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Int. J. Industrial Mathematics Vol. 1, No. 1 (2009) 47-53

Prioritization method for non-extreme efficient unitsin data envelopment analysis

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Abstract

Super efficiency data envelopment analysis(DEA) model can be used in ranking the performance of efficient decision making units(DMUs). In DEA, non-extreme efficient units have a super efficiency score one and the existing super efficiency DEA models do not provide a complete ranking about these units. In this paper, we will propose a method for ranking the performance of the extreme and non-extreme efficient units.

Keywords : Data envelopment analysis, efficiency, super-efficiency.

1 Introduction

In models of data envelopment analysis(DEA), the best performer has full efficiency status denoted by unity, and we know that usually, there are plural decision making units (DMUs) which have this "efficient status". In order to rank efficient units, another approach or modification is required. Often decision makers are interested in a complete ranking, beyond the dichotomized classification, in order to refine the evaluation of the units. Super efficiency DEA models can be used in ranking the performance of efficient units.

In order to discriminate the performance among efficient DMUs, a super efficiency DEA model in which a DMU under consideration is excluded from the reference set was first developed by Andersen and Petersen (1993). During the recent years, the issue of super efficiency in DEA has been extensively studied. By now, many papers on super efficiency (over 50) have been published over the last decade within the DEA context. See, for instances, Torgersen et al.(1996), Mehrabian et al.(1999), Tone(2002), Bogetoft et al.(2004), Chen et al.(2004), Chen (2005), Jahanshahloo et al.(2007), Bernroider (2007) and Shanling (2007).

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As it is, When a DMU under evaluation is excluded from the reference set of the original DEA models, the resulting DEA models are called super efficiency DEA models. In super efficiency DEA models, the efficient units have a super efficiency score greater than or equal to one. Specifically, the extreme efficient units have a super efficiency score greater than unity, and non-extreme efficient units have a score one. Although, the super efficiency DEA models provide a complete ranking on the extreme efficient units, they can not provide more information about the performance of non-extreme efficient units. Our object here is tow-fold. We discuss first the super efficiency issue based on the dominance factors. Secondly, we propose a complete ranking of the extreme and non-extreme efficient units.

The rest of this paper is organized as follows: the next section of the paper presents the various DEA models. Then, we introduce our measure of super efficiency. In section four we will present the general approach. Conclusions appear in section five.

2 Preliminaries

Suppose we have n DMUs $\{DMU_j \ j = 1, 2, \ldots, n\}$, which produce s outputs, $y_{rj}; r = 1, \ldots, s$ by utilizing m inputs, $x_{ij}; i = 1, \ldots, m$. Relative efficiency is defined as the ratio of total weighted outputs to the total weighted inputs. The efficiency measure for DMU_o is defined as

$$e_o = \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}}$$

where the weights u_r and v_i are non-negative. To estimate the DEA efficiency of DMU_o, we use the following original DEA model of Charnes, Cooper and Rhodes (1978):

$$\begin{aligned} & Max \quad e_o = \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} \\ & \text{subject to :} \\ & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1, \quad j = 1, \ \ldots, \ n, \\ & u_r \geq \epsilon, \quad r = 1, \ \ldots, \ s, \\ & v_i \geq \epsilon, \quad i = 1, \ \ldots, \ m. \end{aligned} \tag{2.1}$$

where $\epsilon > 0$ is a non-archimedean construct. The efficiency ratio ranges between zero and one, with DMU_o being considered relatively efficient if it receives a score of one. The fractional program (1) can be translated into a linear programming problem using the

Charnes and Cooper (1962) transformation as

$$\begin{array}{l} Max \quad e_o = \sum_{r=1}^{s} u_r y_{ro} \\ \text{subject to :} \\ \sum_{i=1}^{m} v_i x_{io} = 1, \\ \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, \ \ldots, \ n, \\ u_r \geq \epsilon\rho, \quad r = 1, \ \ldots, \ s, \\ v_i \geq \epsilon\rho, \quad i = 1, \ \ldots, \ m. \end{array} \tag{2.2}$$

