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Numerical solution of fuzzy initial value problems under generalized differentiability by HPM

M. Ghanbari *

Department of Mathematics, Mazandaran University, Babolsar, Iran.

Abstract

In this work, the Homotopy Perturbation Method (HPM) is implemented for finding approximate solution of the Fuzzy Initial Value Problem (FIVP) involving generalized differentiability. This method is based upon homotopy perturbation theory. The comparison of the exact solution with approximate solution obtained by HPM is in detail. The results reveal that the method is very effective and simple.

Keywords : Fuzzy numbers; Fuzzy differential equations; Generalized differentiability; Homotopy perturbation method.

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1 Introduction

The concept of the fuzzy derivative was first introduced by Chang and Zadeh [10]. Later, Dubois and Prade [11] presented a concept of the fuzzy derivative based on the extension principle. Other methods have been discussed by Puri and Ralescu [26], Goetschel and Voxman [14], Seikkala [27] and Friedman et al. [12, 21]. Recently, Bede introduced a strongly generalized differentiability of fuzzy functions in [6, 7] and studied in [8]. The Fuzzy Differential Equation (FDE) and the FIVP were rigorously treated by Kaleva [22, 23], Seikkala [27], He and Yi [15], Kloeden [24] and Menda [25]. The numerical methods for solving FDEs are introduced in [1, 2, 3, 4]. In this paper, the FIVP under generalized differentiability is solved via Homotopy Perturbation Method (HPM). First we replace the FIVP by its parametric form and then solve the new system which consist of two classic ordinary differential equations with initial conditions, then check to see if this solution define a fuzzy function. HPM introduced by He [17, 18, 19, 20] has been used by many mathematicians and engineers to solve various functional equations. In this method the solution is considered as the sum of an infinite series which converges rapidly to the accurate solutions. Using homotopy technique in topology, a homotopy is constructed with

*Email address: m.ghanbari@umz.ac.ir

an embedding parameter $p \in [0,1]$ which is considered as a *small parameter*. The structure of this paper is organized as follows. In Section 2, some basic definitions which will be used later in the paper are provided. In Section 3, we present the different parametric forms of the FIVP by using the strongly generalized differentiability concept. In Section 4, we state the basic concepts of HPM, and apply this method on parametric forms obtained in the previous section. In Section 5, we use the HPM for solve two FIVPs. We conclude in Section 6.

2 Preliminaries

Definition 2.1. *A fuzzy number is a function $u : \Re \longrightarrow [0,1]$ satisfying the following properties:*

(i) *u is normal, i.e. $\exists x_0 \in \Re$ with $u(x_0) = 1$,*

(ii) *u is a convex fuzzy set,*

(iii) *u is upper semi-continuous on $\Re$,*

(iv) *$\overline{\{x \in \Re : u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.*

The set of all these fuzzy numbers is denoted by E. Obviously, $\Re \subset E$. Here $\Re \subset E$ is understood as $\Re = \{\chi_{\{x\}} : x \text{ is usual real number}\}$. For $0 < r \leq 1$, denote $[u]_r = \{x \in \Re : u(x) \geq r\}$ and $[u]_0 = \overline{\{x \in \Re : u(x) > 0\}}$. Then it is well-known that for each $r \in [0,1]$, $[u]_r$ is a bounded closed interval. For $u, v \in E$, and $\lambda \in \Re$, the sum $u \oplus v$ and the product $\lambda \odot u$ are defined by

$$[u \oplus v]_r = [u]_r + [v]_r = \{x + y : x \in [u]_r,\ y \in [v]_r\}, \quad \forall r \in [0,1],$$

$$[\lambda \odot u]_r = \lambda \odot [u]_r = \{\lambda x : x \in [u]_r\}, \quad \forall r \in [0,1].$$

Define $D : E \times E \longrightarrow \Re^+ \bigcup \{0\}$ by the equation

$$D(u,v) = \sup_{0 \leq r \leq 1} \{\max[\ |\underline{u}_r - \underline{v}_r|, |\overline{u}_r - \overline{v}_r|\ ]\}, \tag{2.1}$$

where $[u]_r = [\underline{u}_r, \overline{u}_r],\ [v]_r = [\underline{v}_r, \overline{v}_r]$. Then it is easy to show that D is a metric in E. Using the results in [13], we know that

(i) $D(u \oplus w, v \oplus w) = D(u,v), \quad \forall u, v, w \in E,$

(ii) $D(\lambda \odot u, \lambda \odot v) = |\lambda| D(u,v), \quad \forall \lambda \in \Re,\ u, v \in E,$

(iii) $D(u \oplus v, w \oplus e) \leq D(u,w) + D(v,e), \quad \forall u, v, w, e \in E,$

(iv) (E, D) is a complete metric space.

Theorem 2.1. *[28] If we define $j : E \longrightarrow \overline{C}[0,1] \times \overline{C}[0,1]$ by $j(u) = (\underline{u}, \overline{u})$, where $\underline{u}, \overline{u} : [0,1] \longrightarrow \Re$, $\underline{u}(r) = \underline{u}_r$, $\overline{u}(r) = \overline{u}_r$, then $j(E)$ is a closed convex cone with vertex 0 in $\overline{C}[0,1] \times \overline{C}[0,1]$ (here $\overline{C}[0,1] \times \overline{C}[0,1]$ is a Banach space with the norm $\| (f,g) \| = \max\{\| f \|, \| g \|\}$ where $\| f \| = \sup\{|f(x)| : x \in [0,1]\}$) and j satisfies:*

(i) $j(s \odot u \oplus t \odot v) = s\, j(u) \oplus t\, j(v), \quad \forall u, v \in E,\ s, t \geq 0,$

(ii) $D(u, v) = \| j(u) - j(v) \|$.

Therefore, another definition for a fuzzy number which yields the same E is as follows [22]:

Definition 2.2. *An arbitrary fuzzy number in the parametric form is represented by an ordered pair of functions* $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, *which satisfy the following requirements.*

(i) $\underline{u}(r)$ *is a bounded left continuous nondecreasing function over* $[0, 1]$.

(ii) $\overline{u}(r)$ *is a bounded left continuous nonincreasing function over* $[0, 1]$.

(iii) $\underline{u}(r) \leq \overline{u}(r),\ 0 \leq r \leq 1.$

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha,\ 0 \leq r \leq 1.$
We recall that for arbitrary fuzzy numbers $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and real number k,

(a) $u = v$ if and only if $\underline{u}(r) = \underline{v}(r)$ and $\overline{u}(r) = \overline{v}(r)$.

(b) $u \oplus v = (\underline{u \oplus v}, \overline{u \oplus v}) = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)).$

(c)

$$k \odot u = \begin{cases} (\underline{k \odot u}, \overline{k \odot u}) = (k\, \underline{u}(r), k\, \overline{u}(r)), & k \geq 0, \\ (\underline{k \odot u}, \overline{k \odot u}) = (k\, \overline{u}(r), k\, \underline{u}(r)), & k < 0. \end{cases}$$

Note that $(-1) \odot u$ may not be a fuzzy number when u be a fuzzy number.
Similarly of (1), we define

$$D(u, v) = \sup_{0 \leq r \leq 1} \{\max[\ |\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\ ]\}. \tag{2.2}$$

In this paper, we represent an arbitrary fuzzy number by a pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$.

Theorem 2.2. *[5]*

(i) *If we define* $\widetilde{0} = \chi_{\{0\}}$, *then* $\widetilde{0} \in E$ *is a neutral element with respect to addition, i.e.* $u \oplus \widetilde{0} = \widetilde{0} \oplus u = u$, *for all* $u \in E$.

