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Research Article

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Maximum Likelihood Estimation for Parameters of Extended Burr XII Distribution and Bias Correction Based on K-Records

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Abstract

Our aim at this paper is to estimate the parameters of extended Burr XII (EBXII) based on k-records with corrected bias. To get this goal maximum likelihood estimator (MLE) is applied and bias correction for ML estimation is considered to get better and improved estimator. Numerical results for comparison of performances of estimators are also presented.

Keywords : MLE; Bias corrected estimator; K-Records; Extended Burr XII distribution.

1 Introduction

LEt $X_1, X_2, \cdots$ be a sequence of identically and independently random variables having cumulative distribution function (CDF), $F(x,\theta)$, and probability density function (PDF), $f(x,\theta)$, where θ could be a vector parameter. Record values have been lots of attention since 1952 [5]. There are many references which have pointed out record values such as [1], [2] and [3]. In some special situations, second or third largest values are needed, so usual upper record values are inadequate. Upper k-record values are extension of ordinary upper record values. There are some papers and books which have introduced and investigated k-records such as [3] and [12]. Let $T_{1(k)} = k, R_{1(k)} = X_{1:k}$ and for $n \geq 2$ and let

$$T_{n(k)} =$$

$$min\{j : j > T_{n-1(k)}, X_j > X_{T_{n-1(k)}-k+1:T_{n-1(k)}}, \}$$

where $X_{i:n}$ indicates $i-th$ order statistics in a sample of size n. For $n \geq 1$, the sequence of upper k-records is defined by $R_{n(k)} = X_{T_{n(k)}-k+1:T_{n(k)}}$. Note that for $k = 1$ the ordinary upper record values can be recovered. The PDF of $n-th$ upper k-record value $R_{n(k)}$ for $n \geq 1$ is given by

$$f_{n(k)}(r) = \frac{k^n}{\Gamma(n)}\left[-\ln(\bar{F}(r))\right]^{n-1}\bar{F}^{k-1}(r)f(r), r \geq 0, \quad (1.1)$$

and joint pdf of $R_{1(k)}, R_{2(k)}, ..., R_{n(k)}$ is given by

$$f(r_1, r_2, ..., r_m) = k^m \bar{F}^k(x_m)) \times \prod_{i=1}^{m} \frac{f(x)}{\bar{F}(x)}, r_1 < r_2 < \cdots < r_m. \quad (1.2)$$

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Extended Burr XII has introduced by Shao [10] as generalization of some well-known distributions. Its PDF is as follows

$$\begin{aligned} f(x|\alpha,\lambda,c) &= \frac{c}{\lambda}\left(\frac{x}{\lambda}\right)^{c-1}\left[1-\alpha\left(\frac{x}{\lambda}\right)^{c}\right]^{\frac{1}{\alpha}-1}, \alpha \neq 0 \\ &= \frac{c}{\lambda}\left(\frac{x}{\lambda}\right)^{c-1}\exp\left\{-\left(\frac{x}{\lambda}\right)^{c}\right\}, \quad \alpha = 0, \end{aligned} \tag{1.3}$$

also CDF of EBXII is given by

$$\begin{aligned} F(x|\alpha,\lambda,c) &= 1-\left[1-\alpha\left(\frac{x}{\lambda}\right)^{c}\right]^{\frac{1}{\alpha}}, \quad \alpha \neq 0 \\ &= 1-\exp\left\{-\left(\frac{x}{\lambda}\right)^{c}\right\}, \quad \alpha = 0. \end{aligned} \tag{1.4}$$

EBXII distribution has some interesting properties. As shown in figures 1 and 2, the mentioned distribution has different shapes when parameters take different values, so it yields a wide range of skewness and kurtosis values. Detailed information and surveys are included in [10], also [11] studied different estimation methods for EBXII distribution.

Estimation of unknown parameters is one of the important branches of statistics that contains several methods, such as the method of moments, MLE, Bayesian estimations, and etc. Among these methods, an ML estimator is very common. In many situations, ML estimator cannot be expressed in closed form and its expectation is different from real population parameter and bias appears. Bias correction is a part of improving estimator, and [4] considered analytic approximations to the bias of MLE of a one-dimensional parameter. In 1968 [8] introduced $O(n^{-1})$ bias formula in order to obtain analytic expression, and hence bias-corrected MLE in the multi-parameter case. [7] re-expressed the bias-correction of [8]. More details can be found [6].

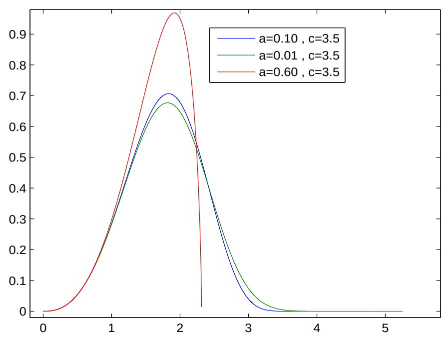

Figure 1: Density Function 1.1.

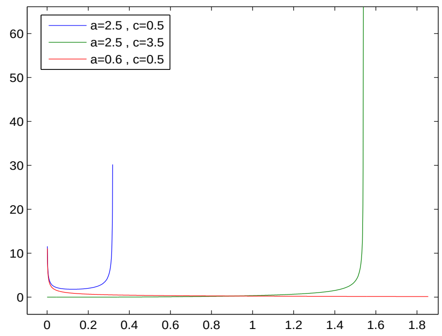

Figure 2: Density Function 1.2.

The structure of the paper is as follows. Section (2) obtains MLE of EBXII and discusses its properties. Section (3) applies the bias correction of MLE based on k-record values and enables us to get the bias-corrected ML estimator. Section (4) uses simulation studies to report the results in the paper and compares them. Also, the appendices contained in the paper included the necessary computational expressions.

2 Maximum Likelihood Estimation

Consider $\mathbf{x} = \{x_1, x_2, ...x_m\}$ be a vector of m upper k-record values from sample of size n of EBXII, then log-likelihood function is obtained as follows:

$$\begin{aligned} L(\mathbf{x};\alpha,\lambda,c) &= m\ln(k) + m\ln(c) - mc\ln(\lambda) \\ &+ (c-1)\sum_{i=1}^{m}\ln(x_i) - \sum_{i=1}^{m}\ln\left(1-\alpha\left(\frac{x_i}{\lambda}\right)^{c}\right) \\ &+ \frac{k}{\alpha}\ln\left(1-\alpha\left(\frac{x_m}{\lambda}\right)^{c}\right), \end{aligned} \tag{2.5}$$

with a *limit*,

$$\begin{aligned} L(\mathbf{x};0,\lambda,c) &= m\ln(k) + m\ln(c) - mc\ln(\lambda) \\ &+ (c-1)\sum_{i=1}^{m}\ln(x_i) - k\left(\frac{x_m}{\lambda}\right)^{c}. \end{aligned} \tag{2.6}$$

Theorem 2.1. *Let L be log-likelihood function of m upper k-record values from EBXII (2.5), with*

Table 1: MLE and Corrected MLE.

