}$ has a Hukuhara derivative $D_H x(t)$ at a point t, if
$$\lim_{\tau\to 0^+} \tau^{-1}\Big(x(t+\tau) - x(t)\Big)$$
and
$$\lim_{\tau\to 0^+} \tau^{-1}\Big(x(t) - x(t-\tau)\Big),$$
exist in the topology of E^{nN} and are equal to $D_H u(t)$. Here limits are taken in the metric space (E^{nN}, d_0):
$$\lim_{\tau\to 0^+} d_0\left[\frac{x(t+\tau) - x(t)}{\tau}, D_H x(t)\right] = 0$$

and

$$\lim_{\tau \to 0^+} d_0 \left[\frac{x(t) - x(t-\tau)}{\tau}, D_H x(t) \right] = 0.$$

3 Main results

Let's the fuzzy integro-differential systems (FIDS) as follows

$$D_H x(t) = f(t, x(t)) + \int_{t_0}^{t} g(t, \eta, x(\eta)) d\eta, \quad (3.5)$$

$$x(t_0) = x_0 \in E^{nN},$$

where $f \in C[\mathbb{R}^+ \times E^{nN}, E^{nN}]$, $g \in C[\mathbb{R}^+ \times \mathbb{R}^+ \times E^{nN}, E^{nN}]$, state $x(t) \in E^{nN}$. The mapping $x \in C^1[J, E^{nN}]$ is said to be a solution of (3.5) on J. The solution of (3.5) is written in the form

$$x(t) = x_0 + \int_{t_0}^{t} \Big[f(\eta, x(\eta)) + \int_{t_0}^{t} g(\sigma, \eta, x(\eta)) d\sigma \Big] d\eta, \quad (3.6)$$

$$t \in J,$$

where the integral is the Hukuhara integral.

Utilizing the properties of the Hausdorff metric and the integral, and employing the known theory of the differential and integral inequalies for ordinary differential equations, we shall first establish the following comparison principle, which we need for later discussion.

Theorem 3.1 Assume that $f \in C[R^+ \times E^{nN}, E^{nN}]$, $g \in C[\mathbb{R}^+ \times \mathbb{R}^+ \times E^{nN}, E^{nN}]$ and for $t \in \mathbb{R}^+$; $x, y \in E^{nN}$;

$$d_0 \Bigg[f(t, x(t)) + \int_{t_0}^{t} g(t, \eta, x(\eta)) d\eta, f(t, y(t)) + \int_{t_0}^{t} g(t, \eta, y(\eta)) d\eta \Bigg] \le g_1(t, d_0[x, y]) + \int_{t_0}^{t} G(t, \eta, d_0[x, y]) d\eta$$

where $g_1 \in C[\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+]$ and $G \in C[\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+]$. Moreover, we require that there exists the maximal solution $r(t, t_0.w_0)$ of the scalar integro-differential equation

$$w'(t) = g_1(t, w(t)) + \int_{t_0}^{t} G(t, \eta, w(\eta)) d\eta,$$

$$w(t_0) = w_0 \ge 0, \ t \ge t_0.$$

Then, if $x(t) = x(t, t_0, x_0), y(t) = y(t, t_0, y_0)$ is any solution of FIDS (3.5) such that $x_0, y_0 \in E^{nN}$ exists for $t \ge t_0$ and $x(t_0) = x_0$, $y(t_0) = y_0$, we have $d_0[x(t), y(t)] \le r(t, t_0, w_0)$, $t \ge t_0$ provided that $d_0[x_0, y_0] \le w_0$.

Since $x(t), y(t)$ are solutions of FIDS (3.5), the Hukuhara difference $x(t+h) - x(t), y(t+h) - y(t)$ exist for small $h > 0$. Set $m(t) = d_0[x(t), y(t)]$. we have

$$m(t+h) - m(t) = \quad (3.7)$$

$$\begin{aligned}
& d_0[x(t+h), y(t+h)] - d_0[x(t), y(t)] \le \\
& \quad d_0 \Big[x(t+h), x(t) + h \Big\{ f(t, x(t)) \\
& \qquad + \int_{t_0}^{t} g(t, \eta, x(\eta)) d\eta \Big\} \Big] \\
& \qquad + d_0 \Big[x(t) + h \Big\{ f(t, x(t)) + \int_{t_0}^{t} g(t, \eta, x(\eta)) d\eta \Big\}, \\
& \quad y(t) + h \Big\{ f(t, y(t)) + \int_{t_0}^{t} g(t, \eta, y(\eta)) d\eta \Big\} \Big] \\
& \qquad + d_0 \Big[y(t) + h \Big\{ g(t, y(t)) \\
& \qquad + \int_{t_0}^{t} g(t, \eta, y(\eta)) d\eta \Big\}, y(t+h) \Big] - d_0[x(t), y(t)]
\end{aligned}$$

Also, we observe that

$$\begin{aligned}
& d_0 \Big[x(t+h), x(t) + h \Big\{ f(t, x(t)) + \textstyle\int_{t_0}^{t} g(t, \eta, x(\eta)) d\eta \Big\} \Big] \\
& = d_0 \Big[x(t+h) - x(t), h \Big\{ f(t, x(t)) + \int_{t_0}^{t} g(t, \eta, x(\eta)) d\eta \Big\} \Big] \\
& \quad = \frac{1}{h} d_0 \Big[\frac{x(t+h) - x(t)}{h}, f(t, x(t)) + \int_{t_0}^{t} g(t, \eta, x(\eta)) d\eta \Big]
\end{aligned} \quad (3.8)$$