(ii) *With respect to* $\widetilde{0}$, *none of* $u \in E \setminus \Re$, *has inverse in* E *(with respect to* $\oplus$*).*

(iii) *For any* $a, b \in \Re$ *with* $a, b \geq 0$ *or* $a, b \leq 0$ *and any* $u \in E$, *we have* $(a + b) \odot u = a \odot u \oplus b \odot u$, *For general* $a, b \in \Re$, *the above property does not hold.*

(iv) *For any* $\lambda \in \Re$ *and any* $u, v \in E$, *we have* $\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot v$.

(v) *For any* λ, $\mu \in \Re$ *and any* $u \in E$, *we have* $\lambda \odot (\mu \odot u) = (\lambda.\mu) \odot u$.

Definition 2.3. *Let* E *be a set of all fuzzy numbers, we say that* f *is a fuzzy function if* $f : \Re \longrightarrow E$.

Definition 2.4. *Consider $u, v \in E$. If there exists $w \in E$ such that $u = v \oplus w$, then w is called the Hukuhara difference of u and v and it is denoted by $u \ominus v$.*

In this paper the "$\ominus$" sign stands always for Hukuhara difference and note that $u \ominus v \neq u \oplus (-1) \odot v$.

Definition 2.5. *[6] Let $f : (a, b) \longrightarrow E$ and $x_0 \in (a, b)$. We say that f is strongly generalized differentiable on x_0 (Bede differentiable), if there exists an element $f'(x_0) \in E$, such that*

- **(i)** *for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $f(x_0) \ominus f(x_0 - h)$ and the limits (in the metric D)*

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0),$$

or

- **(ii)** *for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h)$, $f(x_0 - h) \ominus f(x_0)$ and the limits*

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{(-h)} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{(-h)} = f'(x_0),$$

or

- **(iii)** *for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $f(x_0 - h) \ominus f(x_0)$ and the limits*

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{(-h)} = f'(x_0),$$

or

- **(iv)** *for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h)$, $f(x_0) \ominus f(x_0 - h)$ and the limits*

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{(-h)} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0),$$

(h and $(-h)$ at denominators mean $\frac{1}{h}\odot$ and $-\frac{1}{h}\odot$, respectively).

Theorem 2.3. *[9] Let $f : \Re \longrightarrow E$ be a fuzzy function and denote $f(t) = (\underline{f}(t; r), \overline{f}(t; r))$, for each $r \in [0, 1]$. Then*

1. *If f is differentiable in the first form* **(i)**, *then $\underline{f}(t; r)$ and $\overline{f}(t; r)$ are differentiable functions and $f'(t) = (\underline{f}'(t; r), \overline{f}'(t; r))$.*
2. *If f is differentiable in the second form* **(ii)**, *then $\underline{f}(t; r)$ and $\overline{f}(t; r)$ are differentiable functions and $f'(t) = (\overline{f}'(t; r), \underline{f}'(t; r))$.*

Theorem 2.4. *[8] Let $f : (a, b) \longrightarrow E$ be strongly generalized differentiable on each point $x \in (a, b)$ in the sense of Definition 2.7***(iii)** *or 2.7***(iv)**. *Then $f'(x) \in \Re$ for all $x \in (a, b)$.*

3 Fuzzy initial value problem

Consider the FDE $y' = f(t, y)$ where y is a fuzzy function of t, $f(t, y)$ is a fuzzy function of crisp variable t and fuzzy variable y, and y' is generalized differential (Bede differential) of y. If an initial value $y(t_0) = y_0$ is given, a FIVP will be obtained as follows:

$$\begin{cases} y' = f(t, y), \qquad t_0 \leq t \leq T, \\ \\ y(t_0) = y_0 \in E. \end{cases} \tag{3.3}$$

Lemma 3.1. *[8] For $x_0 \in \Re$, the FIVP $y' = f(t, y)$, $y(t_0) = y_0 \in E$ where $f : R \times E \longrightarrow E$ is supposed to be continuous, is equivalent to one of the integral equations:*

$$y(x) = y_0 \oplus \int_{t_0}^{t} f(s, y(s))\, ds, \qquad \forall t \in [t_0, t_1],$$

or

$$y_0 = y(x) \oplus (-1) \odot \int_{t_0}^{t} f(s, y(s)\, ds, \qquad \forall t \in [t_0, t_1],$$

on some interval $(t_0, t_1) \subset \Re$, depending on the strongly differentiability considered, **(i)** *or* **(ii)**, *respectively.*
Here the equivalence between two equations means that any solution of an equation is a solution too for the other one.

Remark 3.1. *In the case of strongly generalized differentiability, to the FDE $y' = f(t, y)$ we may attach two different integral equations, while in the case of differentiability in the sense of the Definition of H-differentiable, we may attach only one. The second integral equation in Lemma (3.1) can be written in the form $y(x) = y_0 \ominus (-1) \odot \int_{t_0}^{t} f(s, y(s))\, ds$.*

The following theorem concern the existence of solutions of a FIVP under generalized differentiability (see [8]).

Theorem 3.1. *Let us suppose that the following conditions hold:*

(a) *Let $R_0 = [t_0, t_0 + p] \times \overline{B}(y_0, q)$, $p, q > 0$, $y_0 \in E$, where $\overline{B}(y_0, q) = \{y \in E : D(y, y_0) \leq q\}$ denote a closed ball in E and let $f : R_0 \longrightarrow E$ be a continuous function such that $D(\widetilde{0}, f(t, y)) = \| f(t, y) \| \leq M$ for all $(t, y) \in R_0$.*

(b) *Let $g : [t_0, t_0 + p] \times [0, q] \longrightarrow \Re$, such that $g(t, 0) \equiv 0$ and $0 \leq g(t, s) \leq M_1$, $\forall t \in [t_0, t_0 + p]$, $s \in [0, q]$, such that $g(t, s)$ is nondecreasing in s and g is such that the initial value problem $u'(t) = g(t, u(t))$, $u(t_0) = 0$ has only the solution $u(t) \equiv 0$ on $[t_0, t_0 + p]$.*

(c) *We have $D(f(t, y), f(t, z)) \leq g(t, D(y, z))$, $\forall (t, y), (t, z) \in R_0$ and $D(y, z) \leq q$.*

(d) *There exists $d > 0$ such that for $t \in [t_0, t_0 + d]$ the sequence $y^*_n : [t_0, t_0 + d] \longrightarrow E$ given by $y^*_0(t) = y_0$, $y^*_{n+1}(t) = y_0 \ominus (-1) \odot \int_{t_0}^{t} f(s, y^*_n(s))\, ds$, is defined for any $n \in N$.*

Then the FIVP $y' = f(t, y)$, $y(t_0) = y_0$ has two solutions (one differentiable as in Definition 2.7(**i**) *and the other one differentiable as in Definition 2.7*(**ii**)*) $y, y^* : [t_0, t_0 + r] \longrightarrow B(y_0, q)$ where $r = \min\{p, \frac{q}{M}, \frac{q}{M_1}, d\}$ and the successive iterations*

$$y_0(t) = y_0, \qquad y_{n+1}(t) = y_0 \oplus \int_{t_0}^{t} f(s, y_n(s))\, ds,$$

and

$$y_0^*(t) = y_0, \qquad y_{n+1}^*(t) = y_0 \ominus (-1) \odot \int_{t_0}^{t} f(s, y_n^*(s))\, ds,$$

converge to these two solutions, respectively.