n	k	r	MLE $\hat{\alpha}$ (MSE)	MLE $\hat{\lambda}$ (MSE)	MLE $\hat{c}$ (MSE)	CMLE $\hat{\alpha}$ (MSE)	CMLE $\hat{\lambda}$ (MSE)	CMLE $\hat{c}$ (MSE)
10	2	2	2.5837 (0.5362)	1.9225 (0.4111)	3.5786 (0.4675)	2.4347 (0.0261)	1.7817 (0.1091)	4.2383 (0.2109)
		5	2.5368 (0.5138)	2.0223 (0.5413)	3.485 (0.5357)	2.6067 (0.0427)	2.0302 (0.0151)	3.4845 (0.0044)
	6	2	2.4779 (0.7462)	1.9944 (0.4519)	3.3714 (0.4599)	2.9521 (0.7808)	3.6673 (0.8336)	4.3135 (0.2324)
		5	2.3883 (1.1132)	2.0029 (0.4098)	3.447 (0.8005)	2.0383 (0.1846)	1.9134 (0.0432)	3.4506 (0.014)
50	2	2	2.4958 (0.2214)	1.9936 (0.1215)	3.4697 (0.2638)	2.4307 (0.0277)	1.9607 (0.0196)	3.635 (0.0385)
		5	2.4658 (0.234)	1.9923 (0.1382)	3.501 (0.2988)	2.4693 (0.0122)	1.993 (0.0034)	3.4968 (0.0009)
	6	2	2.5164 (0.2278)	2.0056 (0.1194)	3.5031 (0.3101)	3.6747 (0.4699)	2.2919 (0.1459)	3.1888 (0.0888)
		5	2.5173 (0.2371)	2.0024 (0.1443)	3.5602 (0.3376)	2.5289 (0.8115)	2.7247 (0.8623)	3.3901 (0.8257)
1000	2	2	2.5195 (0.0465)	1.999 (0.0242)	3.4916 (0.0492)	2.4619 (0.0152)	1.9709 (0.0145)	3.6291 (0.0369)
		5	2.5008 (0.0533)	1.9976 (0.0322)	3.5118 (0.0689)	2.5036 (0.0014)	1.9983 (0.0008)	3.5076 (0.0021)
	6	2	2.4856 (0.0548)	2.0013 (0.0298)	3.4935 (0.0635)	3.2636 (0.3054)	2.1986 (0.0993)	3.2309 (0.0768)
		5	2.4987 (0.0594)	2.004 (0.0337)	3.5094 (0.0645)	2.9119 (0.7647)	2.1602 (0.0801)	3.9501 (0.7003)

CDF (1.4) then as $\alpha \to 0$ we get,

$$\lim_{\alpha\to 0}\frac{\partial L}{\partial \alpha} = \sum_{i=1}^{m}\left(\frac{x_i}{\lambda}\right)^c - \frac{k}{2}\left(\frac{x_m}{\lambda}\right)^{2c}.$$

Proof. The proof is available at the appendix (A). □

$L(x; 0, \lambda, c)$ is log-likelihood function of Weibull distribution from k-records. As we have seen the range of X varies according to the sign of α, so it is important to determine the sign of it. In order to set the sign, a criterion defined by [9] and [13] is being used. The Maclaurin expansion of (2.5) is also applied to get this goal as follows. Let

$$L(X;\Theta) = L(x; 0, \lambda, c) + \Delta(x, \lambda, c)\alpha + O\{k^2\},$$

where, $\Delta(x; \lambda, c) = \sum_{i=1}^{m}\left(\frac{x_i}{\lambda}\right)^c - \frac{k}{2}\left(\frac{x_m}{\lambda}\right)^{2c}$. Assuming that $(\hat{\lambda}, \hat{c})$ be ML estimates of Weibull distribution, defining $\Delta(x; \hat{\lambda}, \hat{c}) = \Delta$, it can be concluded, if $\Delta > 0$, then $\hat{\alpha} > 0$, and if $\Delta < 0$, then $\hat{\alpha} < 0$, however in the case of $\Delta = 0$, indicates the Weibull fitting.

Using (B.14-B.16) from appendix (B) and equating them to the zero, ML estimators will be obtained. However exact solutions are not reachable so we apply Newton-Raphson as numerical method. Also using expressions (B.17-B.22) in the appendix (B) expected Fisher information matrix is given by

$$\mathbf{I}(\Theta|x) = n\begin{bmatrix} (B.17) & (B.20) & (B.21) \\ (B.20) & (B.18) & (B.22) \\ (B.21) & (B.22) & (B.19) \end{bmatrix}. \quad (2.7)$$

ML estimations well behaved for sample sizes sufficiently large, but in the case of smaller sample size, we will faced with biased estimator. To reduce the bias of estimator a biased-correction method is employed.

3 Biased Correction of MLE

Let $l(\Theta)$ be log-likelihood function of p-dimensional unknown parameters $\Theta = (\theta_1, \theta_2, \cdots, \theta_p)'$ based on n observations. Assuming $\mathbb{E}$ be expectation operator, the joint cumulants of the derivatives of $l(\Theta)$ are as follows

$$E_{ij} = \mathbb{E}\left[\frac{\partial^2 l}{\partial\theta_i \partial\theta_j}\right], \; for\ i,j = 1,2,\cdots,p, \quad (3.8)$$

$$E_{ijv} = \mathbb{E}\left[\frac{\partial^3 l}{\partial\theta_i \partial\theta_j \partial\theta_v}\right], \; for\ i,j,v = 1,2,\cdots,p, \quad (3.9)$$

$$E_{ij,v} = \mathbb{E}\left[\left(\frac{\partial^2 l}{\partial\theta_i \partial\theta_j}\right)\left(\frac{\partial l}{\partial\theta_v}\right)\right], \; for\ i,j,v = 1,2,\cdots,p, \quad (3.10)$$

and the derivatives of cumulates are given by

$$E_{ij}^{v} = \frac{\partial \mathbb{E}_{ij}}{\partial\theta_v}, \; for\ i,j,v = 1,2,\cdots,p. \quad (3.11)$$

All the E expressions are assumed to be $O(n)$. Fisher information matrix of Θ for $i,j = 1,2,\cdots,p$ are indicated as $\mathbb{K} = [-E_{ij}]$. [8] showed that for independent observations but not necessarily identically distributed, the bias of $s-th$ element of MLE of Θ, $\left(\hat{\Theta}\right)$ could be written as

$$\textbf{Bias}\left(\hat{\Theta}_s\right) = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{v=1}^{p} E^{si}E^{jv}\left(\frac{1}{2}E_{ijv} + E_{ij,v}\right) + O(n^{-2}),\; s = 1,2,\cdots,p, \quad (3.12)$$

where E^{ij} is $(i,j)th$ element of inverse Fisher information matrix. Also [7] showed that (3.12) continues to be valid for non-independent observations provided that all E's are $O(n)$, and reconstructed (3.12) up to order $O(n^{-1})$ as

$$\textbf{Bias}\left(\hat{\Theta}_s\right) = \mathbb{K}^{-1}.\mathcal{A}.\textbf{vec}\left(\mathbb{K}^{-1}\right) + O(n^{-2}), \quad (3.13)$$

where $\textbf{vec}\left(\mathbb{K}^{-1}\right)$ is an operator that creates a column vector obtained by stacking the columns of $\mathbb{K}^{-1}$ and where $\mathcal{A} = \left[\mathcal{A}^{(1)}|\mathcal{A}^{(2)}|\cdots|\mathcal{A}^{(p)}\right]$, with $\mathcal{A}^{(v)} = \left[a_{ij}^{(v)}\right]$ having its $(i,j)-th$ element defined by $a_{ij}^{(v)} = E_{ij}^{(v)} - \frac{1}{2}E_{ijv}$ for $i,j,v = 1,2,\cdots,p$. The biased-corrected MLE which is denoted by $\hat{\Theta}^{CMLE}$ is given by $\hat{\Theta}^{CMLE} = \hat{\Theta} - \hat{\mathbb{K}}^{-1}.\hat{\mathcal{A}}.\textbf{vec}\left(\hat{\mathbb{K}}^{-1}\right)$, where $\hat{\Theta}$ is the MLE of the unknown parameter vector Θ, $\hat{\mathbb{K}} = \mathbb{K}|_{\Theta=\hat{\Theta}}$, $\hat{\mathcal{A}} = \mathcal{A}|_{\Theta=\hat{\Theta}}$. The derivatives of log-likelihood function with respect to unknown parameters α, λ, c are given in appendix (B), also using (3.8-3.11) and relations (D.35-D.37) in appendix (D), another terms needed to obtain bias reduction expression are presented in appendix (C).