Let an optimal solution of (2) be (u^*, v^*). Then, we have an optimal solution of (1) as expressed by $(\bar{u}^*, \bar{v}^*) = (\frac{u^*}{\rho}, \frac{v^*}{\rho})$.
The set of DMUs can be partitioned into three sets: E (the set of extreme efficient units), NE (the set of non-extreme efficient units) and F (the set of inefficient units). However, this model does not provide more information about the units in $E \bigcup NE$.
Andersen and Petersen[1] developed a procedure for ranking efficient units. Their methodology enables an extreme efficient unit o to achieve an efficiency score greater than one by removing the *o-th* constraint in (2) as shown in model (3):

$$\begin{array}{l} Max \quad \phi_o = \sum_{r=1}^{s} u_r y_{ro} \\ \text{subject to :} \\ \sum_{i=1}^{m} v_i x_{io} = 1, \\ \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \ j = 1, \ \ldots, \ n, \ j \neq o, \\ u_r \geq \epsilon, \quad r = 1, \ \ldots, \ s, \\ v_i \geq \epsilon, \quad i = 1, \ \ldots, \ m. \end{array} \tag{2.3}$$

Let the optimal objective value of super-CCR be ϕ^*. For an efficient DMU_o, ϕ_o is not less than unity and this value indicates *super-efficiency* of DMU_o. Tone (2002) has defined the super SBM efficiency of DMU_o as the optimal objective function value δ_o of the following program:

$$\begin{array}{l} Min \quad \delta_o = \frac{\frac{1}{m} \sum_{i=1}^{m} \frac{\bar{x}_i}{x_{io}}}{\frac{1}{s} \sum_{r=1}^{s} \frac{\bar{y}_r}{y_{ro}}} \\ \text{subject to :} \\ \sum_{j=1, j \neq o}^{n} \lambda_j x_{ij} \leq \bar{x}_i, \ i = 1, \ldots, m, \\ \sum_{j=1, j \neq o}^{n} \lambda_j y_{rj} \geq \bar{y}_r, \ r = 1, \ldots, s, \\ \bar{x}_i \geq x_{io}, \ i = 1, \ldots, m, \\ 0 \leq \bar{y}_r \leq y_{ro}, \ r = 1, \ldots, s, \\ \lambda_j \geq 0, \ j = 1, \ldots, n. \end{array} \tag{2.4}$$

For an efficient DMU_o, δ_o is not less than unity. However, in both models (3) and (4), the non-extreme efficient units have a super efficiency score one and these models do not provide a complete ranking about the efficient units.

3 A Measure of Super efficiency

In this section, we discuss the super-efficiency issue under the assumption that the $DMU(x_o, y_o)$ is a non-extreme efficient unit. As we know, the super efficiency score of an extreme efficient unit is greater than one, whereas, this score is equal to one for a non-extreme efficient unit. This means that DMUs in E are the top-ranked units as compared with the units in NE. Hence, the ranking procedure is focused on the units in NE. Without lose of generality, we assume that all units in NE are extreme efficient in P_{NE}, in which

$$P_{NE} = \{(x,y) : x \geq \sum_{j \in NE} \lambda_j x_j,\ y \leq \sum_{j \in NE} \lambda_j y_j,\ \lambda_j \geq 0,\ j \in NE\} \quad (3.5)$$

(This assumption will be relaxed in section 4.)
Consider the subset $P_{NE}^{(x_o,y_o)}$ of the set P_{NE} spanned by $(x_j, y_j) :\ j \in NE, j \neq o$ as

$$P_{NE}^{(x_o,y_o)} = \{(\bar{x},\ \bar{y}) : \bar{x} \geq \sum_{j \in NE, j \neq o} \lambda_j x_j,\ \bar{y} \leq \sum_{j \in NE, j \neq o} \lambda_j y_j,\ \lambda_j \geq 0,\ j \in NE, j \neq o\} \quad (3.6)$$

