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Complex Fuzzy Linear Systems

M. A. Jahantigh *[a] , S. Khezerloo [b], M. Khezerloo [b]

(a) *Department of Mathematics, Zahedan Branch, Islamic Azad University, Zahedan, Iran.*

(b) *Islamic Azad University, Young Researcher Club, Ardabil Branch, Ardabil, Iran.*

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Abstract

In this paper, a general complex fuzzy linear system is introduced and a numerical procedure for calculating solution is proposed. Finally, some numerical examples are given to illustrate the mentioned method.

Keywords : Fuzzy linear system; Fuzzy solution ; Fuzzy number vector; Complex fuzzy linear system

1 Introduction

Systems of simulations linear equations play major roles in various areas such as mathematics, physics, statistics, engineering and social sciences. Since in many applications at least some of the systems parameters and measurements are represented by fuzzy rather than crisp numbers, it is important to develop mathematical models and numerical procedures that would appropriately treat general fuzzy linear systems and solve them. The concept of fuzzy numbers and arithmetic operations with these numbers were first introduced and investigated by Zadeh [12] and, etc. One of the major applications using fuzzy number arithmetic is treating linear systems in which parameters are all partially respected by fuzzy number [7] and, etc. This paper is organized as following:

In Section 2, the basic concept of fuzzy number operation is brought. In Section 3, the main section of the paper, complex fuzzy linear system (CFLS)is solved. The proposed idea is illustrated by some examples in the Section 4. Finally conclusion is drawn in Section 5.

2 Basic concepts

There are various definitions for the concept of fuzzy numbers ([8, 10]).

*Corresponding author. Email address: mohamadjahantig@yahoo.com

Definition 2.1. *An arbitrary fuzzy number u in the parametric form is represented by an ordered pair of functions (u_r^-, u_r^+) which satisfy the following requirements:*

1. *u_r^- is a bounded left-continuous non-decreasing function over $[0,1]$.*
2. *u_r^+ is a bounded left-continuous non-increasing function over $[0,1]$.*
3. *$u_r^- \leq u_r^+, \quad 0 \leq r \leq 1$.*

A crisp number m is simply represented by $u_r^- = u_r^+ = m, \quad 0 \leq r \leq 1$. If $u_1^- < u_1^+$, we have a fuzzy interval and if $u_1^- = u_1^+$, we have a fuzzy number. In this paper, we do not distinguish between numbers or intervals and for simplicity we refer to fuzzy numbers as interval. We also use the notation $u_r = [u_r^-, u_r^+]$ to denote the r-cut of arbitrary fuzzy number u. If $u = (u_r^-, u_r^+)$ and $v = (v_r^-, v_r^+)$ are two arbitrary fuzzy numbers, the arithmetic operations are defined as follows:

Definition 2.2. *(Addition)*

$$u + v = (u_r^- + v_r^-, u_r^+ + v_r^+) \tag{2.1}$$

and in the terms of r-cuts

$$(u + v)_r = [u_r^- + v_r^-, u_r^+ + v_r^+], \quad r \in [0,1] \tag{2.2}$$

Definition 2.3. *(Subtraction)*

$$u - v = (u_r^- - v_r^+, u_r^+ - v_r^-) \tag{2.3}$$

and in the terms of r-cuts

$$(u - v)_r = [u_r^- - v_r^+, u_r^+ - v_r^-], \quad r \in [0,1] \tag{2.4}$$

Definition 2.4. *(Scalar multiplication)*
For given $k \in \Re$

$$ku = \begin{cases} (ku_r^-, ku_r^+), & k > 0 \\ (ku_r^+, ku_r^-), & k < 0 \end{cases} \tag{2.5}$$

and

$$(ku)_r = [\min\{ku_r^-, ku_r^+\}, \max\{ku_r^-, ku_r^+\}] \tag{2.6}$$

In particular, if $k = 1$, we have

$$-u = (-u_r^+, -u_r^-)$$

and with α-cuts

$$(-u)_r = [-u_r^+, -u_r^-], \quad r \in [0,1]$$

Definition 2.5. *(Multiplication)*

$$uv = ((uv)_r^-, (uv)_r^+) \tag{2.7}$$

and

$$\begin{aligned} (uv)_r^- &= \min\{u_r^- v_r^-, u_r^- v_r^+, u_r^+ v_r^-, u_r^+ v_r^+\} \\ (uv)_r^+ &= \max\{u_r^- v_r^-, u_r^- v_r^+, u_r^+ v_r^-, u_r^+ v_r^+\}, \quad r \in [0,1] \end{aligned} \tag{2.8}$$

