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Solving Fuzzy Integral Equations of the Second Kind by Fuzzy Laplace Transform Method

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Abstract

Using fuzzy Laplace transform method, the solution of fuzzy convolution Volterra integral equation (FCVIE) of the second kind with convolution fuzzy and crisp kernel is investigated. So, fuzzy convolution operator is proposed and related theorem is submitted which is useful for solving FCVIEs. Finally, several illustrative examples with fuzzy and crisp convolution kernels are given to show the ability of the proposed method.

Keywords : Fuzzy convolution integral equation of the second kind; Fuzzy Laplace transform method; Fuzzy convolution.

1 Introduction

The topic of fuzzy integral equations which has attracted growing interest for some time, in particular in relation to fuzzy control, has been developed in recent years.

Abbasbandy et. al [1] proposed a numerical algorithm for solving linear Fredholm fuzzy integral equations of the second kind by using parametric form of fuzzy number and converting a linear fuzzy Fredholm integral equation to two linear systems of integral equation of the second kind in crisp case. Babolian et. al [3] proposed another numerical procedure for solving fuzzy linear Fredholm integral of the second kind using Adomian method. Moreover, Friedman et. al [8] proposed an embedding method to solve fuzzy Volterra and Fredholm integral equations. However, there are several research papers about obtaining the numerical integration of fuzzy-valued functions and solving fuzzy Volterra and Fredholm integral equations [4, 6, 7, 9, 10, 12, 13, 14, 17, 18].

The fuzzy Laplace transform method solves FDEs and corresponding fuzzy initial and boundary value problems. In this way fuzzy Laplace transforms reduce the problem of

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solving a FDE to an algebraic problem. This switching from operations of calculus to algebraic operations on transforms is called operational calculus, a very important area of applied mathematics, and for engineers, the fuzzy Laplace transform method is practically the most important operational method.
The technique of direct and inverse F-transform and approximating properties of them are described in [15, 16]. Recently, Allahviranloo and Barkhordari in [2] proposed fuzzy Laplace transforms for solving first order fuzzy differential equations under generalized H-differentiability.
By such benefits, we develop fuzzy Laplace transform method to solve fuzzy convolution Volterra integral equation of the second kind. So, the original FCVIE is converted to two crisp convolution integral equations in order to determine the lower and upper function of solution, using fuzzy convolution operator.
The paper is organized as follows:

In section 2, some basic definitions which will be used later in the paper are provided. In section 3, the fuzzy Laplace transform is studied. In section 4, the fuzzy convolution Volterra integral equation of the second kind with fuzzy convolution kernel is studied. Then, the fuzzy Laplace transforms are applied to solve such special fuzzy integral equation. Illustrative examples are also considered to show the ability of the proposed method in section 5, and the conclusion is drawn in section 6.

2 Preliminaries

The basic definition of fuzzy numbers is given in [10, 11].
We denote the set of all real numbers by $\mathbb{R}$. A fuzzy number is a mapping $u : \mathbb{R} \to [0,1]$ with the following properties:

(a) u is upper semi-continuous,

(b) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}, \lambda \in [0,1]$,

(c) u is normal, i.e.,$\exists x_0 \in \mathbb{R}$ for which $u(x_0) = 1$,

(d) supp $u = \{x \in R \mid u(x) > 0\}$ is the support of the u, and its closure cl(supp u) is compact.

Let $\mathbb{E}$ be the set of all fuzzy numbers on $\mathbb{R}$. The α-level set of a fuzzy number $u \in \mathbb{E}$, $0 \leq \alpha \leq 1$, denoted by $[u]_\alpha$, is defined as

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R} \mid u(x) \geq \alpha\} & if \quad 0 < \alpha \leq 1 \\ cl(supp\ u) & if \quad \alpha = 0 \end{cases}$$

