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Exact and Numerical Solutions for Nonlinear Differential Equation of Jeffrey-Hamel Flow

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Abstract

This paper looks at the analysis of Jeffery Hamel flow. The investigation is mainly aimed to determine an exact analytic solution for a nonlinear problem. To the best of my knowledge, no such analysis is available in the literature which can describe the exact solution of the Jeffrey-Hamel flow. Besides this a comparative study between the numerical and exact solutions is presented. The effects of the various parameters intrinsic to the problems are analyzed and depicted via graphs.

Keywords : Nonlinear problem; Exact analytic solution; Numerical solution Jeffrey-Hamel flow.

1 Introduction

The flow between two planes which meet at an angle was first analyzed by Jeffery [7] and Hamel [6]. Under suitable assumptions, the problem can be reduced to an ordinary differential equation. The incompressible viscous fluid flow through convergent–divergent channels is one of the most applicable cases in fluid mechanics, civil, environmental, mechanical and bio-mechanical engineering. A lot of information and references about Jeffery Hamel flow can be found in the refs. [1, 4, 5, 13, 14]. Most scientific problems such as Jeffery−Hamel flows are inherently of nonlinearity. Except a limited number of these problem, most of them do not have exact analytical solution. Therefore, these nonlinear equations have been solved either numerically [2, 15] or by perturbation methods [10, 11, 12]. Very little [3, 8, 9] has been yet said in the regime of exact solutions for nonlinear problems. The convergence of the solution and the large parameter are deficiencies of numerical and the perturbation methods respectively. We confine ourselves here to present a general exact analytical solution and the comparison of numerical solution as

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well. It is also worth mentioning that our exact analytical solution is not only valid for small but also for large values of emerging parameters.

2 Problem formulation

Consider the steady two-dimensional flow of an incompressible viscous fluid from a source or sink at the intersection between two rigid plane walls that the angle between them is 2α as shown in Fig. 1 given below:

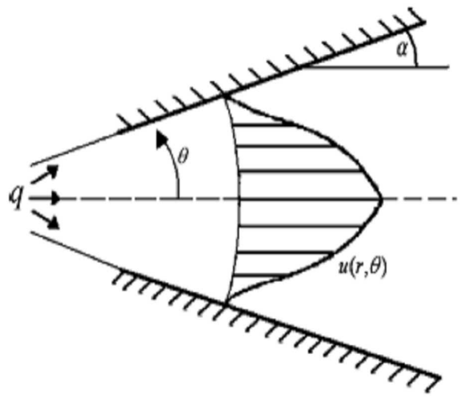

Fig. 1. Geometry of the problem.

We assume that the velocity is only along radial direction and depend upon r and h, i.e., $V(u(r,h),0)$ [10, 11]. Using continuity and the Navier–Stokes equations in polar coordinates we have

$$\frac{\rho\partial}{r\partial r}(ru(r,\theta))=0, \tag{2.1}$$

$$u(r,\theta)\frac{\partial u(r,\theta)}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left[\frac{\partial^2 u(r,\theta)}{\partial r^2}+\frac{1}{r}\frac{\partial u(r,\theta)}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial \theta^2}-\frac{u(r,\theta)}{r^2}\right] \tag{2.2}$$

$$-\frac{1}{\rho r}\frac{\partial p}{\partial \theta}+\frac{2\nu}{r^2}\frac{\partial u(r,\theta)}{\partial \theta}=0.$$

Equation (2.1) yields

$$f(\theta)\equiv ru(r,\theta). \tag{2.3}$$

Introducing

$$F(\eta)\equiv\frac{f(\theta)}{f_{\max}},\eta=\frac{\theta}{\alpha} \tag{2.4}$$

and eliminating p in Eqs. (2.2) and (2.3), we obtain the following equation for the normalized function $F(\eta)$ as [11]

$$F'''(\eta)+2\alpha ReF(\eta)F'(\eta)+4\alpha^2F'(\eta)=0. \tag{2.5}$$

The subjected corresponding boundary conditions are

$$F(0)=1,F'(0)=0,F(1)=0, \tag{2.6}$$

in which Re and α are the constants. The constant $\alpha>0$ gives a divergent channel and for convergent channel the condition $\alpha<0$ holds.