Definition 2.6. *(Division)*
If $0 \notin [v_0^-, v_0^+]$

$$\frac{u}{v} = ((\frac{u}{v})_r^-, (\frac{u}{v})_r^+) \tag{2.9}$$

and

$$\begin{aligned} (\tfrac{u}{v})_r^- &= \min\{\tfrac{u_r^-}{v_r^-}, \tfrac{u_r^-}{v_r^+}, \tfrac{u_r^+}{v_r^-}, \tfrac{u_r^+}{v_r^+}\} \\ (\tfrac{u}{v})_r^+ &= \max\{\tfrac{u_r^-}{v_r^-}, \tfrac{u_r^-}{v_r^+}, \tfrac{u_r^+}{v_r^-}, \tfrac{u_r^+}{v_r^+}\}, \quad r \in [0,1] \end{aligned} \tag{2.10}$$

Definition 2.7. *[9], The fuzzy linear system,*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_n \end{cases} \tag{2.11}$$

is called a fuzzy linear system where $A = [a_{ij}]_{i,j=1}^n$ *is crisp coefficient matrix and* y_i *is a fuzzy number.*

Consider the fuzzy linear system (2.11). Transform its $n \times n$ coefficient matrix A into $(2n) \times (2n)$ matrix as in the following:

$$\begin{aligned} s_{11}x_1^- + \cdots + s_{1n}x_n^- + s_{1,n+1}(-x_1^+) + \cdots + s_{1,2n}(-x_n^+) &= y_1^- \\ &\vdots \\ s_{n1}x_1^- + \cdots + s_{nn}x_n^- + s_{n,n+1}(-x_1^+) + \cdots + s_{n,2n}(-x_n^+) &= y_n^- \\ s_{n+1,1}x_1^- + \cdots + s_{n+1,n}x_n^- + s_{n+1,n+1}(-x_1^+) + \cdots + s_{n+1,2n}(-x_n^+) &= -y_1^+ \\ &\vdots \\ s_{2n,1}x_1^- + \cdots + s_{2n,n}x_n^- + s_{2n,n+1}(-x_1^+) + \cdots + s_{2n,2n}(-x_n^+) &= -y_n^+ \end{aligned} \tag{2.12}$$

where s_{ij} is determined as follows:

$$\begin{aligned} s_{ij} = s_{i+n,j+n} &= a_{ij}, \quad if \;\; a_{ij} \geq 0 \\ s_{i,j+n} = s_{i+n,j} &= -a_{ij}, \quad if \;\; a_{ij} < 0 \end{aligned} \tag{2.13}$$

and any s_{ij} which is not determined by Eq. (2.13) is zero. Using matrix notation,

$$SX = Y$$

where

$$X = (x_1^-, \ldots, x_n^-, -x_1^+, \ldots, -x_n^+)^t, \qquad Y = (y_1^-, \ldots, y_n^-, -y_1^+, \ldots, -y_n^+)^t$$

The structure of S implies that s_{ij}, $1 \leq i, j \leq n$, and that

$$S = \begin{bmatrix} B & C \\ C & B \end{bmatrix}$$

where B contains the positive entries of A, C the absolute values of the negative entries of A and $A = B - C$. Now we must calculate S^{-1} (whenever it exists) and then we obtain

$$X = S^{-1}Y \tag{2.14}$$

Theorem 2.1. *[9], If S^{-1} exists it must have the same structure as S, i.e.*

$$S^{-1} = \begin{bmatrix} D & E \\ E & D \end{bmatrix}$$

The following theorem guarantees the existence of a fuzzy solution for a general case.