It is clear that the α-level set of a fuzzy number is a closed and bounded interval $[\underline{u}(\alpha), \overline{u}(\alpha)]$, where $\underline{u}(\alpha)$ denotes the left-hand endpoint of $[u]_\alpha$ and $\overline{u}(\alpha)$ denotes the right-hand endpoint of $[u]_\alpha$. Since each $y \in \mathbb{R}$ can be regarded as a fuzzy number $\widetilde{y}$ defined by

$$\widetilde{y}(t) = \begin{cases} 1 & if \quad t = y \\ 0 & if \quad t \neq y \end{cases}$$

An equivalent parametric definition is also given in [8] as:

Definition 2.1. *A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(\alpha)$, $\overline{u}(\alpha),\ 0 \leq \alpha \leq 1$, which satisfy the following requirements:*

1. *$\underline{u}(\alpha)$ is a bounded non-decreasing left continuous function in $(0,1]$, and right continuous at 0,*
2. *$\overline{u}(\alpha)$ is a bounded non-increasing left continuous function in $(0,1]$, and right continuous at 0,*
3. *$\underline{u}(\alpha) \leq \overline{u}(\alpha),\ 0 \leq \alpha \leq 1$.*

A crisp number α is simply represented by $\underline{u}(\alpha) = \overline{u}(\alpha) = \alpha,\ 0 \leq \alpha \leq 1$. We recall that for $a < b < c$ which $a, b, c \in R$, the triangular fuzzy number $u = (a, b, c)$ determined by a, b, c is given such that $\underline{u}(\alpha) = a + (b-a)\alpha$ and $\overline{u}(\alpha) = c - (c-b)\alpha$ are the endpoints of the α-level sets, for all $\alpha \in [0,1]$.
The Hausdorff distance between fuzzy numbers given by $d : \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{R}_+ \bigcup \{0\}$,

$$d(u, v) = \sup_{\alpha \in [0,1]} \max\{|\underline{u}(\alpha) - \underline{v}(\alpha)|, |\overline{u}(\alpha) - \overline{v}(\alpha)|\},$$

where $u = (\underline{u}(\alpha), \overline{u}(\alpha))$, $v = (\underline{v}(\alpha), \overline{v}(\alpha)) \subset \mathbb{R}$ is utilized in [5]. Then, it is easy to see that d is a metric in E and has the following properties (see [11])

(i) $d(u + w, v + w) = d(u, v), \quad \forall u, v, w \in \mathbb{E}$,

(ii) $d(ku, kv) = |k| d(u, v), \quad \forall k \in \mathbb{R}, u, v \in \mathbb{E}$,

(iii) $d(u + v, w + e) \leq d(u, w) + d(v, e), \quad \forall u, v, w, e \in \mathbb{E}$,

(iv) $(d, \mathbb{E})$ is a complete metric space.

Definition 2.2. *[8], Let $f : \mathbb{R} \to \mathbb{E}$ be a fuzzy valued function. If for arbitrary fixed $t_0 \in R$ and $\epsilon > 0$, a $\delta > 0$ such that*

$$|t - t_0| < \delta \Rightarrow d(f(t), f(t_0)) < \epsilon,$$

f is said to be continuous.

Theorem 2.1. *[17], Let $f(x)$ be a fuzzy-valued function on $[a, \infty)$ and it is represented by $(\underline{f}(x, \alpha), \overline{f}(x, \alpha))$. For any fixed $r \in [0,1]$, assume $\underline{f}(x, \alpha)$ and $\overline{f}(x, \alpha)$ are Riemann-integrable on $[a, b]$ for every $b \geq a$, and assume there are two positive $\underline{M}(\alpha)$ and $\overline{M}(\alpha)$ such that $\int_a^b |\underline{f}(x, \alpha)| dx \leq \underline{M}(\alpha)$ and $\int_a^b |\overline{f}(x, \alpha)| dx \leq \overline{M}(\alpha)$ for every $b \geq a$. Then $f(x)$ is improper fuzzy Riemann-integrable on $[a, \infty)$ and the improper fuzzy Riemann-integral is a fuzzy number. Further more, we have:*

$$\int_a^\infty f(x) dx = (\int_a^\infty \underline{f}(x, \alpha) dx, \int_a^\infty \overline{f}(x, \alpha) dx).$$