$$\begin{aligned}
& d_0 \Big[x(t) + h \Big\{ f(t, x(t)) + \int_{t_0}^{t} g(t, \eta, x(\eta)) d\eta \Big\}, y(t) \\
& \quad + h \Big\{ f(t, y(t)) + \int_{t_0}^{t} g(t, \eta, y(\eta)) d\eta \Big\} \Big] \\
& \qquad = d_0 \Big[h \Big\{ f(t, x(t)) \\
& \qquad \quad + \int_{t_0}^{t} g(t, \eta, x(\eta)) d\eta \Big\}, h \Big\{ f(t, y(t)) \\
& \qquad \quad + \int_{t_0}^{t} g(t, \eta, y(\eta)) d\eta \Big\} \Big] + d_0[x(t), y(t)] \\
& \qquad = h d_0 \Big[f(t, x(t)) + \int_{t_0}^{t} g(t, \eta, x(\eta)) d\eta, f(t, y(t)) \\
& \qquad \quad + \int_{t_0}^{t} g(t, \eta, y(\eta)) d\eta \Big] + d_0[x(t), y(t)]
\end{aligned} \quad (3.9)$$

$$d_0\Big[y(t)+h\Big\{f(t,y(t))+\int_{t_0}^t g(t,\eta,y(\eta))d\eta\Big\},y(t+h)\Big]$$
$$=d_0\Big[h\Big\{f(t,y(t))+\int_{t_0}^t g(t,\eta,y(\eta))d\eta\Big\},y(t+h)-y(t)\Big]$$
$$=\frac{1}{h}d_0\Big[f(t,y(t))+\int_{t_0}^t g(t,\eta,y(\eta))d\eta,\frac{y(t+h)-y(t)}{h}\Big] \tag{3.10}$$

Form (3), (3.8), (3.9) and (3.10), we have

$$\frac{m(t+h)-m(t)}{h}\le\frac{1}{h}d_0\Big[\frac{x(t+h)-x(t)}{h},f(t,x(t))+\int_{t_0}^t g(t,\eta,x(\eta))d\eta\Big]$$
$$+\frac{1}{h}d_0\Big[f(t,y(t))+\int_{t_0}^t g(t,\eta,y(\eta))d\eta,\frac{y(t+h)-y(t)}{h}\Big]$$
$$+d_0\Big[f(t,x(t)))+\int_{t_0}^t g(t,\eta,x(\eta))d\eta,f(t,y(t))+\int_{t_0}^t g(t,\eta,y(\eta))d\eta\Big]$$

Taking $\lim\sup$ as $h\to 0^+$ yields

$$D^+m(t)=\lim_{h\to 0^+}\sup\frac{1}{h}\Big[m(t+h)-m(t)\Big]$$
$$\le d_0\Big[f(t,x(t))+\int_{t_0}^t g(t,\eta,x(\eta))d\eta,f(t,y(t))+\int_{t_0}^t g(t,\eta,y(\eta))d\eta\Big]$$
$$\le g_1(t,d_0[x,y])+\int_{t_0}^t G(t,\eta,d_0[x,y])d\eta$$
$$\le g_1(t,d_0[x_0,y_0])+\int_{t_0}^t G(t,\eta,d_0[x_0,y_0])d\eta$$

Which together with the fact that $d_0[x_0,y_0]\le w_0$ and by the comparison theorem for ordinary integro-differential equations [5] gives

$$d_0[x(t),y(t)]\le r(t,t_0.w_0)\quad t\ge t_0$$

This completes the proof of the theorem.

We shall begin by proving the existence and uniqueness results under assumptions more general than the Lipschitz type condition, which exhibits the idea of the comparison principle.

Theorem 3.2 Assume that

(ca1) $f\in C[J\times B(x_0,b),E^{nN}],g\in C[J\times J\times B(x_0,b),E^{nN}]$, where $B(x_0,b)=\{x\in E^{nN}:d_0[x,x_0]\le b\}$ and $d_0[f(t,x),\theta^{nN}]\le M_0$ on $J\times B(x_0,b)$, $\int_\eta^t d_0[g(\sigma,\eta,x(\eta)),\theta^{nN}]d\sigma\le N_0$ on $J\times J\times B(x_0,b)$, where θ^{nN} is zero element of E^{nN} regarded as a point set.

(ca2) $d_0[f(t,x),f(t,y)]\le g_1(t,d_0[x,y])$ on $J\times B(x_0,b)$ and
$d_0[g(t,\eta,x),g(t,\eta,y)]\le G(t,\eta,d_0[x,y])$ on $J\times B(x_0,b)$, where $g_1\in C[J\times[0,2b],\mathbb{R}^+]$, $G\in C[J\times J\times[0,2b],\mathbb{R}^+]$, $g_1(t,w)\le M_1$ on $J\times[0,2b]$, $G(t,\eta,w)\le N_1$ on $J\times J\times[0,2b]$, $g_1(t,0)=0,G(t,\eta,0)=0$, $g_1(t,w)$ and $G(t,\eta,w)$ are non-decreasing in w for each $t\in J$, $(t,\eta)\in J\times J$.

(ca3) $w(t)=0$ is the only solution of

$$w'(t)=g_1(t,w(t))+\int_{t_0}^t G(t,\eta,w(\eta))d\eta, \tag{3.11}$$
$$w(t_0)=w_0.$$

Then the successive approximations defined by

$$x_{n+1}(t)=x_0+\int_{t_0}^t\Big[f(\eta,x_n(\eta))+\int_{t_0}^t g(\sigma,\eta,x_n(\eta))d\sigma\Big]d\eta\quad t\in J,$$

exists on $J_0\equiv[t_0,t+\alpha]$, where $\alpha=\min\Big(a,\dfrac{b}{N+M}\Big)$, $M=\max\{M_0,M_1\},N=\max\{N_0,N_1\}$, as continuous functions and converge uniformly to the unique solution $x(t)$ of FIDS (3.5) on J_0.