Obviously, $P_{NE}^{(x_o,y_o)}$ is not empty. Let $\xi_1 = \{i :\ x_{io} > 0\}$ and $\xi_2 = \{r :\ y_{ro} > 0\}$. When $x_{io} = 0$, DMU_o has no function as to the input i and we can exclude these inputs from the analysis. This is true when $y_{ro} = 0$. Consider an expression for DMU_o as

$$\begin{aligned}
&\sum_{j \in NE, j \neq o} \lambda_j x_{ij} \leq \theta_i x_{io}, \quad \theta_i x_{io} \geq x_{io},\ i \in \xi_1, \\
&\sum_{j \in NE, j \neq o} \lambda_j y_{rj} \geq \phi_r y_o, \quad \phi_r y_{ro} \leq y_{ro},\ r \in \xi_2, \\
&\lambda_j \geq 0,\ j \in NE, j \neq o.
\end{aligned}$$

We expand the i-th input of x_o by θ_i and simultaneously contract the r-th output of y_o by ϕ_r, to meet the frontier of $P_{NE}^{(x_o,y_o)}$. Using this expression, we define the super efficiency index ψ_o as

$$\psi_o = \frac{[\frac{1}{m}\sum_{i \in \xi_1} \theta_i^* + \frac{1}{s}\sum_{r \in \xi_2} \phi_r^*]}{2} \quad (3.7)$$

in which θ_i^* and ϕ_r^* are the optimal solution of the following program

$$\begin{aligned}
&\psi_o = Min \ \ \frac{z_o}{\rho_o} \\
&\text{subject } to : \\
&\sum_{j \in NE, j \neq o}^{n} \lambda_j x_{ij} \leq \theta_i x_{io}, \quad i \in \xi_1, \\
&\sum_{j \in NE, j \neq o}^{n} \lambda_j y_{rj} \geq \phi_r y_{ro}, \quad r \in \xi_2, \\
&\theta_i x_{io} \geq x_{io}, \quad i \in \xi_1, \qquad\qquad (3.8) \\
&\phi_r y_{ro} \leq y_{ro}, \quad r \in \xi_2, \\
&z_o \geq \theta_i x_{io}, \quad i \in \xi_1, \\
&\rho_o \leq \phi_r y_{ro}, \quad r \in \xi_2, \\
&\lambda_j \geq 0, \quad j \in NE, j \neq o.
\end{aligned}$$

Minimizing ψ_o in (8) means that z_o is minimized and simultaneously, ρ_o is maximized. In other word, we minimize the maximum relative values of the input variables and maximize

the minimum relative values of the output variables. Hence (8) measures how far DMU_o is from the frontier.
In this program, we look for a virtual DMU on the frontier of $P_{NE}^{(x_o,y_o)}$ so that the weighted distance from x_o to frontier is minimized and simultaneously, the weighted distance from y_o to frontier is maximized.
The fractional program (8) can be translated into a linear programming problem as

$$\begin{aligned}
&\psi_o = Min \quad \bar{z}_o \\
&\text{subject } to: \\
&\textstyle\sum_{j\in NE, j\neq o} \bar{\lambda}_j x_{ij} \leq \bar{\theta}_i x_{io}, \quad i \in \xi_1, \\
&\textstyle\sum_{j\in NE, j\neq o} \bar{\lambda}_j y_{rj} \geq \bar{\phi}_r y_{ro}, \quad r \in \xi_2, \\
&\bar{\theta}_i x_{io} \geq t x_{io}, \quad i \in \xi_1, \\
&\bar{\phi}_r y_{ro} \leq t y_{ro}, \quad r \in \xi_2, \\
&\bar{z}_o \geq \bar{\theta}_i x_{io}, \quad i \in \xi_1, \\
&\bar{\phi}_r y_{ro} \geq 1, \quad r \in \xi_2, \\
&\bar{\lambda}_j \geq 0, \quad j \in NE, j \neq o.
\end{aligned} \tag{3.9}$$