4 Simulation Study

At this section simulation study is employed to compare performances of presented methods by numerical results. Different values of sample size, record number, K value have been used to obtain several results for comparison of performances of MLE and bias corrected results. Therefore sample sizes n=10, 50 , 1000 is used. Also K=2,6 and r=2, 5 is applied as k values and record numbers. In order to get precise results, the simulation has replicated 10000 times. For simulation MATLAB 2019b is used. Based on the results in the table 1 corrected MLE method performs better than MLE method. As in most cases the MSE of corrected MLE is lower than the MSE of MLE.

Appendices

A Proof of Theorem

Defining $\overset{H}{=}$ as l'Hospital's rule equality symbol. By differentiating with respect to α from (2.5) and then calculating its limit as $\alpha \to 0$ we get

$$\begin{aligned}\lim_{\alpha\to 0}\frac{\partial L}{\partial\alpha} &= \sum_{i=0}^{m}\left(\frac{x_i}{\lambda}\right)^c \\ &- k\lim_{\alpha\to 0}\frac{\frac{\ln\left(1-\alpha\left(\frac{x_m}{\lambda}\right)^c\right)}{\alpha} + \frac{\left(\frac{x_m}{\lambda}\right)^c}{1-\alpha\left(\frac{x_m}{\lambda}\right)^c}}{\alpha} \\ &= \sum_{i=0}^{m}\left(\frac{x_i}{\lambda}\right)^c - \frac{0}{0},\end{aligned}$$

since,

$$\lim_{\alpha\to 0}\frac{\ln\left(1-\alpha\left(\frac{x_m}{\lambda}\right)^c\right)}{\alpha} \overset{H}{=} -\left(\frac{x_m}{\lambda}\right)^c.$$

Using L'Hospital's rule we get

$$\lim_{\alpha\to 0} \frac{\frac{\ln\left(1-\alpha\left(\frac{x_m}{\lambda}\right)^c\right)}{\alpha} + \frac{\left(\frac{x_m}{\lambda}\right)^c}{1-\alpha\left(\frac{x_m}{\lambda}\right)^c}}{\alpha} \overset{H}{=} \lim_{\alpha\to 0} \frac{\frac{-\alpha\left(\frac{x_m}{\lambda}\right)^c}{1-\alpha\left(\frac{x_m}{\lambda}\right)^c} - \ln\left(1-\alpha\left(\frac{x_m}{\lambda}\right)^c\right)}{\alpha^2} + \left(\frac{x_m}{\lambda}\right)^{2c}$$

desired result provided by using L'Hospital's rule twice again.

B Derivatives

We define $G := (\lambda, c)$, $H := H(X_i, \lambda) = \frac{X_i}{\lambda}$, $W := W(X_i, G) = H^c$ and $D := D(X_i, G, \alpha) = 1 - \alpha W$, and suppose t and s be integers. First order derivatives of log-likelihood function with respect to unknown parameters is calculated as

$$\frac{\partial l}{\partial \alpha} = \sum_{i=1}^{m} \frac{W_i}{D_i} - \frac{k}{\alpha^2}\ln(D_m) - \frac{k}{\alpha}\frac{W_m}{D_m}, \quad \text{(B.14)}$$

$$\frac{\partial l}{\partial \lambda} = -\frac{mc}{\lambda} - \frac{\alpha c}{\lambda}\sum_{i=1}^{m}\frac{W_i}{D_i} + \frac{ck}{\lambda}\frac{W_m}{D_m}, \quad \text{(B.15)}$$

$$\frac{\partial l}{\partial c} = \frac{m}{c} - m\ln(\lambda) + \sum_{i=1}^{m}\ln(x_i) + \alpha\sum_{i=1}^{m}\frac{W_i\ln(H_i)}{D_i} - k\frac{W_m\ln(H_m)}{D_m}. \quad \text{(B.16)}$$

Higher order partial derivatives of log-likelihood function are also computed as follows

$$\frac{\partial^2 l}{\partial \alpha^2} = \sum_{i=1}^{m}\left(\frac{W_i}{D_i}\right)^2 - \frac{2k}{\alpha^3}\ln(D_m) + \frac{2k}{\alpha^2}\frac{W_m}{D_m} - \frac{k}{\alpha}\left(\frac{W_m}{D_m}\right)^2, \quad \text{(B.17)}$$

$$\frac{\partial^2 l}{\partial \lambda^2} = \frac{mc}{\lambda^2} + \frac{\alpha c}{\lambda^2}\sum_{i=1}^{m}\frac{W_i}{D_i} + \frac{\alpha c^2}{\lambda^2}\sum_{i=1}^{m}\frac{W_i}{D_i^2} - \frac{ck}{\lambda^2}\frac{W_m}{D_m} - \frac{c^2k}{\lambda^2}\frac{W_m}{D_m^2}, \quad \text{(B.18)}$$

$$\frac{\partial^2 l}{\partial c^2} = -\frac{m}{c^2} + \alpha\sum_{i=1}^{m}W_i\left(\frac{\ln(H_i)}{D_i}\right)^2 - kW_m\left(\frac{\ln(H_m)}{D_m}\right)^2, \quad \text{(B.19)}$$

$$\frac{\partial^2 l}{\partial \alpha\partial\lambda} = -\frac{c}{\lambda}\sum_{i=1}^{m}\frac{W_i}{D_i} - \frac{\alpha c}{\lambda}\sum_{i=1}^{m}\left(\frac{W_i}{D_i}\right)^2 + \frac{ck}{\lambda}\left(\frac{W_m}{D_m}\right)^2, \quad \text{(B.20)}$$

$$\frac{\partial^2 l}{\partial \alpha\partial c} = \sum_{i=1}^{m}\frac{W_i\ln(H_i)}{D_i^2} - k\left(\frac{W_m}{D_m}\right)^2\ln(H_m), \quad \text{(B.21)}$$

$$\frac{\partial^2 l}{\partial \lambda\partial c} = -\frac{m}{\lambda} - \frac{\alpha}{\lambda}\sum_{i=1}^{m}\frac{W_i}{D_i} - \frac{\alpha c}{\lambda}\sum_{i=1}^{m}\frac{W_i\ln(H_i)}{D_i^2} + \frac{kW_m}{\lambda D_m} + \frac{ck}{\lambda}\frac{W_m\ln(H_m)}{D_m^2}, \quad \text{(B.22)}$$

$$\frac{\partial^3 l}{\partial \alpha^3} = 2\sum_{i=1}^{m}\left(\frac{W_i}{D_i}\right)^3 - \frac{6k}{\alpha^4}\ln(D_m) + \frac{4k}{\alpha^3}\frac{W_m}{D_m} + \frac{3k}{\alpha^2}\left(\frac{W_m}{D_m}\right)^2 - \frac{2kW_m}{\alpha^3 D_m^2} + \frac{2kW_m^2}{\alpha^2 D_m^3} - \frac{4k}{\alpha}\left(\frac{W_m}{D_m}\right)^3, \quad \text{(B.23)}$$