Remark 3.2. *For FIVP with strongly generalized differentiability, the existence of two solutions in a neighborhood of a point t_0 generates a way of choosing which kind of differentiability is expected for the solution, as follows. If on an interval we expect solution with increasing support then we find* **(i)***-differentiable solution. If we expect decreasing support then we find* (**ii**)*-differentiable solution.*

According to theorem 3.3, we restrict our attention to functions which are **(i)**- or **(ii)**-differentiable on their domain except for a finite number of points.

We consider the following FIVP

$$\begin{cases} y' = f(t, y), & t_0 \le t \le T, \\ y(t_0) = y_0 \in E. \end{cases} \tag{3.4}$$

So , if we consider derivative form **(i)** or **(ii)** we may replace the FIVP by the equivalent system

$$\begin{cases} \underline{y}'(t; r) = g(t, \underline{y}, \overline{y}; r), & \underline{y}(t_0; r) = \underline{y_0}(r), \\ \overline{y}'(t; r) = h(t, \underline{y}, \overline{y}; r), & \overline{y}(t_0; r) = \overline{y_0}(r), \end{cases} \tag{3.5}$$

or

$$\begin{cases} \overline{y}'(t; r) = g(t, \underline{y}, \overline{y}; r), & \overline{y}(t_0; r) = \overline{y_0}(r), \\ \underline{y}'(t; r) = h(t, \underline{y}, \overline{y}; r), & \underline{y}(t_0; r) = \underline{y_0}(r), \end{cases} \tag{3.6}$$

$$g(t, \underline{y}, \overline{y}; r) = \min\{g(t, u, v) | u, v \in [\underline{y}_r, \overline{y}_r]\},$$

$$h(t, \underline{y}, \overline{y}; r) = \max\{g(t, u, v) | u, v \in [\underline{y}_r, \overline{y}_r]\},$$

respectively. For every prefixed $r \in [0, 1]$, the system represents an ordinary initial value problem for which any converging classical numerical procedure can be applied.
To simplify, we consider the following FIVP

$$\begin{cases} y'(t) = a \odot y(t) \oplus f(t), & t_0 \le t \le T, \\ y(t_0) = y_0 \in E, \end{cases} \tag{3.7}$$

where $a \in \Re$, $a \neq 0$, f is a fuzzy-valued function or real-valued function. Since $\Re \subset E$, any real-valued function is a fuzzy-valued function, i.e. if $f : [t_0, T] \longrightarrow \Re$ then we define

$$\underline{f}(t; r) = \overline{f}(t; r) = f(t).$$

Then, $f(t) = (\underline{f}(t; r), \overline{f}(t; r))$ can be considered as a fuzzy-valued function.
In Eq. (7), if we choose derivative form **(i)**, the parametric form will obtain as follows:

$$\begin{cases} \underline{y}' = g(\underline{y}, \overline{y}) + \underline{f}, & \underline{y}(t_0; r) = \underline{y_0}(r), \\ \overline{y}' = h(\underline{y}, \overline{y}) + \overline{f}, & \overline{y}(t_0; r) = \overline{y_0}(r), \end{cases} \tag{3.8}$$

and if choose derivative form **(ii)**, the parametric form of Eq. (7) will obtain as follows:

$$\begin{cases} \overline{y}' = g(\underline{y}, \overline{y}) + \underline{f}, & \overline{y}(t_0; r) = \overline{y_0}(r), \\ \\ \underline{y}' = h(\underline{y}, \overline{y}) + \overline{f}, & \underline{y}(t_0; r) = \underline{y_0}(r), \end{cases} \tag{3.9}$$

where

$$g(\underline{y}, \overline{y}) = \begin{cases} a\,\underline{y}, & a > 0, \\ a\,\overline{y}, & a < 0, \end{cases} \tag{3.10}$$

and

$$h(\underline{y}, \overline{y}) = \begin{cases} a\,\overline{y}, & a > 0, \\ a\,\underline{y}, & a < 0. \end{cases} \tag{3.11}$$

In the crisp case, if Eq. (7) has a solution with increasing support, then, we choose derivative form **(i)** and if has a solution with decreasing support, then, we choose derivative form **(ii)**, (see [8]).
In the next section HPM is applied for Eqs. (8) and (9).

4 Homotopy perturbation method

To illustrate the basic ideas of this method, we consider the following equation:

$$A(u) - f(r) = 0, \qquad r \in \Omega, \tag{4.12}$$

with the boundary condition

$$B(u, \frac{\partial u}{\partial n}) = 0, \qquad t \in \Gamma, \tag{4.13}$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω. A can be divided into two parts which are L and N, where L is linear and N is nonlinear. Eq. (12) can therefore be rewritten as follows:

$$L(u) + N(u) - f(r) = 0, \qquad r \in \Omega. \tag{4.14}$$

By the homotopy technique, we construct a homotopy $U(r,p) : \Omega \times [0,1] \longrightarrow \Re$, which satisfies:

$$H(U,p) = (1-p)[L(U) - L(u_0)] + p\,[A(U) - f(r)] = 0, \quad p \in [0,1],\ r \in \Omega, \tag{4.15}$$

or

$$H(U,p) = L(U) - L(u_0) + p\,L(u_0) + p\,[N(U) - f(r)] = 0, \tag{4.16}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of Eq. (12), which satisfies the boundary conditions. Obviously, from Eqs. (15) and (16) we will have

$$H(U,0) = L(U) - L(u_0) = 0, \tag{4.17}$$

$$H(U,1) = A(U) - f(r) = 0. \tag{4.18}$$

The changing process of p form zero to unity is just that of $U(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called *homotopy*. According to the HPM, we can first use the embedding parameter p as a *small parameter*, and assume that the solution of Eqs. (15) and (16) can be written as a power series in p :

$$U = u_0 + p\,u_1 + p^2 u_2 + p^3 u_3 + \cdots\,, \tag{4.19}$$

and the exact solution is obtained as follows:

$$u = \lim_{p\to 1} U = \lim_{p\to 1}(u_0 + p\,u_1 + p^2 u_2 + p^3 u_3 + \cdots) = \sum_{j=0}^{\infty} u_j\,. \tag{4.20}$$

The above convergence is discussed in [16]. For later numerical computation, we let the expression

$$\Phi_n(t) = \sum_{j=0}^{n-1} u_j, \tag{4.21}$$

to denote the n-term approximation to u.

Now, we consider Eq. (8) as follows:

$$\begin{cases} L\,\underline{y}(t;r) = g(\underline{y}(t;r), \overline{y}(t;r)) + \underline{f}(t;r), & \underline{y}(t_0;r) = \underline{y_0}(r), \\ L\,\overline{y}(t;r) = h(\underline{y}(t;r), \overline{y}(t;r)) + \overline{f}(t;r), & \overline{y}(t_0;r) = \overline{y_0}(r), \end{cases} \tag{4.22}$$

where $L \equiv \frac{d}{dt}$.
For solving Eq. (22), by homotopy perturbation method we construct homotopies as follows:

$$\begin{cases} H_1(\underline{Y}, p) = (1-p)\,[L(\underline{Y}) - L(\underline{y}_0)] + p\,[L(\underline{Y}) - g(\underline{Y}, \overline{Y}) - \underline{f}] = 0, \\ H_2(\overline{Y}, p) = (1-p)\,[L(\overline{Y}) - L(\overline{y}_0)] + p\,[L(\overline{Y}) - h(\underline{Y}, \overline{Y}) - \overline{f}] = 0, \end{cases} \tag{4.23}$$

by considering $\underline{y}_0 = \underline{y_0}(r)$, $\overline{y}_0 = \overline{y_0}(r)$, we have

$$L(\underline{y}_0) = 0, \qquad L(\overline{y}_0) = 0. \tag{4.24}$$

Therefore, by Eqs. (23) and (24), we have

$$\begin{cases} L(\underline{Y}(t;r)) = p\;g(\underline{Y}(t;r), \overline{Y}(t;r)) + p\,\underline{f}(t;r), \\ L(\overline{Y}(t;r)) = p\;h(\underline{Y}(t;r), \overline{Y}(t;r)) + p\,\overline{f}(t;r). \end{cases} \tag{4.25}$$