3 Exact solution

A first integral of equation (2.5) is

$$F'' + \alpha Re F^2 + 4\alpha^2 F = c_1, \tag{3.7}$$

where c_1 is an arbitrary constant. Equation (3.7) has the translational symmetry in η and its order can be reduced as

$$\frac{1}{2}\frac{dF'^2}{dF} + \alpha Re F^2 + 4\alpha^2 F = c_1. \tag{3.8}$$

Hence we have

$$F' = \pm\sqrt{2c_1 F - \frac{2}{3}\alpha Re F^3 - 4\alpha^2 F^2 + 2c_2}, \tag{3.9}$$

where c_2 is a further constant. The boundary conditions ((2.6),b,c) then require that

$$c_1 + c_2 = \frac{1}{3}\alpha Re + 2\alpha^2. \tag{3.10}$$

Since we want $F' > 0$, we omit the negative sign in (3.9). Thus

$$\int_1^F \frac{dE}{\sqrt{2c_1 E - \frac{2}{3}\alpha Re E^3 - 4\alpha^2 E^2 + \frac{2}{3}\alpha Re + 4\alpha^2 - 2c_1}} = \eta \tag{3.11}$$

subject to

$$F(1) = 0. \tag{3.12}$$

is an exact solution of Jeffery Hamel flow.

4 Numerical Solution

In this section we present the numerical solution of Jeffery Hamel flow by a so called method "Shooting method". To apply shooting method on Eqs. (2.5) and (2.6), we write our third order equation in three first order equations

$$F'(\eta) = v(\eta) \tag{4.13}$$

$$F''(\eta) = u(\eta) = v'(\eta) \tag{4.14}$$

$$u'(\eta) = -\left(2\alpha Re F(\eta) v(\eta) + 4\alpha^2 v(\eta)\right) \tag{4.15}$$

$$F(0) = 1, v(0) = 0, F(1) = 0 \tag{4.16}$$

and missing condition is

$$F''(0) = s \; or \; u(0) = s \tag{4.17}$$

Equations (4.13)-(4.17) can be differentiated with respect to s to obtain

$$F^{*\prime}(\eta) = V(\eta) \tag{4.18}$$

$$F^{*\prime\prime}(\eta) = U(\eta) = V'(\eta) \tag{4.19}$$

$$U'(\eta) = -\left(2\alpha Re\left(F^*(\eta)\, v(\eta) + F(\eta)\, V(\eta)\right) + 4\alpha^2 V(\eta)\right) \tag{4.20}$$

$$F^*(0) = 0, V(0) = 0, U(0) = 1 \tag{4.21}$$

where

$$F^* = \frac{\partial F}{\partial s}, V = \frac{\partial v}{\partial s}, U = \frac{\partial u}{\partial s} \tag{4.22}$$

and s is an initial guess and change iteratively after each step by Newton's formula

$$s^{n+1} = s^n - \frac{F(L, s^n) - A}{\frac{\partial F(L, s^n)}{\partial s}} \tag{4.23}$$

or here $A = 0, L = 1$ and $\partial F/\partial s = F^*$ then the equation is

$$s^{n+1} = s^n - \frac{F(1, s^n)}{F^*(1, s^n)} \tag{4.24}$$

where $s^1$is taken to be -1.

5 Graphs and comparison of results

In order to illustrate the influences of R_e and α on F, we have plotted the Figures 2 and 3 respectively. The obtained analytical solution is compared with numerical solution in Figures 4 and 5 respectively.