$$\frac{\partial^3 l}{\partial \lambda^3} = \frac{-2mc}{\lambda^3} - \frac{2\alpha c}{\lambda^3}\sum_{i=1}^{m}\frac{W_i}{D_i} - \frac{3\alpha c^2}{\lambda^3}\sum_{i=1}^{m}\frac{W_i}{D_i^2} - \frac{\alpha c^3}{\lambda^3}\sum_{i=1}^{m}\frac{W_m+\alpha W_m^2}{D_m^3} + \frac{2c2k}{\lambda^3}\frac{W_m}{D_m} + \frac{3c^2kW_m}{\lambda^3 D_m^2} + \frac{c^3k}{\lambda^3}\frac{W_m+\alpha W_m^2}{D_m^3}, \quad \text{(B.24)}$$

$$\frac{\partial^3 l}{\partial c^3} = \frac{2m}{c^3} + \alpha\sum_{i=1}^{m}W_i\frac{\ln^3(H_i)}{D_i^2} + 2\alpha^2\sum_{i=1}^{m}W_i^2\left(\frac{\ln(H_i)}{D_i}\right)^3 - kW_m\frac{\ln^3(H_m)}{D_m^2} - 2\alpha kW_m^2\left(\frac{\ln(H_m)}{D_m}\right)^3, \quad \text{(B.25)}$$

$$\frac{\partial^3 l}{\partial \alpha^2\partial\lambda} = -2\frac{c}{\lambda}\left(\sum_{i=1}^{m}\left(\frac{W_i}{D_i}\right)^2 + \alpha\sum_{i=1}^{m}\left(\frac{W_i}{D_i}\right)^3 + \frac{2ck}{\lambda}\left(\frac{W_m}{D_m}\right)^3\right), \quad \text{(B.26)}$$

$$2\sum_{i=1}^{m}\frac{W_i^2}{D_i^3}\ln(H_i) - 2k\left(\frac{W_m}{D_m}\right)^3\ln(H_m), \quad \text{(B.27)}$$

$$\frac{\partial^3 l}{\partial\alpha\partial\lambda^2} = \frac{c}{\lambda^2}\left(\sum_{i=1}^{m}\frac{W_i}{D_i} + \alpha\sum_{i=1}^{m}\left(\frac{W_i}{D_i}\right)^2 - k\left(\frac{W_m}{D_m}\right)^2\right) \times \frac{c^2}{\lambda^2}\left(\sum_{i=1}^{m}\frac{W_i}{D_i^2} + 2\alpha\sum_{i=1}^{m}\frac{W_i^2}{D_i^3} - 2k\frac{W_m^2}{D_m^3}\right), \quad \text{(B.28)}$$

$$\frac{\partial^3 l}{\partial\alpha\partial\lambda\partial c} = -\frac{1}{\lambda}\sum_{i=1}^{m}\frac{W_i}{D_i} - \frac{\alpha}{\lambda}\sum_{i=1}^{m}\left(\frac{W_i}{D_i}\right)^2 + \frac{k}{\lambda}\left(\frac{W_m}{D_m}\right)^2 - \frac{c}{\lambda}\sum_{i=1}^{m}\frac{W_i}{D_i^2}\ln(H_i) - \frac{2\alpha c}{\lambda}\sum_{i=1}^{m}\frac{W_i^2}{D_i^3}\ln(H_i) + \frac{2ck}{\lambda}\frac{W_m^2}{D_m^3}\ln(H_m), \quad \text{(B.29)}$$

$$\frac{\partial^3 l}{\partial\alpha\partial c^2} = \sum_{i=1}^{m}\frac{W_i}{D_i^2}\ln^2(H_i) + 2\alpha\sum_{i=1}^{m}\frac{W_i^2}{D_i^3}\ln^2(H_i) - 2k\frac{W_m^2}{D_m^3}, \quad \text{(B.30)}$$

$$\frac{\partial^3 l}{\partial\lambda^2\partial c} = \frac{m}{\lambda^2} + \frac{\alpha}{\lambda^2}\sum_{i=1}^{m}\frac{W_i}{D_i} + \frac{\alpha c}{\lambda^2}\sum_{i=1}^{m}\frac{W_i}{D_i}\ln(H_i) + \frac{\alpha^2 c}{\lambda^2}\sum_{i=1}^{m}\left(\frac{W_i}{D_i}\right)^2 \times \ln(H_i) + \frac{2\alpha c}{\lambda^2}\sum_{i=1}^{m}\frac{W_i}{D_i^2} + \frac{\alpha c^2}{\lambda^2}\sum_{i=1}^{m}\frac{W_i}{D_i^2} \times \ln(H_i) + \frac{2\alpha^2 c^2}{\lambda^2}\sum_{i=1}^{m}\frac{W_i^2}{D_i^3}\ln(H_i) - \frac{kW_m}{\lambda^2 D_m} - \frac{ckW_m}{\lambda^2 D_m}\ln(H_m) - \frac{\alpha ck}{\lambda^2}\left(\frac{W_m}{D_m}\right)^2\ln(H_m) - \frac{2ckW_m}{\lambda^2 D_m^2} - \frac{c^2 kW_m}{\lambda^2 D_m^2}\ln(H_m) - \frac{2c^2\alpha k}{\lambda^2}\frac{W_m^2}{D_m^3}\ln(H_m), \quad \text{(B.31)}$$

$$\frac{\partial^3 l}{\partial\lambda\partial c^2} = -\frac{\alpha c}{\lambda}\sum_{i=1}^{m}\frac{W_i}{D_i^2}\ln^2(H_i) - \frac{2\alpha}{\lambda}\sum_{i=1}^{m}\frac{W_i}{D_i^2}\ln(H_i) - \frac{2\alpha^2 c}{\lambda}\sum_{i=1}^{m}\frac{W_i^2}{D_i^3} \times \ln^2(H_i) + \frac{kc}{\lambda}\frac{W_m}{D_m^2}\ln^2(H_m) + \frac{2k}{\lambda}\frac{W_m}{D_m^2}\ln(H_m) + \frac{2k\alpha c}{\lambda}\frac{W_m^2}{D_m^3} \times \ln^2(H_m). \quad \text{(B.32)}$$

C Cumulants

$$E_{11} = \frac{k^n(-1)^{n-1}}{\alpha^3\Gamma(n)} \times \left\{\left(m-\frac{k}{\alpha}\right)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-2\right) + \frac{2k}{\alpha}\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right)\right\} - \frac{2n}{\alpha^2}$$

$$E_{12} = -\frac{c}{\lambda}\frac{k^n(-1)^{n-1}}{\alpha^2\Gamma(n)} \times \left\{m\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right) + \left(m-\frac{k}{\alpha}\right)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-2\right)\right\}$$

$$E_{13} = \frac{1}{c}\frac{k^n(-1)^{n-1}}{\alpha^2\Gamma(n)}\left\{m\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right) \times \left[\Psi(2) - \Psi\left(\frac{k}{\alpha}\right)\right] - m\ln(\alpha)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right) - \frac{k}{\alpha}\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-2\right)\left[\Psi(3) - \Psi\left(\frac{k}{\alpha}+1\right)\right] + \frac{k\ln(\alpha)}{\alpha}\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-2\right)\right\}$$

$$E_{22} = \frac{mc}{\lambda^2} + \frac{c}{\lambda^2}\frac{k^n(-1)^{n-1}}{\alpha\Gamma(n)}\left(m-\frac{k}{\alpha}\right) \times \left\{\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right) + c\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)\right\}$$

