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Comment on" Frames of Subspaces"

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Abstract

In this paper we give a counter example for one of the lemmas of the paper" Frame of subspace in Wavelets, frames and operator theory" by P.G. Casazza and G. Kutyniok .

Keywords : Hilbert space; Fusion Frame; Orthogonal Projection; Orthonormal basis.

1 Introduction

Frames were first introduced by Duffin and Schaeffer[4] in the context of nonharmonic Fourier series, and today frames play important roles in many applications in mathematics, science, and engineering, including time-frequency analysis [5], internet coding [6], speech and music processing[11], communication [9], multiple antenna coding [8], medicine [10], quantum computing [7], and many other areas.

For the discussion of the following section, we state here some definitions, notations and known results. For convenience of readers, we suggest that one refer to [1, 2, 3] for details.

Let H be a separable Hilbert space and let I be a countable (or finite) index set. If W is a closed subspace of H , we denote the orthogonal projection of H onto W by π_W .

A sequence $F = \{f_i\}_{i\in I}$ in H is a frame for H if there exist constants $0 < A \leq B < \infty$ such that $A\|f\|^2 \leq \sum_{i\in I} | \langle f, f_i\rangle|^2 \leq B \parallel f \parallel^2$ for all $f \in H$. The numbers A, B are called lower and upper frame bounds, respectively. The family F is called a tight frame if $A = B$, it is a Parseval frame if $A = B = 1$, it is a x-uniform frame if $\parallel f_i \parallel = \parallel f_j \parallel = x$ for all $i, j \in I$ and an exact frame if it ceases to be a frame when any one of its elements is removed. If the right-handed of mentioned inequality holds, then we say that F is a Bessel sequence and call B the Bessel bound. The operator $S_F : H \to H$ is called frame

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operator and defined by $S_F(f) = \sum_{i\in I}\langle f, f_i\rangle f_i$ which leads us to reconstruction formula $f = \sum_{i\in I}\langle f, f_i\rangle S_F^{-1} f_i = \sum_{i\in I}\langle f, S_F^{-1} f_i\rangle f_i$ for all $f \in H$ and also we have $AId \leq S_F \leq BID$ for frame operator.

Let $\{W_i\}$ be a family of closed subspaces of H and let $\{v_i\}$ be a family of weights, i.e. $v_i > 0$ for all $i \in I$. Then $W = \{(W_i, v_i)\}_{i\in I}$ is a fusion frame for H if there exist constants $0 < C \leq D < \infty$ such that $C < \|f\|^2 \leq \sum_{i\in I} v_i^2 \|\pi_{w_i}(f)\|^2 \leq D\|f\|^2$ for all $f \in H$. The numbers C, D are called lower and upper fusion frames bounds, respectively. The family W is called a tight fusion frame if $C = D$, it is a Parseval fusion frame if $C = D = 1$, it is a v-uniform fusion frame if $v_i = v_j = v$ for all $i, j \in I$ and an orthonormal fusion basis for H if $H = \oplus_{i\in I} W_i$. If the right-handed of mentioned inequality holds, then we say that W is a fusion Bessel sequence and call B the fusion Bessel bound.The operator $S_F : H \longrightarrow H$ defined by $S_F(f) = \sum_{i\in I}\langle f, f_i\rangle f_i$ is called the fusion frame operator which leads us to reconstruction formula $f = \sum_{i\in I} v_i^2 S_W^{-1}\pi_{W_i}(f) = \sum_{i\in I} v_i^2 \pi_{W_i} S_W^{-1}(f)$ for all $f \in H$. Also we have $CId \leq S_W \leq DId$ for frame operator.

2 Main Results

In the following Lemma [1] Casazza and Kutyniok have proved if $W = \{(W_i, v_i)\}_{i\in I}$ is a fusion frame for H, then the intersection of a closed subspace V of H with the family of subspaces $\{W_i\}_{i\in I}$ of H which have the same weights, i.e. $W_v = \{(W_i \cap V, v_i)\}_{i\in I}$,is a fusion frame .

Lemma 2.1. *Let V be a subspace of H and $W = \{W_i, v_i\}_{i\in I}$ be a fusion frame forH with bounds C, D then $W_V = \{(W_i \cap V, v_i)\}_{i\in I}$ is a fusion frame for V with bounds C, D*

Remark 2.1. *The authors have used the following equation in proof of the above Lemma*

$$\sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 = \sum_{i\in I} v_i^2 \|\pi_{W_i\cap V}(f)\|^2 \qquad (for\ all\ f \in H) \tag{2.1}$$

But this equation is not correct in general. Actually, the right-hand side of Eq. (2.1) could be equal to zero. In fact, if $W_i \cap V = \{o\}, (\forall i \in I)$ then $\pi_{W_{i\cap V}}(f) = 0$ for all $f \in H$. Therefore $\sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 = \sum_{i\in I} v_i^2 \|\pi_{W_i\cap V}(f)\|^2 = 0$. Thus $C\|f\|^2 \leq o$ which is in contradiction with $o < C \leq D < \infty$

Now, we state a counter example as follow.