Let us define a sequence $x_n(t):J\to E^{nN},n=1,2,\dots$ of successive approximations as follows $x(t_0)=x_0$ for every J and

$$x_{n+1}(t)=x_0+\int_{t_0}^t\Big[f(\eta,x_n(\eta))+\int_{t_0}^t g(\sigma,\eta,x_n(\eta))d\sigma\Big]d\eta,\ t\in J.$$

We have

$$d_0[x_{n+1},x_0]=d_0\Bigg[x_0+\int_{t_0}^t\Big[f(\eta,x_n(\eta))+\int_{t_0}^t g(\sigma,\eta,x_n(\eta))d\sigma\Big]d\eta,x_0\Bigg]$$
$$\le d_0\Big[\int_{t_0}^t f(\eta,x_n(\eta))d\eta,\theta^{nN}\Big]+d_0\Big[\int_{t_0}^t\int_{t_0}^t g(\sigma,\eta,x_n(\eta))d\sigma d\eta,\theta^{nN}\Big]$$

Using the assumption (ca1), we get

$$\begin{aligned} d_0[x_{n+1}, x_0] &\le \int_{t_0}^{t} d_0\Big[f(\eta, x_n(\eta)), \theta^{nN}\Big]d\eta \\ &\quad + \int_{t_0}^{t} d_0\Big[\int_{t_0}^{t} g(\sigma, \eta, x_n(\eta))d\sigma, \theta^{nN}\Big]d\eta \\ &\le (t-t_0)(N_0+M_0) \\ &\le t(N_0+M_0) \\ &< b \end{aligned}$$

Similar, we define the successive approximations of (3.11) as follows

$$w_0(t) = (t-t_0)(N_1+M_1),\ w_{n+1}(t) = \int_{t_0}^{t} g_1(\eta, w_n(\eta))d\eta + \int_{t_0}^{t}\int_{t_0}^{t} G(\sigma, \eta, w(\eta))d\sigma d\eta$$

where $t \in J,\ n = 1, 2, \ldots$

An easy induction proves that $\{w_m(t)\}$ as well defined and

$$0 < w_{n+1}(t) < w_n(t),\ t \in J$$

Since $|w_n'(t)| \le g(t, w_{n-1}(t)) \le N_1 + M_1$, we conclude from the Ascoli - Arzela theorem and the monotonicity of the sequence $\{w_n(t)\}$, that $w_n(t) \to w(t)$ as $n \to \infty$ uniformly on J. It is also clear that $w(t)$ satisfies (3.11) and hence by conditions (ca2)

$$w(t) \ge 0,\ t \in J$$

We see that

$$\begin{aligned} d_0[x_1, x_0] &\le \int_{t_0}^{t} d_0\Big[f(\eta, x_0(\eta)), \theta^{nN}\Big]d\eta \\ &\quad + \int_{t_0}^{t} d_0\Big[\int_{t_0}^{t} g(\sigma, \eta, x_0(\eta))d\sigma, \theta^{nN}\Big]d\eta \\ &\le (t-t_0)(N_1+M_1) \\ &= w_0(t) \end{aligned}$$

Observe that for $n = 2, 3, \ldots$ one has

$$\begin{aligned} &d_0[x_{n+1}(t), x_n(t)] \\ &= d_0\Big[x_0 + \int_{t_0}^{t}\Big[f(\eta, x_n(\eta)) + \int_{t_0}^{t} g(\sigma, \eta, x_n(\eta))d\sigma\Big]d\eta, \\ &x_0 + \int_{t_0}^{t}\Big[f(\eta, x_{n-1}(\eta)) + \int_{t_0}^{t} g(\sigma, \eta, x_{n-1}(\eta))d\sigma\Big]d\eta\Big] \\ &\le \int_{t_0}^{t} d_0\Big[f(\eta, x_n(\eta)), f(\eta, x_{n-1}(\eta))\Big]d\eta \\ &\quad + \int_{t_0}^{t} d_0\Big[\int_{t_0}^{t} g(\sigma, \eta, x_n(\eta))d\sigma, \int_{t_0}^{t} g(\sigma, \eta, x_{n-1}(\eta))d\sigma\Big] \\ &\le \int_{t_0}^{t} d_0\Big[f(\eta, x_n(\eta)), f(\eta, x_{n-1}(\eta))\Big]d\eta \\ &\quad + \int_{t_0}^{t}\int_{t_0}^{t} d_0\Big[g(\sigma, \eta, x_n(\eta)), g(\sigma, \eta, x_{n-1}(\eta))\Big]d\sigma d\eta \end{aligned}$$

Using the assumption (ca2), we have

$$\begin{aligned} &d_0[x_{n+1}(t), x_n(t)] \\ &\le \int_{t_0}^{t} g_1(\eta, d_0[x_{n+1}(\eta), x_n(\eta)])d\eta \\ &\quad + \int_{t_0}^{t}\int_{t_0}^{t} G(\sigma, \eta, d_0[x_{n+1}(\eta), x_n(\eta)])d\sigma d\eta \\ &= w_n(t) \end{aligned}$$

Thus, we have the estimate

$$d_0[x_{n+1}(t), x_n(t)] \le w_n(t)$$

Let $v(t) = d_0[x_{n+1}(t), x_n(t)],\ t \in J$. The proof of Theorem 3.2 yields, for $t \in J$,

$$\begin{aligned} D^+x(t) &\le g_1(t, d_0[x_{n+1}(t), x_n(t)]) \\ &\quad + \int_{t_0}^{t} G(t, \eta, d_0[x_{n+1}(\eta), x_n(\eta)])d\eta \\ &\le g(t, w_{n-1}(t)) + \int_{t_0}^{t} G(t, \eta, w_{n-1}(\eta))d\eta \end{aligned}$$

Let $n > m$. The we obtain

$$\begin{aligned} &d_0[D_Hx_n(t), D_Hx_m(t)] \\ &= d_0\Big[f(t, x_n(t)) + \int_{t_0}^{t} g(t, \eta, x_n(\eta))d\eta, f(t, x_m(t)) \\ &\quad + \int_{t_0}^{t} g(t, \eta, x_m(\eta))d\eta\Big] \\ &\le d_0\Big[f(t, x_n(t)) + \int_{t_0}^{t} g(t, \eta, x_n(\eta))d\eta, f(t, x_{n-1}(t)) \\ &\quad + \int_{t_0}^{t} g(t, \eta, x_{n-1}(\eta))d\eta\Big] \\ &\quad + d_0\Big[f(t, x_{n-1}(t)) \\ &\quad + \int_{t_0}^{t} g(t, \eta, x_{n-1}(\eta))d\eta, f(t, x_{m-1}(t)) \\ &\quad + \int_{t_0}^{t} g(t, \eta, x_{m-1}(\eta))d\eta\Big] \\ &\quad + d_0\Big[f(t, x_{m-1}(t)) \\ &\quad + \int_{t_0}^{t} g(t, \eta, x_{m-1}(\eta))d\eta, f(t, x_m(t)) \\ &\quad + \int_{t_0}^{t} g(t, \eta, x_m(\eta))d\eta\Big] \\ &\le g_1(t, w_{n-1}(t)) + g_1(t, w_{m-1}(t)) \\ &\quad + g_1(t, d_0[x_n(t), x_m(t)]) \\ &\quad + \int_{t_0}^{t} G(t, \eta, w_{n-1}(\eta))d\eta + \int_{t_0}^{t} G(t, \eta, w_{m-1}(\eta))d\eta \\ &\quad + \int_{t_0}^{t} G(t, \eta, d_0[x_n(t), x_m(t)])d\eta \end{aligned}$$