By applying the inverse operator

$$L^{-1}(\cdot) = \int_{t_0}^{t} (\cdot)ds,$$

on both sides of (25) and by considering

$$\underline{Y}(t_0;r) = \underline{y}(t_0;r) = \underline{y_0}(r), \qquad \overline{Y}(t_0;r) = \overline{y}(t_0;r) = \overline{y_0}(r),$$

we obtain

$$\begin{cases} Y(t;r)) = \underline{y_0}(r) + p\int_{t_0}^{t} \underline{f}(s;r)\,ds + p\int_{t_0}^{t} g(\underline{Y}(s;r),\overline{Y}(s;r)\,ds, \\ Y(t;r)) = \overline{y_0}(r) + p\int_{t_0}^{t} \overline{f}(s;r)\,ds + p\int_{t_0}^{t} h(\underline{Y}(s;r),\overline{Y}(s;r)\,ds. \end{cases} \tag{4.26}$$

We can assume that the solution of (26) can be expressed as a series in p, as follows:

$$\begin{cases} \underline{Y}(t;r) = \underline{y}_0(t;r) + p\,\underline{y}_1(t;r) + p^2\,\underline{y}_2(t;r) + p^3\,\underline{y}_3(t;r) + \cdots, \\ \overline{Y}(t;r) = \overline{y}_0(t;r) + p\,\overline{y}_1(t;r) + p^2\,\overline{y}_2(t;r) + p^3\,\overline{y}_3(t;r) + \cdots. \end{cases} \tag{4.27}$$

Substituting (27) into (26) and equating the terms with identical powers of p, we have

$$p^0 : \begin{cases} \underline{y}_0(t;r) = \underline{y_0}(r), \\ \overline{y}_0(t;r) = \overline{y_0}(r), \end{cases} \tag{4.28}$$

$$p^1 : \begin{cases} \underline{y}_1(t;r) = \int_{t_0}^{t} \underline{f}(s;r)\,ds + \int_{t_0}^{t} g(\underline{y}_0(s;r),\overline{y}_0(s;r))\,ds, \\ \overline{y}_1(t;r) = \int_{t_0}^{t} \overline{f}(s;r)\,ds + \int_{t_0}^{t} h(\underline{y}_0(s;r),\overline{y}_0(s;r))\,ds, \end{cases} \tag{4.29}$$

and for $k \geq 1$, we have

$$p^{k+1} : \begin{cases} \underline{y}_{k+1}(t;r) = \int_{t_0}^{t} g(\underline{y}_k(s;r),\overline{y}_k(s;r))\,ds, \\ \overline{y}_{k+1}(t;r) = \int_{t_0}^{t} h(\underline{y}_k(s;r),\overline{y}_k(s;r))\,ds. \end{cases} \tag{4.30}$$

The exact solutions of (22) or (9) , therefore, can be obtained by setting $p = 1$, i.e.

$$\underline{y}(t;r) = \lim_{p\to 1} \underline{Y}(t;r) = \lim_{p\to 1}(\underline{y}_0(t;r) + p\,\underline{y}_1(t;r) + p^2\,\underline{y}_2(t;r) + \cdots) = \sum_{j=0}^{\infty} \underline{y}_j(t;r), \tag{4.31}$$

$$\overline{y}(t;r) = \lim_{p\to 1} \overline{Y}(t;r) = \lim_{p\to 1}(\overline{y}_0(t;r) + p\,\overline{y}_1(t;r) + p^2\,\overline{y}_2(t;r) + \cdots) = \sum_{j=0}^{\infty} \overline{y}_j(t;r). \tag{4.32}$$

Similarly as before, we can obtain for Eq. (9) the following results:

$$p^0 : \begin{cases} \overline{y}_0(t;r) = \overline{y_0}(r), \\ \underline{y}_0(t;r) = \underline{y_0}(r), \end{cases} \tag{4.33}$$

$$p^1 : \begin{cases} \overline{y}_1(t;r) = \int_{t_0}^{t} \underline{f}(s;r)\,ds + \int_{t_0}^{t} g(\underline{y}_0(s;r),\overline{y}_0(s;r))\,ds, \\ \underline{y}_1(t;r) = \int_{t_0}^{t} \overline{f}(s;r)\,ds + \int_{t_0}^{t} h(\underline{y}_0(s;r),\overline{y}_0(s;r))\,ds, \end{cases} \tag{4.34}$$

and for $k \geq 1$, we have

$$p^{k+1} : \begin{cases} \overline{y}_{k+1}(t;r) = \int_{t_0}^{t} g(\underline{y}_k(s;r),\overline{y}_k(s;r))\,ds, \\ \underline{y}_{k+1}(t;r) = \int_{t_0}^{t} h(\underline{y}_k(s;r),\overline{y}_k(s;r))\,ds. \end{cases} \tag{4.35}$$

The exact solutions of (9) , therefore, can be obtained by setting $p = 1$, i.e.

$$\underline{y}(t;r) = \lim_{p\to 1} \underline{Y}(t;r) = \sum_{j=0}^{\infty} \underline{y}_j(t;r), \tag{4.36}$$

$$\overline{y}(t;r) = \lim_{p\to 1} \overline{Y}(t;r) = \sum_{j=0}^{\infty} \overline{y}_j(t;r). \tag{4.37}$$

5 Numerical results

In this section, we apply HPM to two examples. We Compare results with exact solutions in Tables 1-4 for a fixed t, so, approximate solutions and exact solutions are compared in Figs. 1, 3, 5 and 7. The three-dimensional plot of the error between the exact solutions and the approximate solutions obtained by HPM is shown in Fig. 2, 3, 4 and 6. We use MATLAB software in all the calculations done in this section.

Example 4.1. Consider the following FIVP

$$\begin{cases} y'(t) = 2 \odot y(t) \oplus (t^2 + 1), \\ y(0) = (r, 2 - r). \end{cases} \tag{5.38}$$

Note that this problem has two solutions By theorem 3.3 depending on How we write the two crisp equations and then how we can fuzzify them. Then, for solving Eq. (38), we have two different cases.
Case (1): If we consider $y'(t)$ in the first form ((**i**)-differentiable), we have to solve the following differential system:

$$\begin{cases} \underline{y}'(t;r) = 2\,\underline{y}(t;r) + t^2 + 1, & \underline{y}(0;r) = r, \\ \overline{y}'(t;r) = 2\,\overline{y}(t;r) + t^2 + 1, & \overline{y}(0;r) = 2 - r. \end{cases} \tag{5.39}$$

The exact solution of the system is given by

$$\begin{cases} \underline{y}(t;r) = (r + \frac{3}{4})e^{2t} - \frac{1}{4}(2t^2 + 2t + 3), \\ \overline{y}(t;r) = (\frac{11}{4} - r)e^{2t} - \frac{1}{4}(2t^2 + 2t + 3). \end{cases} \tag{5.40}$$

Now, we solve Eq. (39) via HPM and compare approximate solution with exact solution (40).