Proposition 2.1. *[18], If each of $f(x)$ and $g(x)$ is fuzzy-valued function and fuzzy Rieman integrable on $I = [a, \infty)$ then $f(x) + g(x)$ is fuzzy Riemman integrable on I. Moreover, we have*

$$\int_I (f(x) + g(x)) dx = \int_I f(x) dx + \int_I g(x) dx.$$

3 The Fuzzy Laplace transforms

Suppose that f is a fuzzy-valued function and s is a real parameter. We define the fuzzy Laplace transform of f as following:

Definition 3.1. *The fuzzy Laplace transform of fuzzy-valued function $f(t)$ is defined as following*

$$F(s) = \mathbf{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt = \lim_{\tau \longrightarrow \infty} \int_0^\tau e^{-st} f(t)dt \tag{3.1}$$

whenever the limits exist.

The notation $\mathbf{L}(f)$ be also used to denote the fuzzy Laplace transform of fuzzy-valued function $f(t)$, and the integral is the fuzzy Riemann improper integral.

The symbol $\mathbf{L}$ is the fuzzy Laplace transformation, which acts on fuzzy-valued function $f = f(t)$ and generates a new fuzzy-valued function, $\mathbf{F}(s) = \mathbf{L}(f(t))$.

Consider fuzzy-valued function f, then the lower and upper fuzzy Lapalce transform of this function are denoted, based on the lower and upper of fuzzy-valued function f as following:

$$\mathbf{F}(s,\alpha) = \mathbf{L}(f(t,\alpha)) = [l(\underline{f}(t,\alpha)), l(\overline{f}(t,\alpha))]$$

where,

$$l(\underline{f}(t,\alpha)) = \int_0^\infty e^{-st}\underline{f}(t,\alpha)dt = \lim_{\tau \longrightarrow \infty} \int_0^\tau e^{-st}\underline{f}(t,\alpha)dt,\ 0 \le \alpha \le 1$$

$$l(\overline{f}(t,\alpha)) = \int_0^\infty e^{-st}\overline{f}(t,\alpha)dt = \lim_{\tau \longrightarrow \infty} \int_0^\tau e^{-st}\overline{f}(t,\alpha)dt,\ 0 \le \alpha \le 1.$$

4 Fuzzy convolution Volterra integral equation of the second kind

In this section, we investigate the solution of FCVI equation of the second kind.

The fuzzy convolution Volterra integral equation of the second kind is defined as

$$x(t) = f(t) + \int_0^t k(s-t)x(s)ds, \quad t \in [0,T],\ T < \infty \tag{4.2}$$

where $k(s-t)$ is an arbitrary given fuzzy-valued convolution kernel function and f is a continuous fuzzy-valued function.

We adopt fuzzy Laplace transform to solve Eq.(4.2) such that by taking fuzzy Laplace transform on both sides of Eq.(4.2) and using fuzzy convolution, we get solution of Eq.(4.2) directly. So, the concept of fuzzy convolution must be introduced.

4.1 Fuzzy convolution

The convolution of two fuzzy-valued functions f and g defined for $t > 0$ by

$$(f * g)(t) = \int_0^t f(\tau).g(t-\tau)d\tau \tag{4.3}$$

which of course exists if f and g are, say, piece-wise continuous. Substituting $u = t - \tau$ gives

$$(f * g)(t) = \int_0^t g(u).f(t-u)du = (g * f)(t), \tag{4.4}$$

that is, the fuzzy convolution is *commutative*.
Other basic properties of the fuzzy convolution are as follows:

$$\begin{array}{ll} (i) & c(f*g) = cf*g = f*cg, \quad c \text{ is constant} \\ (ii) & f*(g*h) = (f*g)*h \quad (\text{associative property}) \end{array}$$

Property (i) is routine to verify. As for (ii)

$$\begin{aligned} [f*(g*h)](t) &= \int_0^t f(\tau).(g*h)(t-\tau)d\tau \\ &= \int_0^t f(\tau)\left(\int_0^{t-\tau} g(x).h(t-\tau-x)dx\right)d\tau \\ &= \int_0^t \left(\int_0^u f(\tau).g(u-\tau)d\tau\right)h(t-u)du \\ &= [(f*g)*h](t) \end{aligned}$$

while having reverse the order of integration.
One of the very significant properties possessed by the fuzzy convolution in connection with the fuzzy Laplace transform is that the fuzzy Laplace transform of the convolution of two fuzzy-valued functions is the product of their fuzzy Laplace transform.