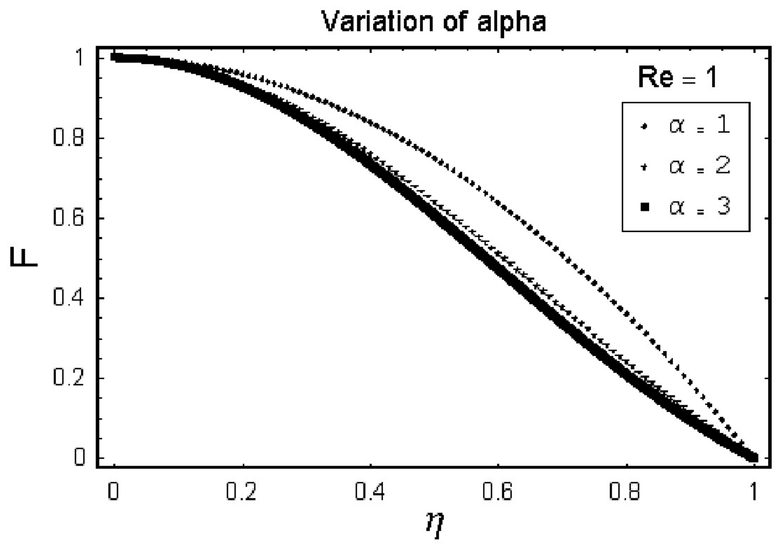

Fig. 2. Profiles of F for various values of α when R_e is fixed.

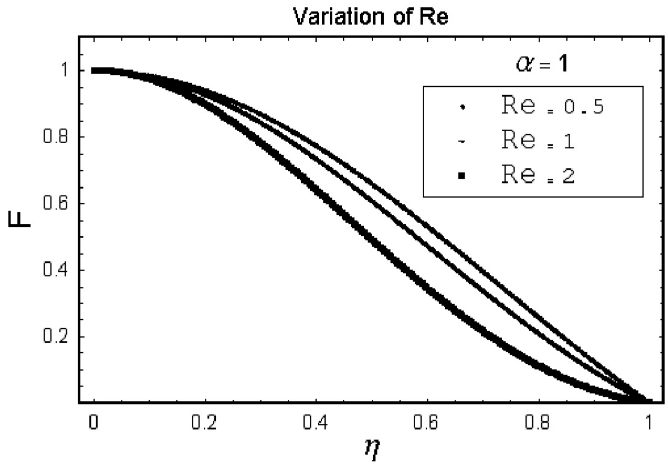

Fig. 3. Profiles of F for various values of R_e when α is fixed.

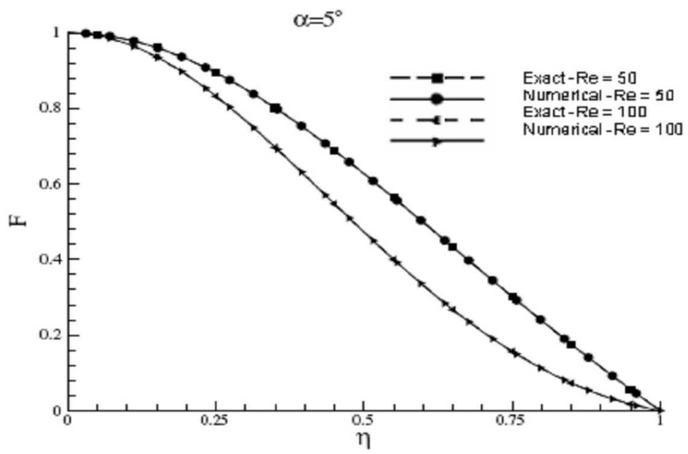

Fig. 4. Profiles of F for Jeffery Hamel flow in divergent channel.