Setting $v(t) = d_0[x_n(t), x_m(t)]$, the proof Theorem 3.2 shows that

$$\begin{aligned} D^+v(t) &\le d_0[D_H x_n(t), D_H x_m(t)] \\ &\le 2g_1(t, w_{n-1}(t)) + g_1(t, v(t)) \\ &\quad + 2\int_{t_0}^{t} G(t,\eta,w_{n-1}(\eta))d\eta \\ &\quad + \int_{t_0}^{t} G(t,\eta,v(\eta))d\eta \end{aligned}$$

in view of the monotone nature of $g_1(t,w)$ and $G(t,\eta,w)$ are nondecreasing in w for each $t \in J$, $(t,\eta) \in J \times J$ and the fact that $w_{m-1}(t) \le w_{n-1}(t)$ since $n \ge m$ and $w_n(t)$ is decreasing sequence. The comparison theorem for integro-differential equations [15] gives

$$v(t) \le r_n(t),\ \ r_n(t) = 0$$

where $r_n(t)$ is the maximal solution of

$$\begin{aligned} r_n'(t) &= 2g_1(t, w_{n-1}(t)) + g_1(t, r_n(t)) \\ &\quad + 2\int_{t_0}^{t} G(t,\eta,w_{n-1}(\eta))d\eta \\ &\quad + \int_{t_0}^{t} G(t,\eta,r_n(\eta))d\eta \end{aligned} \qquad (3.12)$$

Since $g_1(t, w_{n-1}(t)) \to 0$ and $G(t,\eta,w_{n-1}(\eta)) \to 0$ as $n \to \infty$ uniformly on J, it follows by Theorem 1.4.1 in [15] that $r_n(t) \to 0$ uniformly on J. This implies from (3.12) and definition of $v(t)$ that $x_n(t)$ converge uniformly to $x(t)$ and it easy to show that $x(t)$ is a solution of (3.5).

To show uniqueness, let $x_0(t)$ be another solution of (3.5). Then setting $m(t) = d_0[x(t), x_0(t)]$ and noting that $m(t_0) = 0$, we get $D^+m(t) \le g_1(t, m(t)) + \int_{t_0}^{t} G(t,\eta,m(\eta))d\eta,\ t \in J$ and $m(t) \le r(t,t_0,0),\ t \in J$ be Theorem 3.2. By the assumptions $r(t,t_0,0) \equiv 0$ and therefore, we obtain $x(t_0) = x_0,\ t \in J$.

This completes the proof of the theorem.

The second, we have the stability criteria of FIDS (3.5) as below.

Definition 3.1 The trivial set solution of (3.5) is said to be

(S1) equi-stable of for each $\varepsilon > 0$ and $t_0 > 0$, there exists a $\delta = \delta(t_0, \varepsilon)$ such that $d_0[x_0, \theta^{nN}] < \delta$ implies $d_0[x(t), \theta^{nN}] < \varepsilon$, for $t \ge t_0$;

(S2) uniformly stable, if the δ in (S1) is independent of t_0;

(S3) quasi-equi-asymptotically stable, if for each $\varepsilon > 0$, $t_0 > 0$, there exist a $T = T(t_0, \varepsilon)$ and $\delta_0 = \delta_0(t_0)$ such that $d_0[x_0, \theta^{nN}] < \delta_0$ implies $d_0[x(t), \theta^{nN}] < \varepsilon$, for all $t > t_0 + T$;

(S4) quasi-uniformly asymptotically stable, if δ_0 and T in (S3) are independent of t_0;

(S5) equi-asymptotically stable, if (S1) and (S3) hold simultaneously;

(S6) uniformly asymptotically stable, if (S2) and (S4) hold simultaneously;

(S7) exponentially asymptotically stable, if there exist constants $\lambda, \beta > 0$ such that

$$d_0[x(t), \theta^{nN}] \le \beta(d_0[x_0, \theta^{nN}], t_0)\exp[-\lambda(t - t_0)], t > t_0.$$

Theorem 3.3 Assume that

(cc1) $V \in C[\mathbb{R}^+ \times E^{nN}, \times E^{nN}]$ and $|V(t,x) - V(t,y)| \le Ld_0[x,y]$ where L is the local Lipschitz constant, $x, y \in E^{nN}$;

(cc2) $g_1 \in C[\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}]$, $G \in C[\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}]$ and for $x, y \in E^{nN}$, $t \in \mathbb{R}^+$,

$$\begin{aligned} D^+V(t,x) &\equiv \lim_{h\to 0^+} \sup \frac{1}{h}\Big[V(t+h, x(t) \\ &\quad + h\Big\{f(t,x(t)) \\ &\quad + \int_{t_0}^{t} G(t,\eta,x(\eta))d\eta\Big\} \\ &\quad - V(t,x(t))\Big] \\ &\le g_1(t, V(t,x)) \\ &\quad + \int_{t_0}^{t} G(t,\eta,V(\eta,x(\eta)))d\eta \end{aligned}$$

Then, if $x(t) = x(t,t_0,x_0)$ is any solution of FIDS (3.5) existing on $[t_0,\infty)$ such that $V(t,t_0,x_0)$, we have

$$V(t,x(t)) \le r(t,t_0,w_0) \quad t \in [t_0,\infty)$$

where $r(t,t_0,w_0)$ is the maximal solution of

$$\begin{aligned} w'(t) &= g_1(t,w(t)) + \int_{t_0}^{t} G(t,\eta,w(\eta))d\eta \quad w(t_0) \\ &= w_0 \\ &\ge 0, \end{aligned}$$

existing on $[t_0,\infty)$.