$$E_{23} = -\frac{m}{\lambda} - \frac{k^n(-1)^{n-1}}{\alpha\Gamma(n)}\frac{\left(m-\frac{k}{\alpha}\right)}{\lambda} \times \frac{\partial^{n-1}}{\partial k^{n-1}}\left\{\beta\left(2,\frac{k}{\alpha}-1\right)+\beta\left(2,\frac{k}{\alpha}-2\right) \times \left[\Psi(2)-\Psi\left(\frac{k}{\alpha}\right)\right] - \ln(\alpha)\beta\left(2,\frac{k}{\alpha}-2\right)\right\}$$

$$E_{33} = -\frac{m}{c^2} + \frac{k^n(-1)^{n-1}}{\alpha\Gamma(n)}\frac{\left(m-\frac{k}{\alpha}\right)}{c^2} \times \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)\left\{\left[\Psi(2)-\Psi\left(\frac{k}{\alpha}\right)\right]^2 + \left[\Psi'(2)-\Psi'\left(\frac{k}{\alpha}\right)\right] + \left[\Psi'(2)-\Psi'\left(\frac{k}{\alpha}\right)\right] - 2\ln(\alpha)\left[\Psi(2)-\Psi\left(\frac{k}{\alpha}\right)\right] + \ln^2(\alpha)\right\} \tag{C.34}$$

$$E_{11}^1 = \frac{k^n(-1)^{n-1}}{\alpha^5\Gamma(n)} \times \left\{\left(k-3\alpha\left(m-\frac{k}{\alpha}\right)\right)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-2\right) - \left(m-\frac{k}{\alpha}\right)\frac{\partial^{n-1}}{\partial k^{n-1}}k\beta\left(3,\frac{k}{\alpha}-2\right) \times \left[\Psi\left(\frac{k}{\alpha}-2\right)-\Psi\left(\frac{k}{\alpha}+1\right)\right] - 8k \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right) - \frac{2k}{\alpha}\frac{\partial^{n-1}}{\partial k^{n-1}}k\beta\left(2,\frac{k}{\alpha}-1\right) \times \left[\Psi\left(\frac{k}{\alpha}-1\right)-\Psi\left(\frac{k}{\alpha}+1\right)\right]\right\} + \frac{4n}{\alpha^3}$$

$$E_{111} = 2\frac{k^n(-1)^{n-1}}{\alpha^5\Gamma(n)} \times \left\{[\alpha m - 2k]\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(4,\frac{k}{\alpha}-3\right) - 4k\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right) + 3k\frac{\partial^{n-1}}{\partial k^{n-1}} \times \beta\left(3,\frac{k}{\alpha}-2\right) - 2k\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right) + 2k\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-3\right)\right\} + \frac{6n}{\alpha^3}$$

$$E_{11}^2 = 0$$

$$E_{112} = \frac{2c}{\lambda}\frac{k^n(-1)^{n-1}}{\alpha^4\Gamma(n)}\left\{(k-m\alpha) \times \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-3\right) + k\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right) - k\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)\right\}$$

$$E_{11}^3 = 0$$

$$E_{113} =$$

$$\frac{2}{c}\frac{k^n(-1)^{n-1}}{\alpha^3\Gamma(n)}\left\{m\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-3\right) \times \left[\left[\Psi(3)-\Psi\left(\frac{k}{\alpha}\right)\right]-\ln(\alpha)\right] - k\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(4,\frac{k}{\alpha}-3\right) \times \left[\left[\Psi(4)-\Psi\left(\frac{k}{\alpha}+1\right)\right]-\ln(\alpha)\right]\right\}$$

$$E_{12}^1 =$$

$$\frac{c}{\lambda}\frac{k^n(-1)^{n-1}}{\alpha^3\Gamma(n)}\left\{2m\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right) + \frac{m}{\alpha}\frac{\partial^{n-1}}{\partial k^{n-1}}k\beta\left(2,\frac{k}{\alpha}-1\right) \times \left[\Psi\left(\frac{k}{\alpha}-1\right)-\Psi\left(\frac{k}{\alpha}+1\right)\right] + 2m\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-2\right) - \frac{3k}{\alpha}\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-2\right) + \frac{\left(m-\frac{k}{\alpha}\right)}{\alpha}\frac{\partial^{n-1}}{\partial k^{n-1}}k\beta\left(3,\frac{k}{\alpha}-2\right) \times \left[\Psi\left(\frac{k}{\alpha}-2\right)-\Psi\left(\frac{k}{\alpha}+1\right)\right]\right\}$$

$$E_{121} = E_{112}$$

$$
\begin{aligned}
E_{12}^2 &= \frac{c}{\lambda^2}\frac{k^n(-1)^{n-1}}{\alpha^2\Gamma(n)}\left[m\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right)\right.\\
&+ \left.\left(m-\frac{k}{\alpha}\right)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-2\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
E_{122} &= \frac{c}{\lambda^2}\frac{k^n(-1)^{n-1}}{\alpha^2\Gamma(n)}\left\{m\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right)\right.\\
&+ \left(m-\frac{k}{\alpha}\right)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-2\right)\\
&+ c\left[m\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)\right.\\
&+ \left.\left.2\left(m-\frac{k}{\alpha}\right)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-3\right)\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
E_{12}^3 &= \frac{-1}{\lambda}\frac{k^n(-1)^{n-1}}{\alpha^2\Gamma(n)}\left[\left(m-\frac{k}{\alpha}\right)\right.\\
&\times \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-2\right)\\
&+ \left.m\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
E_{123} &= \frac{1}{\lambda}\frac{k^n(-1)^{n-1}}{\alpha^2\Gamma(n)}\{m\\
&\times \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right)\\
&+ \left(m-\frac{k}{\alpha}\right)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-2\right)\\
&+m\ \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)\\
&\times \left[\Psi(2)-\Psi\left(\frac{k}{\alpha}\right)\right]\\
&- m\ln\alpha\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)\\
&+ 2\left(m-\frac{k}{\alpha}\right)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-3\right)\\
&\times \left[\Psi(3)-\Psi\left(\frac{k}{\alpha}\right)\right]\\
&- 2\left(m-\frac{k}{\alpha}\right)\ln(\alpha)\\
&\times \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-3\right)\}
\end{aligned}
$$

$$
\begin{aligned}
E_{13}^1 &= \frac{-2}{c}\frac{k^n(-1)^{n-1}}{\alpha^3\Gamma(n)}\{m\\
&\times \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)\\
&\times \left[\Psi(2)-\Psi\left(\frac{k}{\alpha}\right)\right]\\
&- m\ln(\alpha)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)\\
&- \frac{k}{\alpha}\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-2\right)\left[\Psi(3)-\Psi\left(\frac{k}{\alpha}+1\right)\right]\\
&+ \frac{k\ln(\alpha)}{\alpha}\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-2\right)\}\\
&+ \frac{m}{c\alpha^2}\frac{k^n(-1)^{n-1}}{\alpha^2\Gamma(n)}\\
&\times \left\{\frac{\partial^{n-1}}{\partial k^{n-1}}\left[-k\beta\left(2,\frac{k}{\alpha}-2\right)\right.\right.\\
&\times \left[\Psi\left(\frac{k}{\alpha}-2\right)-\Psi\left(\frac{k}{\alpha}\right)\right]\left[\Psi(2)-\Psi\left(\frac{k}{\alpha}\right)\right]\\
&- \left.\beta\left(2,\frac{k}{\alpha}-2\right)\psi'\left(\frac{k}{\alpha}\right)\right]-\frac{m}{\alpha}\\
&\times \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)-m\ln(\alpha)\\
&\times \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)\\
&\times \left[\Psi\left(\frac{k}{\alpha}-2\right)-\Psi\left(\frac{k}{\alpha}\right)\right]+\frac{k}{\alpha^2}\\
&\times \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-2\right)\left[\Psi(3)-\Psi\left(\frac{k}{\alpha}+1\right)\right]\\
&- \frac{k}{\alpha^3}\frac{\partial^{n-1}}{\partial k^{n-1}}k\beta\left(3,\frac{k}{\alpha}-2\right)\\
&\times \left[\Psi\left(\frac{k}{\alpha}-2\right)-\Psi\left(\frac{k}{\alpha}+1\right)\right]\\
&\times \left[\Psi(3)-\Psi\left(\frac{k}{\alpha}+1\right)\right]-\frac{k}{\alpha}\\
&\times \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-2\right)\psi'\left(\frac{k}{\alpha}+1\right)\\
&+ \frac{k(1-\ln(\alpha))}{\alpha^2}\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-2\right)\\
&+ \frac{k\ln(\alpha)}{\alpha}\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-2\right)\\
&\times \left.\left[\Psi\left(\frac{k}{\alpha}-2\right)-\Psi\left(\frac{k}{\alpha}+1\right)\right]\right]\right\}
\end{aligned}
$$