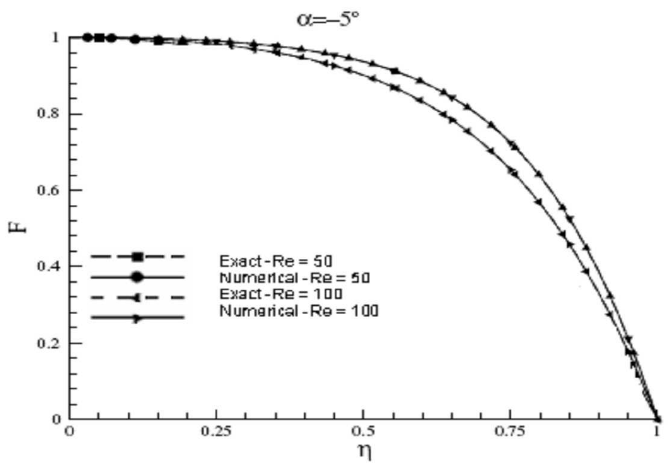

Fig. 5. Profiles of F for Jeffery Hamel flow in convergent channel.

6 Concluding remarks

The present study contributes exact solution for the Jeffrey-Hamel flow. In addition, the numerical analysis is also presented for the Jeffrey-Hamel flow. As a result, the following observations are made.

- Increase in R_e results in the increase of boundary layer.
- An increase in α leads to an decrease in the boundary layer thickness.
- The constant $\alpha > 0$ and $\alpha < 0$ give a divergent and convergent channel respectively.
- It is also worth mentioning that our exact solutions are more general and such solutions have been presented first time in the literature.

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References

[1] K. Batchelor, An introduction to fluid dynamics. Cambridge University Press, (1967).

[2] M. Dehghan, A. Saadatmandi,The numerical solution of a nonlinear system of second-order boundary value problems using the sinc-collocation method, Mathematical and Computer Modelling, 46 (2007) 1434-1441.

[3] C. Fetecau, A. Mahmood, Corina Fetecau and D. Vieru, Some exact solutions for the helical flow of a generalized Oldroyd-B fluid in a circular cylinder, Computers & Mathematics with Applications, 56 (2008) 3096-3108.

[4] L. E. Fraenkel, Laminar flow in symmetrical channels with slightly curved walls. I: On the Jeffery–Hamel solutions for flow between plane walls. Proc R Soc Lond A 267 (1962) 119–38.

[5] M. Hamadiche, J. Scott and D. Jeandel, Temporal stability of Jeffery–Hamel flow. J Fluid Mech, 268 (1994) 71–88.

[6] G. Hamel, Spiralfo¨rmige Bewgungen Za¨her Flu¨ ssigkeiten. Jahresbericht der Deutschen Math Vereinigung, 25 (1916) 34–60.

[7] G. B. Jeffery, The two-dimensional steady motion of a viscous fluid. Philos Mag, 6 (1915) 455–465.

[8] C.M. Khalique, F.M. Mahomed, B.P. Ntsime, Group classification of the generalized Emden–Fowler-type equation, Nonlinear Analysis: Real World Applications, 10 (2009) 3387-3395.

[9] F. M. Mahomed and P. G. L. Leach, Symmetry Lie algebras of nth order ordinary differential equations, Journal of Mathematical Analysis and Applications, 151 (1990) 80-107.

[10] O.D. Makinde, P.Y. Mhone, Hermite–Pade' approximation approach to MHD Jeffery–Hamel flows. Appl Math Comput, 181 (2006) 966–72.

[11] A. McAlpine, P.G. Drazin, On the spatio-temporal development of small perturbations of Jeffery–Hamel flows. Fluid Dyn Res, 22 (1998) 123–38.

[12] R.K. Rathy, An introduction to fluid dynamics. New Delhi: Oxford and IBH Pl, (1976).

[13] L. Rosenhead, The steady two-dimensional radial flow of viscous fluid between two inclined plane walls. Proc Roy Soc A 175 (1940) 436–67.

[14] J. SobeyI, P.G. Drazin, Bifurcations of two-dimensional channel flows. J Fluid Mech, 171 (1986) 263–87.

[15] M.R. Sadri, Channel entrance flow, PhD thesis, Department of Mechanical Engineering, the University of Western Ontario, (1997).