Theorem 2.2. *[9], The unique solution X of fuzzy linear system (2.14) is a fuzzy number vector if and only if the inverse matrix of S exists and nonnegative.*

In [4], Allahviranloo has proved that $S_{ij} \geq 0$ is not a necessary condition for an unique fuzzy solution of the fuzzy linear system.

Definition 2.8. *[9], Let $X = \{(x_{ir}^-, -x_{ir}^+) \mid 1 \leq i \leq n, \quad 0 \leq r \leq 1\}$ denote the unique solution of fuzzy linear system (2.11). The fuzzy number vector $U = \{(u_{ir}^-, u_{ir}^+) \mid 1 \leq i \leq n, \quad 0 \leq r \leq 1\}$ defined by*

$$u_{ir}^- = \min\{x_{ir}^-, x_{ir}^+, x_{i1}^-\}$$

$$u_{ir}^+ = \min\{x_{ir}^-, x_{ir}^+, x_{i1}^+\}$$

is called the fuzzy solution of $SX = Y$.
If (x_{ir}^-, x_{ir}^+), $1 \leq i \leq n$, are all fuzzy numbers then $u_{ir}^- = x_{ir}^-$ and $u_{ir}^+ = x_{ir}^+$, $1 \leq i \leq n$ and U is called a strong fuzzy solution. Otherwise, U is a weak fuzzy solution.

3 Complex fuzzy linear systems

Definition 3.1. *The $n \times n$ linear system*

$$AX = Y \tag{3.15}$$

is called a complex fuzzy linear system where the coefficient matrix $A = [a_{ij}]_{i,j=1}^n$ is a crisp nonsingular matrix and

$$y_k = \tilde{b}_k + i\tilde{c}_k, \qquad 1 \leq k \leq n$$

is a complex fuzzy number.

So, we can rewrite CFLS (3.15) as follows:

$$AX = B + iC \tag{3.16}$$

where B and C are fuzzy number vectors.

Definition 3.2. *A complex fuzzy number vector $X = (x_1, \ldots, x_n)^t$ given by*

$$x_j = (e_j + if_j), \qquad 1 \leq j \leq n$$

is called a fuzzy complex solution of the CFLS (3.16) if

$$E = (e_1, \ldots, e_n), \qquad F = (f_1, \ldots, f_n)$$

are fuzzy solutions of fuzzy linear systems

$$AE = B, \quad AF = C \tag{3.17}$$

respectively.

In the parametric form, we have

$$x_j = (x_{jr}^-, x_{jr}^+) = (e_{jr}^- + if_{jr}^+, e_{jr}^+ + if_{jr}^+), \qquad 1 \leq j \leq n, \quad 0 \leq r \leq 1$$

In order to solve CFLS (3.16) one must solve two $2n \times 2n$ crisp linear systems. Let us rearrange the fuzzy linear systems of (3.17) so that the unknowns are

$$E^{'} = (e_1^-, \dots, e_n^-, -e_1^+, \dots, -e_n^+)^t$$

and

$$F^{'} = (f_1^-, \dots, f_n^-, -f_1^+, \dots, -f_n^+)^t$$

and right-hand side columns are the function vectors

$$B^{'} = (b_1^-, \dots, b_n^-, -b_1^+, \dots, -b_n^+)^t$$

and

$$C^{'} = (c_1^-, \dots, c_n^-, -c_1^+, \dots, -c_n^+)^t$$

respectively. We get two $2n \times 2n$ linear systems

$$SE^{'} = B^{'}, \qquad SF^{'} = C^{'} \tag{3.18}$$

where s_{ij} is determined as follows:

$$\begin{aligned} s_{ij} &= s_{i+n,j+n} = a_{ij}, \qquad if \quad a_{ij} \geq 0 \\ s_{i,j+n} &= s_{i+n,j} = -a_{ij}, \quad if \quad a_{ij} < 0 \end{aligned} \tag{3.19}$$

while all the remaining s_{ij} are taken zero. By solving two linear systems (3.18), we obtain the fuzzy complex solution of CFLS.