$$E_{131} = E_{113},$$

$$E_{13}^2 = 0,$$

$$E_{132} = E_{123}$$

$$\begin{aligned} E_{13}^3 &= \frac{1}{c^2}\frac{k^n(-1)^{n-1}}{\alpha^2\Gamma(n)}\{m \\ &\times \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)[\ln(\alpha) \\ &- \left[\Psi(2)-\Psi\left(\frac{k}{\alpha}\right)\right]\Big]-\frac{k}{\alpha} \\ &\times \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-2\right) \\ &\times \left[\ln(\alpha)-\left[\Psi(3)-\Psi\left(\frac{k}{\frac{k}{\alpha}+1}\right)\right]\right]\Bigg\} \end{aligned}$$

$$\begin{aligned} E_{133} &= \frac{1}{c^2}\frac{k^n(-1)^{n-1}}{\alpha^2\Gamma(n)}\{m \\ &\times \Bigg\{\beta\left(2,\frac{k}{\alpha}-2\right)\left[\left[\Psi(2)-\Psi\left(\frac{k}{\alpha}\right)\right]^2\right. \\ &+ \left.\psi'(2)-\psi'\left(\frac{k}{\alpha}\right)\right]-2\ln(\alpha) \\ &\times \beta\left(2,\frac{k}{\alpha}-2\right)\left[\Psi(2)-\Psi\left(\frac{k}{\alpha}\right)\right] \\ &+ \ln^2(\alpha)\beta\left(2,\frac{k}{\alpha}-2\right)\Bigg\}+2\left(m-\frac{k}{\alpha}\right) \\ &\times \Bigg\{\beta\left(3,\frac{k}{\alpha}-3\right)\left[\left[\Psi(3)-\Psi\left(\frac{k}{\alpha}\right)\right]^2\right. \\ &+ \left.\psi'(3)-\psi'\left(\frac{k}{\alpha}\right)\right] \\ &- 2\ln(\alpha)\beta\left(3,\frac{k}{\alpha}-3\right) \\ &\times \left[\Psi(3)-\Psi\left(\frac{k}{\alpha}\right)\right] \\ &+ \ln^2(\alpha)\beta\left(3,\frac{k}{\alpha}-3\right)\Bigg\}\Bigg\} \end{aligned}$$

$$E_{21}^1 = E_{12}^1,$$

$$E_{211} = E_{121},$$

$$E_{21}^2 = E_{12}^2,$$

$$E_{212} = E_{122},$$

$$E_{21}^3 = E_{12}^3,$$

$$E_{213} = E_{123},$$

$$\begin{aligned} E_{22}^1 &= \frac{c}{\lambda^2}\frac{k^n(-1)^{n-1}}{\alpha^2\Gamma(n)}\Bigg\{-\left(m-\frac{k}{\alpha}\right)\times . \\ &\quad \left[\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right)+c\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)\right] \\ &+ \frac{1}{\alpha}\left[k\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right)\right. \\ &+ \left(m-\frac{k}{\alpha}\right)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right) \\ &\times \left[\Psi\left(\frac{k}{\alpha}-1\right)-\Psi\left(\frac{k}{\alpha}+1\right)\right] \\ &+ kc\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)+\left(m-\frac{k}{\alpha}\right)\times c\frac{\partial^{n-1}}{\partial k^{n-1}} \\ &\quad \left.\beta\left(2,\frac{k}{\alpha}-2\right)\left[\Psi\left(\frac{k}{\alpha}-2\right)-\Psi\left(\frac{k}{\alpha}\right)\right]\right]\Bigg\} \end{aligned}$$

$$E_{221} = E_{122}$$

$$\begin{aligned} E_{22}^2 &= \frac{-2c}{\lambda^3}\Bigg\{m+\frac{k^n(-1)^{n-1}}{\alpha\Gamma(n)}\left(m-\frac{k}{\alpha}\right)\times \\ &\quad \left[\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right)+c\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)\right]\Bigg\} \end{aligned}$$

$$\begin{aligned} E_{222} &= \frac{-2mc}{\lambda^3}-\frac{c}{\lambda^3}\frac{k^n(-1)^{n-1}}{\alpha\Gamma(n)}\Bigg\{2\left(m-\frac{k}{\alpha}\right) \\ &\times \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right)+c^2\left(m-\frac{k}{\alpha}\right) \\ &\times \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-3\right)+m(2c-c^2) \\ &+ 3c\left(m-\frac{k}{\alpha}\right)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right) \\ &+ c^2\left(m-\frac{k}{\alpha}\right)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-3\right)\Bigg\} \end{aligned}$$

$$\begin{aligned} E_{22}^3 &= \frac{m}{\lambda^2}+\frac{1}{\lambda^2}\frac{k^n(-1)^{n-1}}{\alpha\Gamma(n)}\left(m-\frac{k}{\alpha}\right)\times \\ &\quad \left[\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right)+2c\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)\right] \end{aligned}$$