Let $x(t) = x(t,t_0,x_0)$ be any solution of FIDS (3.5) existing on $[t_0,\infty)$. Define $m(t) = V(t,x(t))$ so that $m(t_0) = V(t_0,x_0) \le w_0$. Now for small $h > 0$, we consider

$$\begin{aligned} m(t+h) - m(t) &= V(t+h, x(t+h)) - V(t, x(t)) \\ &\le V(t+h, x(t+h)) \\ &\quad + V(t+h, x(t) + h\Big\{f(t,x(t)) \\ &\qquad + \int_{t_0}^{t} g(t,\eta,x(\eta))d\eta\Big\}) \\ &\quad - V(t+h, x(t) + h\Big\{f(t,x(t)) \\ &\quad + \int_{t_0}^{t} g(t,\eta,x(\eta))d\eta\Big\}) - V(t,x(t)) \\ &\le Ld_0\Big[x(t+h), x(t) + h\Big\{f(t,x(t)) \\ &\quad + \int_{t_0}^{t} g(t,\eta,x(\eta))d\eta\Big\}\Big] \\ &\quad - V(t+h, x(t) + h\Big\{f(t,x(t)) \\ &\quad + \int_{t_0}^{t} g(t,\eta,x(\eta))d\eta\Big\}) - V(t,x(t)) \end{aligned}$$

using the Lipschitz conditions give in (cc1). Thus

$$\begin{aligned} D^+m(t) &\equiv \lim_{h\to 0^+} \sup \frac{1}{h}[m(t+h) - m(t)] \\ &\le D^+V(t,x(t)) + \lim_{h\to 0^+} \sup \frac{1}{h} d_0\Big[x(t+h), x(t) \\ &\quad + h\Big\{f(t,x(t)) + \int_{t_0}^{t} g(t,\eta,x(\eta))d\eta\Big\}\Big] \end{aligned}$$

Since

$$\begin{aligned} &\frac{1}{h} d_0\Big[x(t+h), x(t) + h\Big\{f(t,x(t)) + \int_{t_0}^{t} g(t,\eta,x(\eta))d\eta\Big\}\Big] \\ &= d_0\Big[\frac{m(t+h) - m(t)}{h}, f(t,x(t)) \\ &\quad + \int_{t_0}^{t} g(t,\eta,x(\eta))d\eta\Big] \end{aligned}$$

and $x(t)$ is any solution of FIDS (3.5), we find that

$$\begin{aligned} &\lim_{h\to 0^+} \sup \frac{1}{h} d_0\Big[x(t+h), x(t) \\ &\quad + h\Big\{f(t,x(t)) + \int_{t_0}^{t} g(t,\eta,x(\eta))d\eta\Big\}\Big] \\ &= \lim_{h\to 0^+} \sup d_0\Big[\frac{m(t+h) - m(t)}{h}, f(t,x(t)) \\ &\quad + \int_{t_0}^{t} g(t,\eta,x(\eta))d\eta\Big] \\ &= d_0\Big[D_H x(t), f(t,x(t)) + \int_{t_0}^{t} g(t,\eta,x(\eta))d\eta\Big] \\ &= 0 \end{aligned}$$

We therefore have the scalar integro-differential inequality

$$D^+m(t) \le g_1(t, m(t)) + \int_{t_0}^{t} G(t,\eta,m(\eta))d\eta, \quad m(t_0) \le w_0$$

By the Theorem 1.4.1 in [15], it follows the estimate

$$m(t) \le r(t, t_0, w_0), \quad [t_0, \infty)$$

This proves the assertion of the theorem.

Corollary 3.1 *Assume that the Lyapunov-like function $V(t, x(t))$ satisfies conditions in Theorem 3.3. If functions $g_1(t,w) \equiv 0$ and $G(t,s,w) \equiv 0$ are admissible in Theorem 3.3 to yield the estimate*

$$V(t, x(t)) \le V(t_0, x(t_0)), \forall t \ge t_0 > 0.$$

Putting $\mathcal{S}^\rho(x_0) = \{x(t) \in E^{nN} : d_0[x(t), x_0] < \rho\}$, we have: Assume that for FIDS (3.5) exists the Lyapunov like function $V(t, x(t))$ which satisfies the conditions of Theorem 3.3, and

a) there exist the positive functions $a(\cdot,\cdot), b(\cdot)$ are strictly increasing and $\mu > 0$ such that $\forall t \in [t_0, T], x(t) \in E^{nN} : b\left(d_0[x(t), \theta^{nN}]\right) \le V(t, x(t)) \le a\left(t, d_0[x(t), \theta^{nN}]\right)$. Then,
   - (i) if function $g_1(t, V(t,x)) + \int_{t_0}^{t} G(t,\eta,V(\eta,x(\eta)))d\eta \le 0$ is admissible in Theorem 3.3, the estimate (S1) holds.
   - (ii) if function $g_1(t, V(t,x)) + \int_{t_0}^{t} G(t,\eta,V(\eta,x(\eta)))d\eta \le -\mu$ is admissible in Theorem 3.3, the estimate (S3) holds.
   - (iii) if function $g_1(t, V(t,x)) + \int_{t_0}^{t} G(t,\eta,V(\eta,x(\eta)))d\eta < -\mu$, is admissible in Theorem 3.3, the estimate (S5) holds.

b) there exist the positive functions $a(\cdot,\cdot), b(\cdot)$ are strictly increasing and $\eta > 0$ such that $\forall t \in [t_0; T], x(t) \in S^\rho(x_0) : b\left(d_0[x(t), \theta^{nN}]\right) \le V(t, x(t)) \le a\left(t, d_0[x(t), \theta^{nN}]\right)$. Then,
   - (i) if function $g_1(t, V(t,x)) + \int_{t_0}^{t} G(t,\eta,V(\eta,x(\eta)))d\eta \le 0$ is admissible in Theorem 3.3, the estimate (S2) holds.
   - (ii) if function $g_1(t, V(t,x)) + \int_{t_0}^{t} G(t,\eta,V(\eta,x(\eta)))d\eta \le -\eta V(t,x(t))$ is admissible in Theorem 3.3, the estimate (S4) holds.
   - (iii) if function $g_1(t, V(\cdot,x)) + \int_{t_0}^{t} G(t,\eta,V(\eta,x(\eta)))d\eta < -\eta V(t,x(t))$ is admissible in Theorem 3.3, the estimate (S6) holds.