Theorem 3.1. *The fuzzy complex vector solutions of CFLSs (3.15) and (3.16) are equivalent.*

Proof: It is sufficient to prove that the solutions of $SX^{'} = Y^{'}$ obtained from (3.15) and $SX^{'} = Z^{'}$ obtained from (3.16) are the same where S is defined in Eq. (3.19) and

$$\begin{aligned} X^{'} &= (x_1^-, \dots, x_n^-, -x_1^+, \dots, -x_n^+)^t \\ Z^{'} &= (b_1^-, \dots, b_n^-, -b_1^+, \dots, -b_n^+)^t + i(c_1^-, \dots, c_n^-, -c_1^+, \dots, -c_n^+)^t \\ Y^{'} &= (b_1^- + ic_1^-, \dots, b_n^- + ic_n^-, -(b_1^+ + ic_1^+), \dots, -(b_n^+ + ic_n^+))^t \end{aligned} \tag{3.20}$$

It is clear that

$$\begin{aligned} Z^{'} \quad &= (b_1^-, \dots, b_n^-, -b_1^+, \dots, -b_n^+)^t + i(c_1^-, \dots, c_n^-, -c_1^+, \dots, -c_n^+)^t \\ &= (b_1^- + ic_1^-, \dots, b_n^- + ic_n^-, -b_1^+ - ic_1^+, \dots, -b_n^+ - ic_n^+)^t \\ &= (b_1^- + ic_1^-, \dots, b_n^- + ic_n^-, -(b_1^+ + ic_1^+), \dots, -(b_n^+ + ic_n^+))^t \\ &= Y^{'} \end{aligned} \tag{3.21}$$

Then the solutions are the same and the proof is completed.

4 Numerical examples

Example 4.1. *Consider* 2×2 *complex fuzzy linear system*

$$\begin{cases} x_1 - x_2 = (r + i(1+r), (2-r) + i(3-r)) \\ x_1 + 3x_2 = ((4+r) + i(r-4), (7-2r) + i(-1-2r)) \end{cases}$$

Then we solve

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} e_1^- \\ e_2^- \\ -e_1^+ \\ -e_2^+ \end{bmatrix} = \begin{bmatrix} r \\ 4+r \\ -(2-r) \\ -(7-2r) \end{bmatrix} \tag{4.22}$$

and

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} f_1^- \\ f_2^- \\ -f_1^+ \\ -f_2^+ \end{bmatrix} = \begin{bmatrix} 1+r \\ r-4 \\ -(3-r) \\ -(-1-2r) \end{bmatrix} \tag{4.23}$$

So,

$$e_1^- = 1.375 + 0.625r, \quad e_1^+ = 2.875 - 0.875r$$

$$e_2^- = 0.875 + 0.125r, \quad e_2^+ = 1.375 - 0.375r$$

and

$$f_1^- = 0.125 + 0.625r, \quad f_1^+ = 1.625 - 0.875r$$

$$f_2^- = 1.375 + 0.125r, \quad f_2^+ = 0.875 - 0.375r$$

then

$$x_1^- = (1.375 + 0.625r) + i(0.125 + 0.625r), \quad x_1^+ = (2.875 - 0.875r) + i(1.625 - 0.875r)$$

$$x_2^- = (0.875 + 0.125r) + i(1.375 + 0.125r), \quad x_2^+ = (1.375 - 0.375r) + i(0.875 - 0.375r)$$

Example 4.2. *Consider* 2×2 *complex fuzzy linear system*

$$\begin{cases} x_1 - 2x_2 = (r + i(-2+2r), (2-r) + i(2-2r)) \\ x_1 + 3x_2 = (r + i(r-4), (2.5-1.5r) + i(-1-2r)) \end{cases}$$