$$
\begin{aligned}
E_{223} &= \frac{m}{\lambda^2} + \frac{1}{\lambda^2}\frac{k^n(-1)^{n-1}}{\alpha\Gamma(n)}\left(m-\frac{k}{\alpha}\right) \\
&\times \left\{\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right)\right. \\
&+ 2c\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right) \\
&+ \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right) \\
&\times \left[\Psi(2)-\Psi\left(\frac{k}{\alpha}+1\right)\right] \\
&- \ln(\alpha)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right) \\
&+ \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-2\right) \\
&\times \left[\Psi(3)-\Psi\left(\frac{k}{\alpha}+1\right)\right] \\
&- \ln(\alpha)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-2\right)+c \\
&\times \left[\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)\right. \\
&\times \left[\Psi(2)-\Psi\left(\frac{k}{\alpha}\right)\right] \\
&- \left.\ln(\alpha)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)\right] \\
&+ 2c\left[\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-3\right)\right. \\
&\times \left[\Psi(3)-\Psi\left(\frac{k}{\alpha}\right)\right] \\
&- \left.\left.\ln(\alpha)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-3\right)\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
E_{23}^{1} &= \frac{-1}{\lambda}\frac{k^n(-1)^{n-1}}{\alpha^2\Gamma(n)}\left\{\left(\frac{2k}{\alpha}-m\right)\right. \\
&\times \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right)-\frac{\left(m-\frac{k}{\alpha}\right)}{\alpha} \\
&\times \frac{\partial^{n-1}}{\partial k^{n-1}}k\beta\left(2,\frac{k}{\alpha}\right)\left[\Psi\left(\frac{k}{\alpha}-1\right)-\Psi\left(\frac{k}{\alpha}+1\right)\right] \\
&+ \left(\frac{2k}{\alpha}-m\right)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right) \\
&\times \left[\Psi(2)-\Psi\left(\frac{k}{\alpha}\right)\right]-\left(\frac{2k}{\alpha}-m\right) \\
&\times \ln(\alpha)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right) \\
&+ \left(m-\frac{k}{\alpha}\right)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)-\frac{\left(m-\frac{k}{\alpha}\right)}{\alpha} \\
&\times \frac{\partial^{n-1}}{\partial k^{n-1}}k\beta\left(2,\frac{k}{\alpha}-2\right)\left[\Psi\left(\frac{k}{\alpha}-2\right)-\Psi\left(\frac{k}{\alpha}\right)\right] \\
&\times \left[\Psi(2)-\Psi\left(\frac{k}{\alpha}\right)\right]+\frac{\left(m-\frac{k}{\alpha}\right)}{\alpha} \\
&\times \frac{\partial^{n-1}}{\partial k^{n-1}}k\beta\left(2,\frac{k}{\alpha}-2\right)\psi'\left(\frac{k}{\alpha}\right) \\
&+ \frac{\ln(\alpha)}{\alpha^2}\frac{\partial^{n-1}}{\partial k^{n-1}}k\beta\left(2,\frac{k}{\alpha}-2\right) \\
&\times \left.\left[\Psi\left(\frac{k}{\alpha}-2\right)-\Psi\left(\frac{k}{\alpha}\right)\right]\right\}
\end{aligned}
$$

$$E_{231} = E_{123}$$

$$
\begin{aligned}
E_{23}^{2} &= \frac{m}{\lambda^2}+\frac{\left(m-\frac{k}{\alpha}\right)}{\lambda^2}\frac{k^n(-1)^{n-1}}{\alpha\Gamma(n)} \\
&\times \left\{\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-1\right)\right. \\
&+ \left.\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)\left[\Psi(2)-\Psi\left(\frac{k}{\alpha}\right)\right]\right\} \\
&- \frac{\left(m-\frac{k}{\alpha}\right)\alpha\ln(\alpha)}{\lambda^2}\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)
\end{aligned}
$$

$$
\begin{aligned}
E_{232} &= E_{223}, \\
E_{23}^{3} &= 0,
\end{aligned}
$$

$$
\begin{aligned}
E_{233} &= \frac{-\left(m-\frac{k}{\alpha}\right)}{c\lambda}\frac{k^n(-1)^{n-1}}{\alpha\Gamma(n)} \\
&\times \left\{\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)\right. \\
&\times \left[\Psi(2)-\Psi\left(\frac{k}{\alpha}\right)\right]^2 \\
&+ \left[\Psi'(2)-\Psi'\left(\frac{k}{\alpha}\right)\right] \\
&- 2\ln(\alpha)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right) \\
&\times \left[\Psi(2)-\Psi\left(\frac{k}{\alpha}\right)\right] \\
&+ \ln^2(\alpha)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \; 2\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)\\
&\times \; \left[\Psi\left(2\right)-\Psi\left(\frac{k}{\alpha}\right)\right]\\
&- \; 2\ln(\alpha)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)\\
&+ \; 2\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-3\right)\\
&\times \; \left[\Psi\left(3\right)-\Psi\left(\frac{k}{\alpha}\right)\right]^2\\
&+ \; \left[\Psi'\left(3\right)-\Psi'\left(\frac{k}{\alpha}\right)\right]\\
&- \; 4\ln(\alpha)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-3\right)\\
&\times \; \left[\Psi\left(3\right)-\Psi\left(\frac{k}{\alpha}\right)\right]\\
&+ \; 2\ln^2(\alpha)\frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(3,\frac{k}{\alpha}-3\right)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
E_{31}^1 &= E_{13}^1,\\
E_{311} &= E_{131},\\
E_{31}^2 &= E_{13}^2,\\
E_{312} &= E_{132},\\
E_{31}^3 &= E_{13}^3,\\
E_{313} &= E_{133},\\
E_{32}^1 &= E_{23}^1,\\
E_{321} &= E_{231},\\
E_{32}^2 &= E_{23}^2,\\
E_{322} &= E_{232},\\
E_{32}^3 &= E_{23}^3,\\
E_{323} &= E_{233}
\end{aligned}
$$

$$
\begin{aligned}
E_{33}^1 &= \frac{1}{c^2}\frac{k^n(-1)^{n-1}}{\alpha^3\Gamma(n)}\{(2k-m\alpha)\\
&\times \; \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)\\
&\times \; \left[\left[\left[\Psi\left(2\right)-\Psi\left(\frac{k}{\alpha}\right)\right]^2\right.\right.\\
&+ \; \left.\left.\left[\Psi'\left(2\right)-\Psi'\left(\frac{k}{\alpha}\right)\right]\right]\right.\\
&- \; 2\ln(\alpha)\left[\Psi\left(2\right)-\Psi\left(\frac{k}{\alpha}\right)\right]\\
&+ \; \left.\ln^2(\alpha)\right]-\left(m-\frac{k}{\alpha}\right)\\
&\times \; \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)\\
&\times \; \left[k\left[\Psi\left(\frac{k}{\alpha}-2\right)-\Psi\left(\frac{k}{\alpha}\right)\right]\right.\\
&\times \; \left[\left[\Psi\left(\frac{k}{\alpha}-2\right)-\Psi\left(\frac{k}{\alpha}\right)\right]^2\right.\\
&+ \; \left.\left.\left[\Psi'\left(2\right)-\Psi'\left(\frac{k}{\alpha}\right)\right]\right]\right]\\
&- \; k\left[2\psi'\left(\frac{k}{\alpha}\right)\left[\Psi\left(2\right)-\Psi\left(\frac{k}{\alpha}\right)\right]\right.\\
&+ \; \left.\psi''\left(\frac{k}{\alpha}\right)\right]+2\alpha\left[\Psi\left(2\right)-\Psi\left(\frac{k}{\alpha}\right)\right]\\
&- \; 2\alpha\ln(\alpha)k\left[\left[\Psi\left(\frac{k}{\alpha}-2\right)-\Psi\left(\frac{k}{\alpha}\right)\right]\right.\\
&\times \; \left.\left[\Psi\left(2\right)-\Psi\left(\frac{k}{\alpha}\right)\right]\right]+2\ln(\alpha)k\\
&\times \; \psi'\left(\frac{k}{\alpha}\right)+2\alpha\ln(\alpha)+\ln^2(\alpha)k\\
&\times \; \left.\left[\Psi\left(\frac{k}{\alpha}-2\right)-\Psi\left(\frac{k}{\alpha}\right)\right]\right]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
E_{331} &= E_{133},\\
E_{33}^2 &= 0,\\
E_{332} &= E_{233},
\end{aligned}
$$