Let $\varepsilon > 0$ and t_0 be given, choosing $\delta = \delta(t_0, \varepsilon)$ such that $a(t_0, \varepsilon) < b(\delta)$ with this we have (S1) .

If this is not true, there would exists a the set solution $x(t) \in E^{nN}$ of FIDS (3.5) and $t_1 > t_0$ such that

$$d_0[x(t_1), \theta^{nN}] = \varepsilon \quad \text{and} \quad d_0[x(t), \theta^{nN}] > \delta$$

for $0 \le t_0 < t_1 < t$, and $\delta > \varepsilon$.

By assumptions of theorem 3 show that $V(t, x(t)) \le V(t_0, x_0), \forall t \ge t_0 \ge 0$ and condition $a(t_0, \varepsilon) < b(\delta)$ as result, yield:

$$b(\delta) < b\Big(d_0[x(t), \theta^{nN}]\Big) \le V(t, x(t)) \le V(t_0, x_0) \le a(t_0, d_0[x_0, \theta^{nN}]) \le a(t_0, \varepsilon) < b(\delta)$$

This contradiction proves that (S1) holds.

Next, we have to prove that: $\forall \varepsilon > 0, t_0 \in R_+$ there exists a $B > 0$ and number $T_1(t_0, \varepsilon) > 0$ such that: $d_0[x(T_1), \theta^{nN}] < \varepsilon$ implies $d_0[x(t), \theta^{nN}] < B$ for $t \ge t_0 + T_1 > t_0 \ge 0$. Let $\varepsilon > 0$ and $t_0 > 0$. Choosing $B = B(t_0, \varepsilon)$ such that $a(t_0, \varepsilon) < b(B)$ with this we have (S3).

If this is not true, there would exists a set solution $x(t)$ of FIDS (3.5) and $t \ge t_0 + T_1 > t_0 \ge 0$ such that, $d_0[x(T_1), \theta^{nN}] = \varepsilon$ and $d_0[x(t), \theta^{nN}] > B$, for $t \ge t_0 + T_1 > t_0 \ge 0$ and $B > \varepsilon$.

By assumptions of theorem 3 show that $V(t, x(t)) \le V(t_0, x_0)$, $\forall t \ge t_0 > 0$ and condition a/ii as result, yield:

$$b(B) < b\left(d_0[x(t), \theta^{nN}]\right) \le V(t, x(t)) \le V(t_0, x(t_0)) - \mu_1 \le a\left(t_0, d_0[x_0, \theta^{nN}]\right) - \mu_1 < a(t_0, \varepsilon) < b(B).$$

This contradiction proves that (S3) holds.

The affirmation for (B5) is proved analogous proof of the affirmations for (B1), (B3).
Next, we have to prove that (S2) holds:

Because assumptions (b/(i)) imply that $V(t, x(t)) \le V(t_0, x_0)$ and $\forall t \ge t_0$

$$b(d_0[x(t), \theta^{nN}]) \le V(t, x(t)) \le V(t_0, x_0)$$

$$\le a(t_0, d_0[x_0, \theta^{nN}]).$$

Thus for all $x(t) \in S^\rho(x_0)$ and $\forall t_0 \in R_+$ the affirmation for (S1) holds, that means the affirmation for (S2) holds.

Next, we have to prove that (S4) holds. Also assumption b) of this Theorem to be

i) $b\left(d_0[x(t), \theta^{nN}]\right) \le V(t, x(t)) \le a\left(t_0, d_0[x(t), \theta^{nN}]\right)$

ii) $D^+V \le -\eta V(t, x(t))$

For all $t_0 \in R_+$, we have

$$V(t, x(t)) \le V(t_0, x_0) exp[-\eta(t - t_0)]$$

$$\le a(t, d_0[x_0, \theta^{nN}]) exp[-\eta(t - t_0)], \forall t \ge t_0.$$

As a results

$$b\left(d_0[x(t), \theta^{nN}]\right)$$

$$\le a\left(t_0, d_0[x_0, \theta^{nN}]\right) .exp[-\eta(t - t_0)], \forall t \ge t_0$$

and (S4) holds.

The affirmation for (S6) is proved analogous proof of the affirmations for (S2),(S4).

Corollary 3.2 *Assume that for FIDS (3.5) exists the Lyapunov like function $V(t, x(t))$ which satisfies the conditions of Theorem 3.3, and there exist the positive numbers a, b such that*

$$\forall t \in [t_0, T], x(t) \in E^{nN} : bd_0[x(t), \theta^{nN}]$$

$$\le V(t, x(t)) \le ad_0[x(t), \theta^{nN}].$$

Then , if $D^+V \le -\eta_1 V(t, x(t))$ satisfies , solution of FIDS (3.5) is exponentially asymptotically stable.

The proof of this Corollary is proved analogous to the proof of the affirmations for (S4).