According to Eqs. (28), (29) and (30), we have

$$p^0 : \begin{cases} \underline{y}_0(t;r) = r, \\ \overline{y}_0(t;r) = 2 - r, \end{cases} \tag{5.41}$$

$$p^1 : \begin{cases} \underline{y}_1(t;r) = (1 + 2r)t + \frac{1}{3}t^3, \\ \overline{y}_1(t;r) = (5 - 2r)t + \frac{1}{3}t^3, \end{cases} \tag{5.42}$$

and for $k \geq 1$, we have

$$p^{k+1}: \begin{cases} \underline{y}_{k+1}(t;r) = 2\int_{t_0}^{t} \underline{y}_k(s;r)\,ds, \\ \overline{y}_{k+1}(t;r) = 2\int_{t_0}^{t} \overline{y}_k(s;r)\,ds. \end{cases} \tag{5.43}$$

We approximate $\underline{y}(t;r)$ and $\overline{y}(t;r)$, with $\underline{\Phi}_6(t;r)$ and $\overline{\Phi}_6(t;r)$, respectively, as follows:

$$\begin{aligned} \underline{\Phi}_6(t;r) = \sum_{i=0}^{5} \underline{y}_i(t;r) &= r + (1+2r)t + (1+2r)t^2 + (1+\frac{4}{3}r)t^3 + (\frac{1}{2}+\frac{2}{3}r)t^4 \\ &+ (\frac{1}{5}+\frac{4}{15}r)t^5 + \frac{1}{45}t^6 + \frac{2}{350}t^7, \end{aligned}$$

$$\begin{aligned} \overline{\Phi}_5(t;r) = \sum_{i=0}^{5} \overline{y}_i(t;r) &= (2-r) + (5-2r)t + (5-2r)t^2 + (\frac{11}{3}-\frac{4}{3}r)t^3 + (\frac{11}{6}-\frac{2}{3}r)t^4 \\ &+ (\frac{11}{15}-\frac{4}{15}r)t^5 + \frac{1}{45}t^6 + \frac{2}{350}t^7. \end{aligned}$$

Table 1 show the comparison of the exact solution and the approximate solution obtained by HPM at $t = 0.1$ and $t = 0.3$ for any $r \in [0,1]$. Also, in Fig. 1, we compare the exact solution with the approximate solution. The three-dimensional plot of the error between the exact solution and the approximate solution is shown in Fig. 2.

Table 1
The results for six-term approximate of HPM in Example 4.1 *case 1*.

	$t = 0.1$		$t = 0.3$	
r	$\lvert\underline{y} - \underline{\Phi}_6\rvert$	$\lvert\overline{y} - \overline{\Phi}_6\rvert$	$\lvert\underline{y} - \underline{\Phi}_6\rvert$	$\lvert\overline{y} - \overline{\Phi}_6\rvert$
0	4.5763e-08	2.2875e-07	3.5512e-05	1.7711e-04
0.1	5.4912e-08	2.1960e-07	4.2592e-05	1.7003e-04
0.2	6.4062e-08	2.1045e-07	4.9672e-05	1.6295e-04
0.3	7.3211e-08	2.0130e-07	5.6752e-05	1.5587e-04
0.4	8.2360e-08	1.9215e-07	6.3832e-05	1.4879e-04
0.5	9.1510e-08	1.8300e-07	7.0912e-05	1.4171e-04
0.6	1.0066e-07	1.7385e-07	7.7992e-05	1.3463e-04
0.7	1.0981e-07	1.6470e-07	8.5072e-05	1.2755e-04
0.8	1.1896e-07	1.5556e-07	9.2152e-05	1.2047e-04
0.9	1.2811e-07	1.4641e-07	9.9232e-05	1.1339e-04
1.0	1.3726e-07	1.3726e-07	1.0631e-04	1.0631e-04

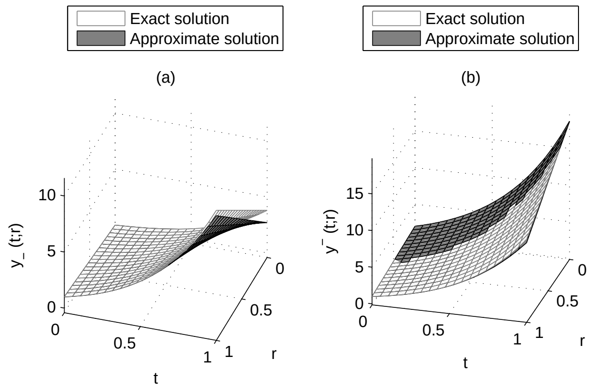

Fig. 1. (a) Comparing $\underline{y}(t;r)$ and $\underline{\phi}_6(t;r)$ in Example 4.1 *case 1*. (b) Comparing $\overline{y}(t;r)$ and $\overline{\phi}_6(t;r)$ in Example 4.1 *case 1*.

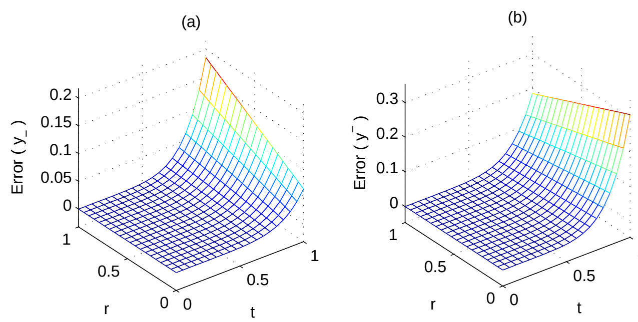

Fig. 2. (a) The error of $\underline{\phi}_6(t;r)$ in Example 4.1 *case 1*. (b) The error of $\overline{\phi}_6(t;r)$ in Example 4.1 *case 1*.

Case (2): If we consider $y'(t)$ in the second form ((**ii**)-differentiable), we have to solve the following differential system:

$$\begin{cases} \overline{y}'(t;r) = 2\,\underline{y}(t;r) + t^2 + 1, & \overline{y}(0;r) = 2 - r, \\ \underline{y}'(t;r) = 2\,\overline{y}(t;r) + t^2 + 1, & \underline{y}(0;r) = r. \end{cases} \tag{5.44}$$

The exact solution is given by

$$\begin{cases} \underline{y}(t;r) = \frac{7}{4}e^{2t} + (r-1)e^{-2t} - \frac{1}{4}(2t^2+2t+3), \\ \overline{y}(t;r) = \frac{7}{4}e^{2t} + (1-r)e^{-2t} - \frac{1}{4}(2t^2+2t+3). \end{cases} \tag{5.45}$$

Now, we solve Eq. (44) via HPM and compare approximate solution with exact solution (45).