$$
\begin{aligned}
E_{33}^3 &= \frac{2m}{c^3}-\frac{2\left(m-\frac{k}{\alpha}\right)}{c^3}\frac{k^n(-1)^{n-1}}{\alpha\Gamma(n)}\\
&\times \; \frac{\partial^{n-1}}{\partial k^{n-1}}\beta\left(2,\frac{k}{\alpha}-2\right)\{-2\ln(\alpha)\\
&\times \; \left[\Psi\left(2\right)-\Psi\left(\frac{k}{\alpha}\right)\right]+\ln^2(\alpha)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
E_{333} &= \frac{2m}{c^3} + \frac{\left(m - \frac{k}{\alpha}\right)}{\lambda} \frac{k^n(-1)^{n-1}}{\alpha\Gamma(n)} \\
&\times \frac{\partial^{n-1}}{\partial k^{n-1}} \beta\left(2, \frac{k}{\alpha} - 2\right) \left\{ \frac{1}{c^3} \right. \\
&\times \left(\left[\Psi(2) - \Psi\left(\frac{k}{\alpha}\right) \right]^3 \right. \\
&+ 3 \left[\Psi(2) - \Psi\left(\frac{k}{\alpha}\right) \right] \\
&\times \left. \left[\Psi'(2) - \Psi'\left(\frac{k}{\alpha}\right) \right] \right) \\
&- \frac{3\ln(\alpha)}{c^2} \left(\left[\Psi(2) - \Psi\left(\frac{k}{\alpha}\right) \right]^2 \right. \\
&+ \left. \left[\Psi'(2) - \Psi'\left(\frac{k}{\alpha}\right) \right] \right) + \frac{3\ln^2(\alpha)}{c} \\
&\times \left. \left[\Psi(2) - \Psi\left(\frac{k}{\alpha}\right) \right] - \ln^3(\alpha) \right\} \\
&+ 2 \frac{\partial^{n-1}}{\partial k^{n-1}} \beta\left(3, \frac{k}{\alpha} - 3\right) \left\{ \frac{1}{c^3} \right. \\
&\times \left(\left[\Psi(3) - \Psi\left(\frac{k}{\alpha}\right) \right]^3 \right. \\
&+ 3 \left[\Psi(3) - \Psi\left(\frac{k}{\alpha}\right) \right] \\
&\times \left. \left[\Psi'(3) - \Psi'\left(\frac{k}{\alpha}\right) \right] \right) \\
&- \frac{3\ln(\alpha)}{c^2} \left(\left[\Psi(3) - \Psi\left(\frac{k}{\alpha}\right) \right]^2 \right. \\
&+ \left. \left[\Psi'(3) - \Psi'\left(\frac{k}{\alpha}\right) \right] \right) + \frac{3\ln^2(\alpha)}{c} \\
&\times \left. \left[\Psi(2) - \Psi\left(\frac{k}{\alpha}\right) \right] - \ln^3(\alpha) \right\}
\end{aligned}
$$

D Expectations

$$
\begin{aligned}
E\left(\frac{W^t}{D^s}\right) &= \frac{k^n(-1)^{n-1}}{\alpha^{t+1}\Gamma(n)} \\
&\times \frac{\partial^{n-1}}{\partial k^{n-1}} \beta\left(t+1, \frac{k}{\alpha} - s\right),
\end{aligned} \tag{D.35}
$$

$$
\begin{aligned}
E\left(\frac{W^t}{D^s}\ln(H)\right) &= \frac{k^n(-1)^{n-1}}{\alpha^{t+1}\Gamma(n)} \\
&\times \frac{\partial^{n-1}}{\partial k^{n-1}} \left\{ \beta\left(t+1, \frac{k}{\alpha} - s\right) \right. \\
&\times \left[\Psi(t+1) - \Psi\left(\frac{k}{\alpha} + t - s + 1\right) \right. \\
&- \left. \ln(\alpha) \right] \Big\},
\end{aligned} \tag{D.36}
$$

$$
\begin{aligned}
&E\left(\frac{W^t}{D^s}\ln^2(H)\right) = c^{-2}\frac{k^n(-1)^{n-1}}{\alpha^{n+t}\Gamma(n)} \\
&\times \frac{\partial^{n-1}}{\partial k^{n-1}} \beta\left(t+1, \frac{k}{\alpha} - s\right) \\
&\times \left\{ \left[\Psi(t+1) - \Psi\left(\frac{k}{\alpha} - s + t + 1\right) \right]^2 \right. \\
&+ \Psi'(t+1) - \Psi'\left(\frac{k}{\alpha} - s + t + 1\right) \\
&- 2\ln(\alpha) \left[\Psi(t+1) - \Psi\left(\frac{k}{\alpha} - s + t + 1\right) \right] \\
&+ \left. \alpha^{n-1}\ln^2(\alpha) \right\}.
\end{aligned} \tag{D.37}
$$

$$
E(X^t_{n(k)}) = \int_0^{\frac{\lambda}{\alpha^{1/c}}} x f_{n(k)}(x) dx = I_1
$$

using change of variable

$$
z = 1 - \alpha\left(\frac{x}{\lambda}\right)^c,
$$

it can be shown that

$$
\begin{aligned}
I_1 &= \frac{(-1)^{n-1}k^n\lambda^t}{\alpha^{n+\frac{t}{c}}\Gamma(n)} \\
&= \frac{(-1)^{n-1}k^n\lambda^t}{\alpha^{n+\frac{t}{c}}\Gamma(n)} \alpha^{n-1} \frac{\partial^{n-1}}{\partial k^{n-1}}
\end{aligned}
$$

$$
E(X^t_{n(k)}) = \frac{(-1)^{n-1}k^n\lambda^t}{\alpha^{1+\frac{t}{c}}\Gamma(n)} \frac{\partial^{n-1}}{\partial k^{n-1}} \beta\left(\frac{k}{\alpha}, \frac{t}{c} + 1\right) \tag{D.38}
$$

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References

[1] J. Ahmadi, Record values, theory and applications, *Mashhad, Iran: Ferdowsi University of Mashhad. Ph. D Thesis*, 2000.

[2] M. Ahsanullah, V. B. Nevzorov, *Records via Probability Theory*, *Atlantis Press*, 2015.

[3] C. Barry Arnold, N. Balakrishnan, H. N. Nagaraja, Records, *John Wiley & Sons,* 1998.

[4] M. S. Bartlett, Approximate Confidence Intervals, *Biometrika* 40 (1953) 12-19.

[5] K. N. Chandler, The distribution and frequency of record values, *Journal of the Royal Statistical Society: Series B (Methodological)* 14 (1952) 220-228.

[6] M. Gauss, Cordeiro and Francisco Cribari-Neto, An Introduction to Bartlett Correction and Bias Reduction, *Springer-Verlag Berlin Heidelberg*, 2014.

[7] M. Gauss, Cordeiro and Ruben Klein, Bias correction in arma models, *Statistics & Probability Letters* 19 (1994) 169-176.

[8] D. R. Cox, E. J. Snell, A general definition of residuals, *Journal of the Royal Statistical Society. Series B (Methodological)* 30 (1968) 248-275.

[9] Q. Shao, Notes on maximum likelihood estimation for the three-parameter burrxii distribution, *Computational Statistics and Data Analysis* 45 (2004) 675-687.

[10] Q. Shao, H. Wong, J. Xia, Wai-Cheung Ip, Models for extremes using the extended three-parameter burr xii system with application to flood frequency analysis / modles dextrmes utilisant le systme burr xii tendu trois paramtres et application lanalyse frquentielle des crues, *Hydrological Sciences Journal* 49 (2004) 685-702.

[11] I. Usta, Different estimation methods for the parameters of the extended burrxii distribution, *Journal of Applied Statistics* 40 (2013) 397-414.

[12] B. Kopociski, W. Dziubdziela, Limiting properties of the k-th record values, *Applicationes Mathematicae* 15 (1976) 187-190.

[13] J. Alan Watkins, An algorithm for maximum likelihood estimation in the three parameter burrxii distribution, *Computational Statistics and Data Analysis* 32 (1999) 19-27.

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