Let an optimal solution of (9) be $(\bar{\theta}^*,\ \bar{\phi}^*,\ \bar{\lambda}^*,\ t^*)$. Then, we have an optimal solution of (8) as $\theta = \frac{\bar{\theta}^*}{t^*},\ \phi = \frac{\bar{\phi}^*}{t^*},\ \lambda = \frac{\bar{\lambda}^*}{t^*}$.
We illustrate the proposed super efficiency measure with a small-scale example consisting of seven DMUs. The DMUs use two inputs to produce a single output whose value is normalized to one for each DMU. The CCR model indicates that all DMUs are efficient and $E = \{1, 2, 3, 4\}$ and $NE = \{5, 6, 7\}$. It can be seen that model (3) yields to a score one to DMU_5, DMU_6 and DMU_7. We have calculated the proposed super efficiency measure for each DMU. The data set, the super efficiency score ϕ_o and the super efficiency measure ψ_o are listed in table 1. Our approach shows that DMU_2 is the top-ranked DMU followed by DMU_1, DMU_3, DMU_4, DMU_5, DMU_7 and DMU_6.

4 General approach

So far we have discussed the super efficiency issue under the assumption that all units in NE are extreme efficient in P_{NE}. In this section, we will relax this assumption and extend our approach to general case. As we know, in P_{NE}, all extreme efficient units are the top-ranked units as compared with the non-extreme units. Hence, first, extreme efficient units in P_{NE} will be ranked, and then we focus on the non-extreme efficient units and the procedure is repeated. In fact, a set of DMUs in P_{NE} can be divided into different levels of efficient frontiers. If we remove the extreme efficient units of P_{NE}, then, the remaining non-extreme efficient units will form a new second level efficient frontier. This frontier

(Table 1. The data and results used in the simple example)

DMU_j	x_1	x_2	y	e_o	ϕ_o	ψ_o
#1	1	10	1	1.00	1.9875(2)	-
#2	10	1	1	1.00	2.5800(1)	-
#3	3	5	1	1.00	1.0732(3)	-
#4	5	3	1	1.00	1.0462(4)	-
#5	2	7.5	1	1.00	1	1.6875(5)
#6	4	4	1	1.00	1	1.0590(7)
#7	6	2.6	1	1.00	1	1.3270(6)

The number in parentheses represents rank.

consists of all non-extreme efficient units. If we remove the new extreme efficient units of P_{NE}, a third level efficient frontier is formed, and so on, until no non-extreme efficient unit is left. Each efficient frontier needs a ranking procedure, separately, for ranking units located on the frontier.
Let $NE^{(1)} = \{DMU_j : \ j = 1, \ldots, n\} - (E \bigcup F)$ be the set of all non-extreme efficient units. We define

$$P_{NE^{(1)}} = \{(x, y) : x \geq \sum_{j \in NE^{(1)}} \lambda_j x_j, \ y \leq \sum_{j \in NE^{(1)}} \lambda_j y_j, \ \lambda_j \geq 0, \ j \in NE^{(1)}\}$$

and

$$NE^{(2)} = NE^{(1)} - E^{(1)}$$

where $E^{(1)}$ is the set of all extreme efficient units in $P_{NE^{(1)}}$. We interactively define

$$P_{NE^{(k)}} = \{(x, y) : x \geq \sum_{j \in NE^{(k)}} \lambda_j x_j, \ y \leq \sum_{j \in NE^{(k)}} \lambda_j y_j, \ \lambda_j \geq 0, \ j \in NE^{(k)}\}$$

and

$$NE^{(k+1)} = NE^{(k)} - E^{(k)}$$

where $E^{(k)}$ is the set of all extreme efficient units in $P_{NE^{(k)}}$.
In this manner, we identify several levels of efficient frontiers, and the proposed ranking procedure can be applied on each levels.

5 Conclusion

In the existing super efficiency DEA models, the non-extreme efficient units have a super efficiency score one and these models do not provide more information about these units. In order to obtain a complete ranking of efficient DMUs when non-extreme efficient units exist, a modified super efficiency DEA model is proposed.

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