Then we solve

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 3 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} e_1^- \\ e_2^- \\ -e_1^+ \\ -e_2^+ \end{bmatrix} = \begin{bmatrix} r \\ r \\ -(2-r) \\ -(2.5-1.5r) \end{bmatrix} \tag{4.24}$$

and

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 3 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} f_1^- \\ f_2^- \\ -f_1^+ \\ -f_2^+ \end{bmatrix} = \begin{bmatrix} -2+2r \\ r-4 \\ -(2-2r) \\ -(-1-2r) \end{bmatrix} \tag{4.25}$$

So,

$$e_1^- = 0.6 + 0.4r, \qquad e_1^+ = 1.6 - 0.6r$$

$$e_2^- = -0.2 + 0.2r, \quad e_2^+ = 0.3 - 0.3r$$

and

$$f_1^- = 2.8r - 4, \quad f_1^+ = 2 - 3.2r$$
$$f_2^- = -0.6r, \qquad f_2^+ = 0.4r - 1$$

It is clear that f_2 is not a fuzzy number. The fuzzy solution in this case is a weak solution given by

$$u_1 = (2.8r - 4, 2 - 3.2r)$$
$$u_2 = (0.4r - 1, -0.6r)$$

then

$$x_1^- = (0.6 + 0.4r) + i(2.8r - 4), \qquad x_1^+ = (1.6 - 0.6r) + i(2 - 3.2r)$$
$$x_2^- = (-0.2 + 0.2r) + i(0.4r - 1), \quad x_2^+ = (0.3 - 0.3r) + i(-0.6r)$$

therefore $X = (x_1, x_2)^t$ is a weak fuzzy complex solution too.

5 Conclusion

In this paper, we introduced the complex fuzzy linear system and discussed the numerical method for solving it. So, CFLS is transformed into two fuzzy linear systems and the proposed method in [9] is used for solving them and we showed that the complex combination of two solutions is the solution of CFLS. This numerical method is illustrated by two numerical examples.

References

[1] T. Allahviranloo, " Successive over relaxation iterative method for fuzzy system of linear equations ", Applied Mathematics and Computation, 162 (2005) 189-196.

[2] T. Allahviranloo, " The Adomian decomposition method for fuzzy system of linear equations ", Applied Mathematics and Computation, 163 (2005) 553-563.

[3] T. Allahviranloo, E. Ahmady, N. Ahmady, Kh. Shams Alketaby, " Block Jacobi two-stage method with Gauss-Sidel inner iterations for fuzzy system of linear equations ", Applied Mathematics and Computation, 175 (2006) 1217-1228.

[4] T. Allahviranloo, " A comment on fuzzy linear system ", Fuzzy Sets and Systems, 140 (2003) 559.

[5] S. Abbasbandy, R. Ezzati, A. Jafarian, " LU decomposition method for solving fuzzy system of linear equations ", Applied Mathematics and Computation, 172 (2006) 633-643.

[6] E. Babolian, H. Sadeghi Goghary, S. Abbasbandy, " Numerical solution of linear Fredholm fuzzy integral equations of the second kind by Adomian method", Applied Mathematics and Computation, 161 (2005) 733-744.

[7] J. J. Bukley, Fuzzy eigenvalue and input-output analysis, Fuzzy sets and systems 34 (1990) 187-195.

[8] D. Dubois, H. Prade, Towards fuzzy differential calculus: Part 3, differentiation, Fuzzy Sets and Systems, 8 (1982) 225-233.

[9] M. Friedman, M. Ming, A. Kandel, " Fuzzy linear systems", Fuzzy Sets and Systems, 96 (1998) 201-209.

[10] S. G. Gal, Approximation theory in fuzzy setting, in: G.A. Anastassiou (Ed.), Handbook of Analytic-Computational Methods in Applied Mathematics, Chapman Hall CRC Press, (2000) 617-666.

[11] K. Wang, B. Zheng, " Symmetric successive overrelaxation methods for fuzzy linear systems", Applied Mathematics and Computation, 175 (2006) 891-901.

[12] L.A.Zadeh, " The concept of a linguistic variable and its application to approximate reasoninig", Information Sciences 8 (1975) 199-249.