Theorem 4.1. *(Convolution Theorem), If f and g are piecewise continuous fuzzy-valued functions on [0,∞] and of exponential order p, then*

$$\boldsymbol{L}\{(f*g)(t)\} = \boldsymbol{L}\{f(t)\}.\boldsymbol{L}\{g(t)\} = F(s).G(s), \quad s > p \tag{4.5}$$

Proof: Let us start with the product

$$\begin{aligned} \mathbf{L}\{f(t)\}.\mathbf{L}\{g(t)\} &= \left(\int_0^\infty e^{-s\tau}f(\tau)d\tau\right).\left(\int_0^\infty e^{-su}g(u)du\right) \\ &= \int_0^\infty \left(\int_0^\infty e^{-s(\tau+u)}f(\tau)g(u)du\right)d\tau \end{aligned}$$

substituting $t = \tau + u$, and noting that τ is fixed in the interior integrals, so that $du = dt$, we have

$$\mathbf{L}\{f(t)\}.\mathbf{L}\{g(t)\} = \int_0^\infty \left(\int_\tau^\infty e^{-st}f(\tau)g(t-\tau)dt\right)d\tau \tag{4.6}$$

If we define $g(t) = \widetilde{0}$ for $t < 0$, then $g(t-\tau) = \widetilde{0}$ for $t < \tau$ and we can write (4.6) as

$$\mathbf{L}\{f(t)\}.\mathbf{L}\{g(t)\} = \int_0^\infty \int_0^\infty e^{-st}f(\tau)g(t-\tau)dtd\tau$$

Due to the hypotheses on f, g, the fuzzy Laplace integrals of f, g converge absolutely and hence

$$\int_0^\infty \int_0^\infty |e^{-st}f(\tau)g(t-\tau)|dtd\tau$$

converges. This fact allows us to reverse the order of integration, so that

$$\begin{aligned}
\mathbf{L}\{f(t)\}.\mathbf{L}\{g(t)\} &= \int_0^\infty \int_0^\infty e^{-st} f(\tau)g(t-\tau)d\tau dt \\
&= \int_0^\infty \left(\int_0^t e^{-st} f(\tau).g(t-\tau)d\tau \right) dt \\
&= \int_0^\infty e^{-st} \left(\int_0^t f(\tau).g(t-\tau)d\tau \right) dt \\
&= L\{(f * g)(t)\}
\end{aligned}$$

Please notice that in the fuzzy case, we should investigate more accurately than the deterministic case. So, mentioned calculation is assumed valid under suitable conditions.

4.2 The method

Here, we shall obtain the solution of fuzzy convolution Volterra integral equation using fuzzy Laplace transform. Indeed, our method is constructed on applying fuzzy convolution. Consider the original Eq.(4.2), then by taking fuzzy Laplace transform on both sides of it we get the following:

$$\mathbf{L}\{x(t)\} = \mathbf{L}\{f(t)\} + \mathbf{L}\left\{ \int_0^t k(s-t)x(s)ds \right\}, \quad t \in [0, T], \; T < \infty,$$

then, we get by using fuzzy convolution and definition of fuzzy Laplace transform:

$$l\{\underline{x}(t;\alpha)\} = l\{\underline{f}(t;\alpha)\} + \underline{l\{k(t;\alpha)\}l\{x(t;\alpha)\}}, \quad 0 \le \alpha \le 1$$

and

$$l\{\overline{x}(t;\alpha)\} = l\{\overline{f}(t;\alpha)\} + \overline{l\{k(t;\alpha)\}l\{x(t;\alpha)\}}, \quad 0 \le \alpha \le 1$$