According to Eqs. (33), (34) and (35), we have

$$p^0 : \begin{cases} \overline{y}_0(t;r) = 2-r, \\ \underline{y}_0(t;r) = r, \end{cases} \tag{5.46}$$

$$p^1 : \begin{cases} \overline{y}_1(t;r) = (1+2r)t + \frac{1}{3}t^3, \\ \underline{y}_1(t;r) = (5-2r)t + \frac{1}{3}t^3, \end{cases} \tag{5.47}$$

and for $k \geq 1$, we have

$$p^{k+1} : \begin{cases} \overline{y}_{k+1}(t;r) = 2\int_{t_0}^{t} \underline{y}_k(s;r)\,ds, \\ \underline{y}_{k+1}(t;r) = 2\int_{t_0}^{t} \overline{y}_k(s;r)\,ds. \end{cases} \tag{5.48}$$

We approximate $\underline{y}(t;r)$ and $\overline{y}(t;r)$, with $\underline{\Phi}_6(t;r)$ and $\overline{\Phi}_6(t;r)$, respectively, as follows:

$$\begin{aligned} \underline{\Phi}_6(t;r) = \sum_{i=0}^{5} \underline{y}_i(t;r) &= r + (5-2r)t + (1+2r)t^2 + (\frac{11}{3} - \frac{4}{3}r)t^3 + (\frac{1}{2} + \frac{2}{3}r)t^4 \\ &+ (\frac{1}{5} + \frac{11}{15}r)t^5 + \frac{1}{45}t^6 + \frac{2}{315}t^7, \end{aligned}$$

$$\begin{aligned} \overline{\Phi}_5(t;r) = \sum_{i=0}^{5} \overline{y}_i(t;r) &= (2-r) + (1+2r)t + (5-2r)t^2 + (1 + \frac{4}{3}r)t^3 + (\frac{11}{6} - \frac{2}{3}r)t^4 \\ &+ (\frac{1}{5} + \frac{4}{15}r)t^5 + \frac{1}{45}t^6 + \frac{2}{350}t^7. \end{aligned}$$

Table 2 show the comparison of the exact solution and the approximate solution obtained by HPM at $t = 0.1$ and $t = 0.3$ for any $r \in [0,1]$. Also, in Fig. 3, we compare the exact solution with the approximate solution. The three-dimensional plot of the error between the exact solution and the approximate solution is shown in Fig. 4.

Table 2
The results for six-term approximate of HPM in Example 4.1 *case 2*.

	$t=0.1$		$t=0.3$	
r	$\lvert \underline{y}-\underline{\Phi}_6 \rvert$	$\lvert \overline{y}-\overline{\Phi}_6 \rvert$	$\lvert \underline{y}-\underline{\Phi}_6 \rvert$	$\lvert \overline{y}-\overline{\Phi}_6 \rvert$
0	5.0845e-08	2.2367e-07	4.6676e-05	1.6595e-04
0.1	5.9486e-08	2.1503e-07	5.2640e-05	1.5698e-04
0.2	6.8127e-08	2.0639e-07	5.8603e-05	1.5402e-04
0.3	7.6769e-08	1.9774e-07	6.4567e-05	1.4806e-04
0.4	8.5410e-08	1.8910e-07	7.0530e-05	1.4209e-04
0.5	9.4051e-08	1.8046e-07	7.6494e-05	1.3613e-04
0.6	1.0269e-07	1.7182e-07	8.2458e-05	1.3017e-04
0.7	1.1133e-07	1.6318e-07	8.8421e-05	1.2420e-04
0.8	1.1997e-07	1.5454e-07	9.4385e-05	1.1824e-04
0.9	1.2862e-07	1.4590e-07	1.0035e-04	1.1228e-04
1.0	1.3726e-07	1.3726e-07	1.0631e-04	1.0361e-04

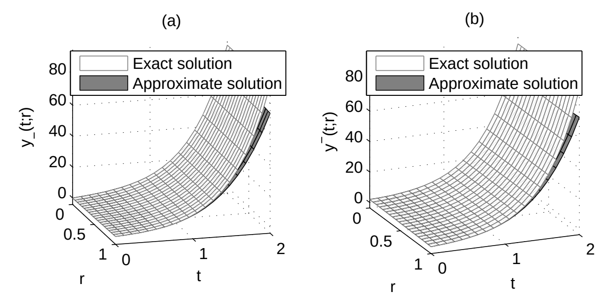

Fig. 3. (a) Comparing $\underline{y}(t;r)$ and $\underline{\phi}_6(t;r)$ in Example 4.1 *case 2*. (b) Comparing $\overline{y}(t;r)$ and $\overline{\phi}_6(t;r)$ in Example 4.1 *case 2*.

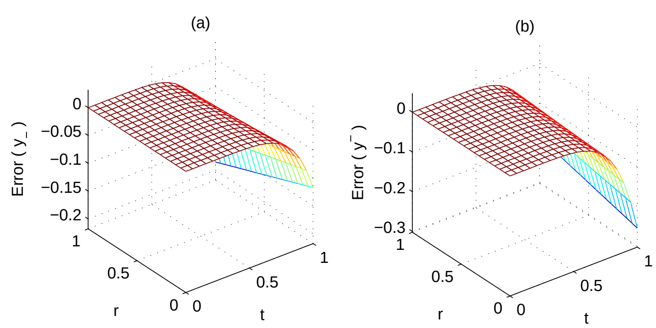

Fig. 4. (a) The error of $\underline{\phi}_6(t;r)$ in Example 4.1 *case 2*. (b) The error of $\overline{\phi}_6(t;r)$ in Example 4.1 *case 2*.

Example 4.2. Consider the following FIVP

$$\begin{cases} y'(t) = -3 \odot y(t) \oplus e^t, \\ y(0) = (r-1, 1-r). \end{cases} \tag{5.49}$$

Similarly as before, we have two different cases.
Case (1): If we consider $y'(t)$ in the first form ((**i**)-differentiable), we have to solve the following differential system:

$$\begin{cases} \underline{y}'(t;r) = -3\,\overline{y}(t;r) + e^t, & \underline{y}(0;r) = r-1, \\ \overline{y}'(t;r) = -3\,\underline{y}(t;r) + e^t, & \overline{y}(0;r) = 1-r. \end{cases} \tag{5.50}$$

The exact solution of the system is given by

$$\begin{cases} \underline{y}(t;r) = (r-1)e^{3t} - \frac{1}{4}e^{-3t} + \frac{1}{4}e^t, \\ \overline{y}(t;r) = (1-r)e^{3t} - \frac{1}{4}e^{-3t} + \frac{1}{4}e^t. \end{cases} \tag{5.51}$$

Now, we solve Eq. (50) via HPM and compare approximate solution with exact solution (51).

According to Eqs. (28), (29) and (30), we have

$$p^0 : \begin{cases} \underline{y}_0(t;r) = r-1, \\ \overline{y}_0(t;r) = 1-r, \end{cases} \tag{5.52}$$

$$p^1 : \begin{cases} \underline{y}_1(t;r) = (e^t-1) - 3(1-r)t, \\ \overline{y}_1(t;r) = (e^t-1) - 3(r-1)t, \end{cases} \tag{5.53}$$

and for $k \geq 1$, we have

$$p^{k+1} : \begin{cases} \underline{y}_{k+1}(t;r) = 2\int_{t_0}^{t} \overline{y}_k(s;r)\,ds, \\ \overline{y}_{k+1}(t;r) = 2\int_{t_0}^{t} \underline{y}_k(s;r)\,ds. \end{cases} \tag{5.54}$$

We approximate $\underline{y}(t;r)$ and $\overline{y}(t;r)$, with $\underline{\Phi}_6(t;r)$ and $\overline{\Phi}_6(t;r)$, respectively, as follows:

$$\begin{aligned} \underline{\Phi}_6(t;r) = \sum_{i=0}^{5} \underline{y}_i(t;r) &= (r-1) + 61(e^t-1) + (3r-63)t + (\frac{9}{2}r - 36)t^2 \\ &+ (\frac{9}{2}r - \frac{27}{2})t^3 + (\frac{27}{8}r - \frac{27}{4})t^4 + (\frac{81}{40}r - \frac{81}{40})t^5, \end{aligned}$$

$$\begin{aligned} \overline{\Phi}_5(t;r) = \sum_{i=0}^{5} \overline{y}_i(t;r) &= (1-r) + 61(e^t-1) - (3r+57)t - (\frac{9}{2}r + 27)t^2 \\ &+ (\frac{9}{2}r + \frac{9}{2})t^3 - \frac{27}{8}rt^4 + (\frac{81}{40} - \frac{81}{40}r)t^5. \end{aligned}$$

Table 3 show the comparison of the exact and the approximate solution obtained by HPM at $t = 0.1$ and $t = 0.3$ for any $r \in [0,1]$. Also, in Fig. 5, we compare the exact solution with the approximate solution. The three-dimensional plot of the error between the exact solution and the approximate solution is shown in Fig. 6.