Example 2.1. *Let $n \in \mathbb{N}$ and $n \geq 2$, set $N = \{1, 2, ..., n\} \subseteq \mathbb{N}$. Then $E = \{e_i\}_{i\in N}$ is a canonical orthonormal basis for $l^2(N)$, where $e_j = \{\delta_{ij}\}_{i\in N}, (\forall j \in N)$. Let $W_1 = \overline{span}\{e_1 + e_2\}$ and for every $i \in N, i \neq 1$, $W_i = \overline{span}\{e_i\}$. Then $\{W_i\}_{i\in N}$ is a family of subspaces of $l^2(N)$. Put $V = \overline{span}\{e_1\}$, we show that there exists a family of weights $\{v_i\}_{i\in I}$ such that $W = \{(W_i, v_i)\}_{i\in I}$.*

Let $\{f_i\}_{i\in I}$ be an orthonormal basis for a subspace W of H then

$$\pi_W f = \sum_{i\in I}\langle \pi_W f, f_i\rangle f_i = \sum_{i\in I}\langle f, \pi_W f_i\rangle f_i = \sum_{i\in I}\langle f, f_i\rangle f_i \qquad (\forall f \in H)$$

so

$$\begin{aligned} \pi_{w_1} f &= \langle f, \tfrac{e_1+e_2}{\sqrt{2}}\rangle \tfrac{e_1+e_2}{\sqrt{2}} \\ &= \tfrac{1}{2}\langle f, e_1 + e_2\rangle (e_1 + e_2) \\ &= \tfrac{1}{2}\langle f, e_1\rangle e_1 + \tfrac{1}{2}\langle f, e_1\rangle e_2 + \tfrac{1}{2}\langle f, e_2\rangle e_1 + \tfrac{1}{2}\langle f, e_2\rangle e_2 \end{aligned}$$

we note $\pi_{W_i} f = \langle f, e_i\rangle e_i, (\forall i \in N, i \neq 1,\ \forall f \in H)$, *so* $\|\pi_{W_i} f\|^2 = |\langle f, e_i\rangle|^2$. *On the other hand*

$$\begin{aligned}\|\pi_W f\|^2 &= \langle \tfrac{1}{2}\langle f, e_1\rangle e_1 + \tfrac{1}{2}\langle f, e_1\rangle e_2 + \tfrac{1}{2}\langle f, e_2\rangle e_1 + \tfrac{1}{2}\langle f, e_2\rangle e_2, \\ &\tfrac{1}{2}\langle f, e_1\rangle e_1 + \tfrac{1}{2}\langle f, e_1\rangle e_2 + \tfrac{1}{2}\langle f, e_2\rangle e_1 + \tfrac{1}{2}\langle f, e_2\rangle e_2\rangle \\ &= \tfrac{1}{2}|\langle f, e_1\rangle|^2 + \tfrac{1}{2}|\langle f, e_2\rangle|^2 + |\langle f, e_1\rangle||\langle f, e_2\rangle|\end{aligned}$$

let $v_1 = \sqrt{2}, v_i = 1$ *where,* $i \neq 1, i \in N$ *we show that* $W = \{(W_i, v_i)\}_{i=I}$ *is a fusion frame of subspaces for* $l^2(N)$. *We have*

$$\begin{aligned}\textstyle\sum_{i=1}^n v_i^2 \|\pi_{W_i}(f)\|^2 &= 2[\tfrac{1}{2}|\langle f, e_1\rangle|^2 + \tfrac{1}{2}|\langle f, e_2\rangle|^2 + |\langle f, e_1\rangle|\,|f, e_2\rangle|] + \textstyle\sum_{i=2}^n |\langle f, e_i\rangle|^2 \\ &= \textstyle\sum_{i=1}^n |\langle f, e_i\rangle|^2 + |\langle f, e_2\rangle|^2 + 2|\langle f, e_1\rangle|\,|\langle f, e_2\rangle| \\ &= \|f\|^2 + |\langle f, e_2\rangle|^2 + 2|\langle f, e1\rangle|\,|\langle f, e_2\rangle|\end{aligned}$$

If $|\langle f, e_1\rangle| \geq |\langle f, e_2\rangle|$, *then*

$$2|\langle f, e_2\rangle|^2 \leq 2|\langle f, e_1\rangle|\,|\langle f, e_2\rangle| \leq 2|\langle f, e_1\rangle|^2$$

Thus

$$\|f\|^2 \leq \|f\|^2 + 3|\langle f, e_2\rangle|^2 \leq \sum_{i=1}^n v_i^2 \parallel \pi_{W_i}(f) \parallel^2 \leq \parallel f \parallel^2 + \mid \langle f, e_2\rangle|^2 + 2|\langle f, e_1\rangle|^2 \leq 3 \parallel f \parallel^2$$

If $\mid \langle f, e_1\rangle| <\mid \langle f, e_2\rangle|$, *then*

$$2|\langle f, e_1\rangle|^2 < 2|\langle f, e_1\rangle|\,|\langle f, e_2\rangle| < 2|\langle f, e_2\rangle|^2$$

so

$$\|f\|^2 \leq \|f\|^2 + |\langle f, e_2\rangle|^2 + |\langle f, e_1\rangle|^2 < \sum_{i=1}^n v_i^2 \|\pi_{w_i}(f)\|^2 < \parallel f \parallel^2 + 3 \mid \langle f, e_2\rangle|^2 \leq 4 \parallel f \parallel^2$$

Then $W = \{W_i, v_i\}_{i\in I}$ *is a fusion frame for* $l^2(N)$ *with bounds* $C = 1$ *and* $D = 4$. *On the other hand,* $W_i \cap V = \{0\}$. *Thus* $W_V = \{(W_i \cap V, v_i)\}_{i=I}$ *could not be fusion frames for* $l^2(N)$.

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