Now, we should discuss $\underline{l\{k(t;\alpha)\}l\{x(t;\alpha)\}}$ and $\overline{l\{k(t;\alpha)\}l\{x(t;\alpha)\}}$. To this end, we have the following cases:
Case 1- if $k(t;\alpha)$ and $x(t;\alpha)$ are positive, then we get

$$\underline{l\{k(t;\alpha)\}l\{x(t;\alpha)\}} = l\{\underline{k(t;\alpha)}\}l\{\underline{x(t;\alpha)}\}, \quad \overline{l\{k(t;\alpha)\}l\{x(t;\alpha)\}} = l\{\overline{k(t;\alpha)}\}l\{\overline{x(t;\alpha)}\}$$

Case 2- if $k(t;\alpha)$ is positive and $x(t;\alpha)$ is negative, then we get

$$\underline{l\{k(t;\alpha)\}l\{x(t;\alpha)\}} = l\{\overline{k(t;\alpha)}\}l\{\underline{x(t;\alpha)}\}, \quad \overline{l\{k(t;\alpha)\}l\{x(t;\alpha)\}} = l\{\underline{k(t;\alpha)}\}l\{\overline{x(t;\alpha)}\}$$

Case 3- if $k(t;\alpha)$ is negative and $x(t;\alpha)$ is positive, then we get

$$\underline{l\{k(t;\alpha)\}l\{x(t;\alpha)\}} = l\{\underline{k(t;\alpha)}\}l\{\overline{x(t;\alpha)}\}, \quad \overline{l\{k(t;\alpha)\}l\{x(t;\alpha)\}} = l\{\overline{k(t;\alpha)}\}l\{\underline{x(t;\alpha)}\}$$

Case 4- if $k(t;\alpha)$ and $x(t;\alpha)$ are negative, then we get

$$\underline{l\{k(t;\alpha)\}l\{x(t;\alpha)\}} = l\{\overline{k(t;\alpha)}\}l\{\overline{x(t;\alpha)}\}, \quad \overline{l\{k(t;\alpha)\}l\{x(t;\alpha)\}} = l\{\underline{k(t;\alpha)}\}l\{\underline{x(t;\alpha)}\}$$

Notice that, it is assumed that zero does not exist in support.
We obtain explicit formula for Case 1 and the others are similar. Indeed, we can write case 1 in a compact form:

$$l\{\underline{x}(t;\alpha)\} = \frac{l\{\underline{f}(t;\alpha)\}}{1 - l\{\underline{k}(t;\alpha)\}}$$

and

$$l\{\overline{x}(t;\alpha)\} = \frac{l\{\overline{f}(t;\alpha)\}}{1 - l\{\overline{k}(t;\alpha)\}}$$

Finally, using the inverse of fuzzy Laplace transform we get the solution:

$$\underline{x}(t;\alpha) = l^{-1}\left(\frac{l\{\underline{f}(t;\alpha)\}}{1 - l\{\underline{k}(t;\alpha)\}}\right)$$

and

$$\overline{x}(t;\alpha) = l^{-1}\left(\frac{l\{\overline{f}(t;\alpha)\}}{1 - l\{\overline{k}(t;\alpha)\}}\right)$$

for all $0 \le \alpha \le 1$, provided that a fuzzy valued function is defined.

5 Examples

In this section, we give some examples to obtain the solution of fuzzy convolution Volterra integral equation of the second kind.