Table 3
The results for six-term approximate of HPM in Example 4.2 *case 1*.

	$t = 0.1$		$t = 0.3$	
r	$\lvert \underline{y} - \underline{\Phi}_6 \rvert$	$\lvert \overline{y} - \overline{\Phi}_6 \rvert$	$\lvert \underline{y} - \underline{\Phi}_6 \rvert$	$\lvert \overline{y} - \overline{\Phi}_6 \rvert$
0	1.3858e-06	7.2931e-07	1.0723e-03	6.1739e-04
0.1	1.2801e-06	6.2355e-07	9.8785e-04	5.3290e-04
0.2	1.1743e-06	5.1779e-07	9.0336e-04	4.4841e-04
0.3	1.0686e-06	4.1204e-07	8.1888e-04	3.6393e-04
0.4	9.6281e-07	5.0628e-07	7.3439e-04	2.7944e-04
0.5	8.5705e-07	2.0052e-07	6.4991e-04	1.9496e-04
0.6	7.5130e-07	9.4764e-08	5.6542e-04	1.1047e-04
0.7	6.4554e-07	1.0993e-08	4.8093e-04	2.5983e-05
0.8	5.3978e-07	1.1675e-07	3.9645e-04	5.8503e-05
0.9	4.3402e-07	2.2251e-07	3.1196e-04	1.4249e-04
1.0	3.2827e-07	3.2827e-07	2.2748e-04	2.2748e-04

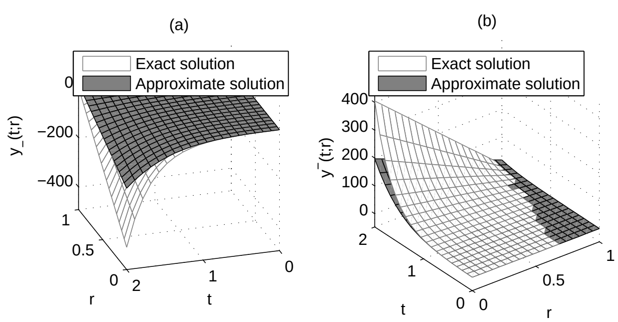

Fig. 5. (a) Comparing $\underline{y}(t;r)$ and $\underline{\phi}_6(t;r)$ in Example 4.2 *case 1*. (b) Comparing $\overline{y}(t;r)$ and $\overline{\phi}_6(t;r)$ in Example 4.2 *case 1*.

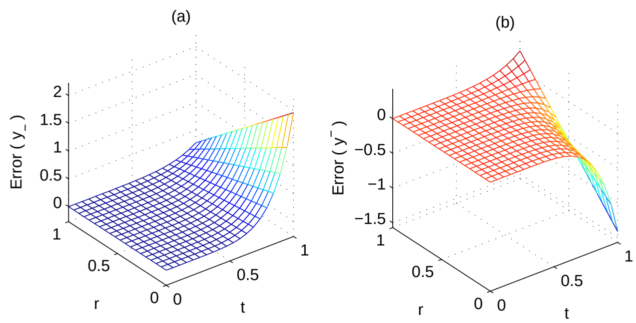

Fig. 6. (a) The error of $\underline{\phi}_6(t;r)$ in Example 4.2 *case 1*. (b) The error of $\overline{\phi}_6(t;r)$ in Example 4.2 *case 1*.

Case (2): If we consider $y'(t)$ in the second form ((**ii**)-differentiable), we have to solve the following differential system:

$$\begin{cases} \overline{y}'(t;r) = -3\,\overline{y}(t;r) + e^t, & \overline{y}(0;r) = 1 - r, \\ \\ \underline{y}'(t;r) = -3\,\underline{y}(t;r) + e^t, & \underline{y}(0;r) = r - 1. \end{cases} \tag{5.55}$$

The exact solution is given by

$$\begin{cases} \underline{y}(t;r) = (r - \frac{5}{4})e^{-3t} + \frac{1}{4}e^t, \\ \\ \overline{y}(t;r) = (\frac{3}{4} - r)e^{-3t} + \frac{1}{4}e^t. \end{cases} \tag{5.56}$$

Now, we solve Eq. (55) via HPM and compare approximate solution with exact solution (56).

According to Eqs. (33), (34) and (35), we have

$$p^0 : \begin{cases} \overline{y}_0(t;r) = 1 - r, \\ \\ \underline{y}_0(t;r) = r - 1, \end{cases} \tag{5.57}$$

$$p^1 : \begin{cases} \overline{y}_1(t;r) = (e^t - 1) - 3(1 - r)t, \\ \\ \underline{y}_1(t;r) = (e^t - 1) - 3(r - 1)t, \end{cases} \tag{5.58}$$

and for $k \geq 1$, we have

$$p^{k+1} : \begin{cases} \overline{y}_{k+1}(t;r) = 2\int_{t_0}^{t} \overline{y}_k(s;r)\,ds, \\ \\ \underline{y}_{k+1}(t;r) = 2\int_{t_0}^{t} \underline{y}_k(s;r)\,ds. \end{cases} \tag{5.59}$$

We approximate $\underline{y}(t;r)$ and $\overline{y}(t;r)$, with $\underline{\Phi}_6(t;r)$ and $\overline{\Phi}_6(t;r)$, respectively, as follows:

$$\begin{aligned}\underline{\Phi}_6(t;r)=\sum_{i=0}^{5}\underline{y}_i(t;r) &= (r-1)+61(e^t-1)-(3r+57)t+(\frac{9}{2}r-36)t^2\\ &- (\frac{9}{2}r+\frac{9}{2})t^3+(\frac{27}{8}r-\frac{27}{4})t^4+(\frac{81}{40}-\frac{81}{40}r)t^5,\end{aligned}$$

$$\begin{aligned}\overline{\Phi}_5(t;r)=\sum_{i=0}^{5}\overline{y}_i(t;r) &= (1-r)+61(e^t-1)+(3r-63)t+(\frac{9}{2}r+27)t^2\\ &+ (\frac{9}{2}r-\frac{27}{2})t^3-\frac{27}{4}rt^4+(\frac{81}{40}r-\frac{81}{40})t^5.\end{aligned}$$

Table 4 show the comparison of the exact solution and the approximate solution obtained by HPM at $t=0.1$ and $t=0.3$ for any $r\in[0,1]$. Also, in Fig. 7, we compare the exact solution with the approximate solution. The three-dimensional plot of the error between the exact solution and the approximate solution is shown in Fig. 8.

Table 4
The results for six-term approximate of HPM in Example 4.2 *case 2*.