Lemma 3.1 *Let $x : J \to E^1$ and put $[x(t)]^\alpha = [\underline{x}_\alpha(t), \overline{x}_\alpha(t)]$ for each $\alpha \in [0, 1]$. If x is Hukuhara derivative then $\underline{x}_\alpha$, $\overline{x}_\alpha$ are differentiable functions and $[D_H x(t)]^\alpha = [\underline{x}'_\alpha(t), \overline{x}'_\alpha(t)]$.*

Example 3.1 *Let us consider the following FIDE in E^1*

$$D_H x(t) = -3x(t) + \int_0^t e^{-(t-s)} x(s)\, ds, \quad (3.13)$$

$$x(0) = x_0 \in E^1,$$

where $f : [0, \infty) \times E^1 \to E^1$ is given by $f(t, x) = -3x(t)$ and $g : [0, \infty) \times [0, \infty) \times E^1 \to E^1$ is given by $g(t, s, x(s)) = e^{-(t-s)} x(s)$.

We put $[x_0]^\alpha = [\underline{x}^0_\alpha, \overline{x}^0_\alpha]$ and $[f(t, x(t))]^\alpha = [\underline{f}(t, \underline{x}_\alpha(t), \overline{x}_\alpha(t)), \overline{f}(t, \underline{x}_\alpha(t), \overline{x}_\alpha(t))]$, $[g(t, s, x(t))]^\alpha = [\underline{g}(t, s, \underline{x}_\alpha(t), \overline{x}_\alpha(t)), \overline{g}(t, s, \underline{x}_\alpha(t), \overline{x}_\alpha(t))]$.

Then, with this notations, the problem (3.13) is *transformed into the following :*

$$\begin{cases} \underline{x}'_\alpha(t) = \underline{f}\,(t, \underline{x}_\alpha\,(t)\,, \overline{x}_\alpha\,(t)) \\ + \int\limits_0^t \underline{g}\,(t, s, \underline{x}_\alpha\,(t)\,, \overline{x}_\alpha\,(t))\,ds, \quad \underline{x}_\alpha\,(0) = \underline{x}_\alpha^0 \\ \overline{x}'_\alpha(t) = \overline{f}\,(t, \underline{x}_\alpha\,(t)\,, \overline{x}_\alpha\,(t)) \\ + \int\limits_0^t \overline{g}\,(t, s, \underline{x}_\alpha\,(t)\,, \overline{x}_\alpha\,(t))\,ds, \quad \overline{x}_\alpha\,(0) = \overline{x}_\alpha^0 \end{cases} \tag{3.14}$$

By solving (3.14), we get $[x(t)]^\alpha = [\underline{x}_\alpha^0, \overline{x}_\alpha^0]$ *where*

$$\underline{x}_\alpha(t) = \frac{\overline{x}_\alpha^0 + \underline{x}_\alpha^0}{16}((2-\sqrt{2})e^{(-2+\sqrt{2})t} + (2+\sqrt{2})e^{-(2+\sqrt{2})t} - \frac{-\overline{x}_\alpha^0 + \underline{x}_\alpha^0}{40}((5+2\sqrt{5})e^{(1+\sqrt{5})t} + (5-2\sqrt{2})e^{(1-\sqrt{5})t})).$$

$$\overline{x}_\alpha(t) = \frac{\overline{x}_\alpha^0 + \underline{x}_\alpha^0}{16}((2-\sqrt{2})e^{(-2+\sqrt{2})t} + (2+\sqrt{2})e^{-(2+\sqrt{2})t} + \frac{-\overline{x}_\alpha^0 + \underline{x}_\alpha^0}{40}((5+2\sqrt{5})e^{(1+\sqrt{5})t} + (5-2\sqrt{2})e^{(1-\sqrt{5})t})).$$

Therefore, the trivial solution of (3.13) is stable.

Example 3.2 *Let us consider the following FIDE in* E^1

$$\begin{cases} D_H x(t) = \int\limits_0^t x(s)ds, \ t \in [0,2] \\ x(0) = (-1, 0, 1) \end{cases}$$

The solution is $[x(t)]^\alpha = \dfrac{e^t - e^{-t}}{2}[\alpha - 1, 1 - \alpha]$, $t \in [0,2]$ *. Futher, given* $\varepsilon > 0$*, we can choose* $\delta = \dfrac{2\varepsilon}{e^2+1}$ *such that for* $d_0[x_0, \theta^1]$ *we have* $d_0[x(t), \theta^1] < \varepsilon$*.*

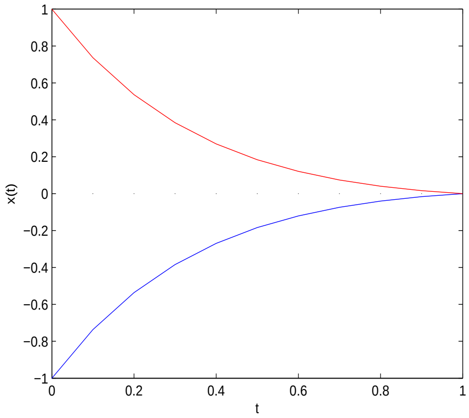

Figure 1: Solution of Example 3.1

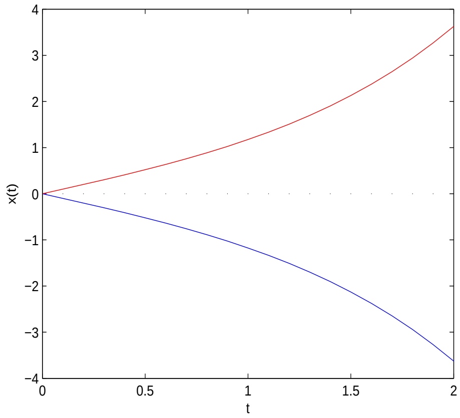

Figure 2: Solution of Example 3.2

Acknowledgements

The authors gratefully acknowledge the referees for their careful reading and many valuable remarks which improved the presentation of the paper. This work was supported by the Vietnam National Research Fund (NAFOSTED).

References

[1] T. Allahviranloo, S. Hajighasemi, M. Khezerloo, M. Khorasany, and S. Salahshour, *Existence and Uniqueness of Solutions of Fuzzy Volterra Integro-differential Equations*, IPMU 2010, Part II, CCIS 81 (2010) 491-500.

[2] T. Allahviranloo, A. Amirteimoori, M. Khezerloo, S. Khezerloo, *A New Method for Solving Fuzzy Volterra Integro-Differential Equations*, Australian Journal of Basic and Applied Sciences 5 (2011) 154 -164, http://dx.doi.org/10.3850/978-981-08-5118-7060/.