Example 5.1. *Consider the following FCVI equation*

$$x(t) = (\alpha, 2-\alpha).e^{-t} + \int_0^t \sin(t-\tau).x(\tau)d\tau, \quad t \in T = [a,b]$$

We apply the fuzzy Laplace transform to both sides of the equation, so that

$$\mathbf{L}\{x(t)\} = \mathbf{L}\{(\alpha, 2-\alpha).e^{-t}\} + \mathbf{L}\{\sin(t)\}.L\{x(t)\}$$

i.e.,

$$l\{\underline{x}(t;\alpha)\} = l\{(\alpha).e^{-t}\} + l\{\sin(t)\}.l\{\underline{x}(t;\alpha)\}, \quad 0 \le \alpha \le 1$$

$$l\{\overline{x}(t;\alpha)\} = l\{(2-\alpha).e^{-t}\} + l\{\sin(t)\}.l\{\overline{x}(t;\alpha)\}, \quad 0 \le \alpha \le 1$$

Hence, we get

$$l\{\underline{x}(t;\alpha)\} = (\alpha)\left(\frac{2}{s+1} + \frac{1}{s^2} - \frac{1}{s}\right), \quad 0 \le \alpha \le 1$$

$$l\{\overline{x}(t;r)\} = (2-\alpha)\left(\frac{2}{s+1} + \frac{1}{s^2} - \frac{1}{s}\right), \quad 0 \le \alpha \le 1$$

By taking the inverse of fuzzy Laplace transform on both sides of above relations, we have the following

$$\underline{x}(t;\alpha) = (\alpha)(2e^{-t} + t - 1), \quad \overline{x}(t;\alpha) = (2-\alpha)(2e^{-t} + t - 1), \quad 0 \le \alpha \le 1$$

Example 5.2. *Consider the following FCVI equation*

$$x(t) = (2+\alpha, 5-\alpha).\cos(t) + \int_0^t e^{t-\tau}x(\tau)d\tau, \quad t \in T = [a,b]$$

Similarly, by taking fuzzy Laplace transform on both sides of equation, we get the following:

$$\mathbf{L}\{x(t)\} = \mathbf{L}\{(2+\alpha, 5-\alpha).\cos(t)\} + \mathbf{L}\{e^t\}.L\{x(t)\}$$

i.e.,

$$l\{\underline{x}(t;\alpha)\} = \frac{(2+\alpha)s(s-1)}{(s^2+1)(s-2)}, \quad 0 \le \alpha \le 1$$

$$l\{\overline{x}(t;\alpha)\} = \frac{(5-\alpha)s(s-1)}{(s^2+1)(s-2)}, \quad 0 \le \alpha \le 1$$

Finally, by applying the inverse of fuzzy Laplace transform, we get the following:

$$\underline{x}(t;\alpha) = (2+\alpha)\left(3/5\cos(t) + 1/5\sin(t) + 2/5\exp(2t)\right), \quad 0 \le \alpha \le 1$$

$$\overline{x}(t;\alpha) = (5-\alpha)\left(3/5\cos(t) + 1/5\sin(t) + 2/5\exp(2t)\right), \quad 0 \le \alpha \le 1$$

Example 5.3. *Consider the following FCVI equation with fuzzy convolution kernel*

$$x(t) = (\alpha, 2-\alpha).t + \int_0^t (\alpha+2, 4-\alpha)\cos(t-\tau).x(\tau)d\tau, \quad t \in T$$

then, using fuzzy Laplace transform on both sides of above equation and similar calculations we get the following:

$$\begin{aligned}
\underline{x}(t;\alpha) &= \alpha\left(1 + \frac{2(\alpha+2)e^{(\frac{\alpha}{2}+1)t}}{(\alpha^2+4\alpha)^{0.5}}\sinh(\frac{t(\alpha^2+4\alpha)^{0.5}}{2})\right), \\
\overline{x}(t;\alpha) &= 2-\alpha)\left(1 - \frac{2(\alpha-4)e^{(\frac{-\alpha}{2}+2)t}}{(\alpha^2-8\alpha+12)^{0.5}}\sinh(\frac{t(\alpha^2-8\alpha+12)^{0.5}}{2})\right).
\end{aligned}$$

6 Conclusion

In this paper, fuzzy convolution Volterra integral equation was solved using fuzzy laplace transform method. To our knowledge, this work is the first attempt to solve such special fuzzy integral equations with fuzzy kernel. For future research, we will solve fuzzy convolution Integro-Differential system of equations using fuzzy Laplace transform method and other fuzzy F-transform techniques.

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