	$t=0.1$		$t=0.3$	
r	$\lvert\underline{y}-\underline{\Phi}_6\rvert$	$\lvert\overline{y}-\overline{\Phi}_6\rvert$	$\lvert\underline{y}-\underline{\Phi}_6\rvert$	$\lvert\overline{y}-\overline{\Phi}_6\rvert$
0	1.2989e-06	6.4242e-07	8.8038e-04	4.2543e-04
0.1	1.2019e-06	5.4535e-07	5.1506e-04	3.6014e-04
0.2	1.1048e-06	4.4828e-07	7.4980e-04	2.9485e-04
0.3	1.0077e-06	3.5121e-07	6.8451e-04	2.2956e-04
0.4	9.1068e-07	2.5414e-07	6.1922e-04	1.6427e-04
0.5	8.1361e-07	1.5707e-07	5.5393e-04	9.8980e-05
0.6	7.1654e-07	6.0007e-07	4.8864e-04	3.3689e-05
0.7	6.1947e-07	3.7062e-07	4.2335e-04	3.1602e-05
0.8	5.2240e-07	1.3413e-07	3.5806e-04	9.6893e-05
0.9	4.2533e-07	2.3120e-07	2.9277e-04	1.6218e-04
1.0	3.2827e-07	3.2827e-07	2.2748e-04	2.2748e-04

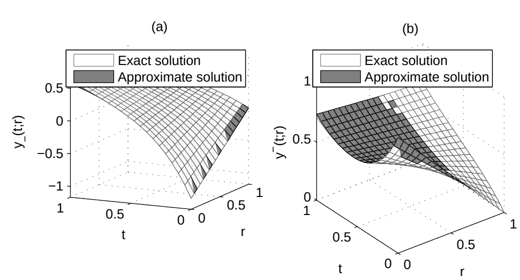

Fig. 7. (a) Comparing $\underline{y}(t;r)$ and $\underline{\phi}_6(t;r)$ in Example 4.2 *case 2*. (b) Comparing $\overline{y}(t;r)$ and $\overline{\phi}_6(t;r)$ in Example 4.2 *case 2*.

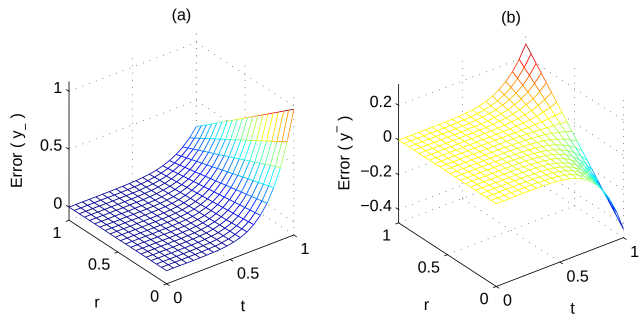

Fig. 8. (a) The error of $\underline{\phi}_6(t;r)$ in Example 4.2 *case 2*. (b) The error of $\overline{\phi}_6(t;r)$ in Example 4.2 *case 2*.

6 Conclusion

In this paper, we applied Homotopy Perturbation Method (HPM) for approximate solving of the FIVP. The original FIVP is replaced by two parametric ordinary differential equations which are then solved approximately using the HPM. HPM provides the components of the exact solution, where these components should follow the summation give in (21). The exact solutions are compared with solutions obtained by means of the HPM. The results show that this method is useful for finding an accurate approximation of the exact solution. Also, this method can be used for solving N-th fuzzy differential equations.

References

[1] S. Abbasbandy, T. Allahviranloo, Numerical solutions of fuzzy differential equations by taylor method, Computational Methods in Applied Mathematics 2 (2002) 113–124.

[2] S. Abbasbandy, T. Allahviranloo, O. Lopez-Pouso, J.J. Nieto, Numerical methods for fuzzy differential inclusions, Computer and Mathematics With Applications 48/10-11 (2004) 1633–1641.

[3] T. Allahviranloo, N. Ahmady, E. Ahmady, Numerical solution of fuzzy differential equations by predictorcorrector method, Information Sciences 177/7 (2007) 1633–1647.

[4] T. Allahviranloo, E. Ahmady, A. Ahmady, N-th fuzzy differential equations , Information Sciences 178 (2008) 1309–1324.

[5] G.A. Anastassiou, S.G. Gal, On a fuzzy trigonometric approximation theorem of Weierstrass-type, J. Fuzzy Math. 9 (3) (2001) 701–708.

[6] B. Bede, S. G. Gal, Almost periodic fuzzy-number-value functions, Fuzzy Sets and Systems 147 (2004) 385–403.

[7] B. Bede, I. Rudas, A. Bencsik, First order linear fuzzy differential equations under generalized differentiablity, information secieness 177 (2006) 3627–3635.

[8] B. Bede, S. G. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, Fuzzy Sets and Systems 151 (2005) 581–599.

[9] Y. Chalco-Cano, H. Roman-Flores, On new solutions of fuzzy differential equations, Chaos, Solitons and Fractals (2006) 1016–1043.

[10] S.S.L. Chang, L. Zadeh, On fuzzy mapping and control, IEEE Trans. Systems Man Cybernet. 2 (1972) 30–34.

[11] D. Dubois, H. Prade, Towards fuzzy differential calculus, Fuzzy Sets and Systems 8 (1982) 1–7 , 105–116, 225–233.

[12] M. Friedmen, M. Ming, A. Kandel, Fuzzy derivatives and fuzzy Cauchy problems using LP metric, in: Da Ruan (ED.), Fuzzy Logic Foundations and Industrial Applications, Kluwer Dordrecht, 1996, pp.57–72.

[13] S.G. Gal, Approximation theory in fuzzy setting, in: G.A.Anastassiou (Ed.), Handbook ofAnalytic-Computational Methods in Applied Mathematics, Chapman & Hall/CRC, Boca Raton, London, NewYork,Washington DC, 2000, pp. 617–666.

[14] R. Goetschel, W. Voxman, Elementary calculus, Fuzzy Sets and Systems 18 (1986) 31–43.

[15] O. He. W. Yi, On fuzzy differential equations, Fuzzy Sets and Systems 24 (1989) 321–325.

[16] J.H. He, A coupling method of a homotopy technique and a perturbation technique for non-linear problems. International Journal of Non-Linear Mechanics 35 (1) (2000), 37–43.

[17] J.H. He, Application of homotopy perturbation method to nonlinear wave equations, Chaos, Solitons and Fractals 26 (2005) 695–700.

[18] J.H. He, Homotopy perturbation method for solving boundary value problems, Physics Letters A 350 (2006)87–88.

[19] J.H. He, Limit cycle and bifurcation of nonlinear problems, Chaos, Solitons and Fractals 26 (3) (2005) 827–833.

[20] J.H. He, The homotopy perturbation method for nonlinear oscillators with discontinuities, Applied Mathematics and Computation 151 (2004) 287–292.

[21] A. Kandel, M. Friedmen, M. Ming, On Fuzzy dynamical processes, Proc. FUZZ-IEEE'96, New Orleans, 8–11 Sept. 1996, pp. 1813–1818.

[22] O. Kaleva, Fuzzy differential equations, Fuzzy Sets Systems 24 (1987) 301–317.

[23] O. Kaleva, The Cauchy problem fo fuzzy differential equations, Fuzzy Sets Systems 35 (1990) 389–396.

[24] P. Kloeden, Remarks on peano-like theorems for fuzzy differential equations, Fuzzy Sets and Systems 44 (1991) 161–164.

[25] W. Menda, Linear fuzzy differential equation systems on R^1, Jounal of Fuzzy Systems Mathematics 2 (1) (1988) 51–56, in Chinese.

[26] M.L. Puri, D. Ralescu, Differential for fuzzy function, J. Math. Anal. Appl. 91 (1983) 552–558.

[27] S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems 24 (1987) 319–330.

[28] C. Wu, S. Song, E. Stanley Lee, Approximate solutions, existence and uniqueness of the Cauchy problem of fuzzy differential equations, J. Math. Anal. Appl. 202 (1996) 629–644.