[3] T. Allahviranloo, S. Salahshour, S. Abbasbandy, *Solving fuzzy fractional differential equations by fuzzy Laplace transforms*, Original Research Article Communications in Nonlinear Science and Numerical Simulation 17 (2012) 1372-1381, http://dx.doi.org/10.1016/j.cnsns.2011.07.005/.

[4] T. Allahviranloo, S. Abbasbandy, O. Sedaghatfar, P. Darabi, *A New Method for Solving Fuzzy Integro-Differential Equation Under Generalized Differentiability*, J. Neural Computing and Applications, http://dx.doi.org/10.1007/s00521-011-0759-3/.

[5] T. Allahviranloo, M. Ghanbari, E. Haghi, A. Hosseinzadeh, R. Nouraei, *"A note on "Fuzzy linear systems"*, Journal of Fuzzy Sets and Systems 177 (2011) 87-92.

[6] R. P. Agarwal, D. ORegan, V. Lakshmikantham, *A stacking theorem approach for fuzzy differential equations*, J. Nonlinear Anal. 55 (2003) 299- 312, http://dx.doi.org/10.1016/S0362-546X(03)00241-4/.

[7] R. P. Agarwal, D. ORegan, V. Lakshmikantham, *Viability theory and fuzzy differential equations*, J. Fuzzy Sets and Systems 151 (2005) 536-580, http://dx.doi.org/10.1016/j.fss.2004.08.008/.

[8] Bashir Ahmad, S. Sivasundaram, *Dynamics and stability of impulsive hybrid setv alued integro-differential equations with delay*, Nonlinear Analysis 65 (2006) 2082-2093.

[9] Bashir Ahmad, S. Sivasundaram, *Some stability criteria for set integro-differential equations*, Mathematical Inequalities & Applications 10 (2007) 597 605.

[10] V. Lakshmikantham, A. A. Tolstonogov, *Existence and interrelation between set and fuzzy differential equations*, Nonlinear Anal. 55 (2003) 255-268.

[11] V. Lakshmikantham, S. Leela, A. S. Vatasala, *Interconnection between set and fuzzy differential equations*, Nonlinear Anal. 54 (2003) 351-360.

[12] V. Lakshmikantham, S. Leela, *Fuzzy Differential Systems and the New Concept of Stability*, Nonlinear Dynamics and Systems Theory 2 (2001) 111-119.

[13] V. Lakshmikantham, T. Gnana Bhashar, *Lyapunov stability for set differential equations*, Dynam.Systems Appl. 13 (2004) 1-10.

[14] V. Lakshmikantham, M. R. M. Rao, *Theory of Integro-differential Equations*, Gordon & Breach, London, 1995.

[15] V. Lakshmikantham, Mohapatra, *Theory of fuzzy differential equations and inclusions*, J. Taylor Francis, London, 2003.

[16] V. Lakshmikantham, T. Gnana Bhaskar, J. Vasundhara Devi, *Theory of set differential equations in metric spaces* , Cambridge Scientiffic Publisher, UK, 2006.

[17] Nguyen Dinh Phu, Tran Thanh Tung, *On the Existence of fuzzy solutions of fuzzy control differential equations*, J. Science and Technology Development 10 (2006) 5-15.

[18] Nguyen Dinh Phu, Lam Quoc Dung, *On the Stability and global controllability for Fuzzy set control differential equations*, International Journal Reliability and safety 5 (2011) 320 - 335.

[19] Nguyen Dinh Phu, Ngo Van Hoa, Ho Vu, *On Comparisons of Set solutions for Fuzzy Control Integro-Differential Systems, Advanced Research in Applied Mathematics*, Institute of Advanced Scientiffic Research, http://dx.doi.org/10.5373/jaram.1081.082011/.

[20] B. G. Pachpatte, *Integral and finite difference inequalies and applications*, Elsevier Pub-lisher, UK, 2006.

[21] Jong Yeoul Park, Jae Ug Jeong, *On existence and uniqueness of solutions of fuzzy integro-differential equations*, Indian J. pure appl. Math. 10 (2003) 1503 -1512.

[22] Ho Vu, Ngo Van Hoa, Nguyen Dinh Phu, *Stability Results of Fuzzy Control Integro-Differential System*, Journal of Advanced Research in Dynamical and Control Systems 4 (2012) 49-59.

[23] X. Xue, Y. Fu, *On the structure of solutions for fuzzy initial value problem*, J. Fuzzy Sets and Systems 157 (2006) 212-219, http://dx.doi.org/10.1016/j.fss.2005.06.009/.

[24] L. A. Zadeh, *Fuzzy sets*, J. Information and Control, 8 (1965) 338-353, http://dx.doi.org/10.1016/S0019-9958(65)90241-X/.

Vu Ho, he has born in Long An, Viet Nam. He is PhD in Mathematics and currently working at Division Applied Mathematics, University of Ton Duc Thang HCM City, Viet Nam. Main research interest includ fuzzy differential equation, fuzzy integro-differential equation, set differential equation, control theory,

Ngo Van Hoa, he has born in Long An, Viet Nam. Got B.Sc degrees in Mathematics - Computer Science in 2009 and now he is PhD student (2012-2017) in Mathematics at University of Natural Sciences Ho Chi Minh City, Viet Nam and and currently working at Division Applied Mathematics, University of Ton Duc Thang HCM. Main research interest includ fuzzy differential equation, interval differential equation, set differential equation, sheaf control,

Vitaly Ivanovych Slyn'ko, he was graduated Cherkassy National University, in 1996 1999, Ukraine. His postgraduate was in S. P. Timoshenko Institute of mechanics, Kyiv, Ukraine, 1999 2002. Ph.D. of physical and mathematical sciences was awarded by the specialized academic council of S.P. Timoshenko Institute of mechanics National Academy of Sciences of Ukraine in 2002. The main research interests in mathematics and mechanics: stability of solutions of differential equations with impulsive action in critical cases; robust stability of linear systems of differential equations with impulsive action;...