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On The Symmetric Crossed Polymodule on A Category of Polymodules

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Abstract
The polygroup theory is a natural generalization of the group theory. In a group the composition of two elements is an element, while in a polygroup the composition of two elements is a set. Polygroups have been applied in many area, such as geometry, lattices, combinatorics, and color scheme. Also, Crossed modules and its applications play very important roles in category theory, homology and cohomology of groups, homotopy theory, algebra, k-theory, etc. In this paper, we have definition of a polyfunctor and transformation for polygroups. Also, we introduce the concept of the symmetric crossed module to the symmetric crossed polymodules. Our results extend the classical results of crossed modules to crossed polymodules of polygroups.

Keywords: Group; Polygroup; Crossed Module; Crossed Polymodule; Symmetric Crossed Polymodule.

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1. Introduction

The Yang-Baxter equation plays a fundamental role in various areas of mathematics. In fact, this equation plays a fundamental role in such apparently distant fields as statistical mechanics, particle physics, quantum field theory and quantum groups. Its solutions, called braidings, are built, among others,

1. from Yetter-Drinfel'd modules over a Hopf algebra,
2. from self-distributive structures,
3. from crossed modules of groups.

Also, Crossed modules and its applications play very important roles in category theory, homology and cohomology of groups, homotopy theory, algebra, k-theory etc. Therefore, study crossed modules and it's all kinds automorphisms at least through this is very important. This is in fact one of the motivations of recent half-century studies in this field. Crossed modules was defined by Whitehead in [1]. So many mathematicians work on this subject. There are many application of crossed module such as Actor crossed module, Pullback crossed module, Pushout crossed module and Induced crossed module, etc [2,3]. Nilpotent, Solvable, n-Complete and Representations of crossed modules were studied by Dehghanizadeh and Davvaz [4,5,6,7,8]. Polygroups were studied by Comer [9], also see in [10]. Specially, Comer and Davvaz developed the algebraic theory for polygroups. Alp and Davvaz in [11] introduced the notion of crossed polymodule of polygroups and they give some of its properties. Also, they introduced new important classes by the fundamental relations. Alp and Davvaz, introduce the concept of pullback and pushout crossed polymodules and describe the construction of pullback and pushout crossed polymodules. Arvasi, Porter and Onarh in [12,13] introduce the notion of an (co)-induced 2-crossed module, which extends the notion of an (co)-induced crossed module which were defined by Brown, Gilbert, Loday and Mosa [14,15,16]. The notion of crossed polysquares was introduced by Dehghanizadeh, Davvaz and Alp in [17]. They introduce the notion of crossed polysquare of polygroups and gave of its propertiesa Also, we extend the classical results of crossed squares to crossed polysquares [18].

2. Polygroups and crossed polymodules

The polygroup theory is a natural generalization of the group theory. In a group the composition of two elements is an element, while in a polygroup the composition of two elements is a set. Polygroups have been applied in many areas, such as geometry, lattices, combinatorics, and color scheme. There exists a rich bibliography: publications appeared with in 2012 can be found in "Polygroup Theory and Related Systems" by Davvaz [10]. This book containsthe principle definitions endowed with examples and the basic results of the theory. Applications of hypergroups appear in special subclasses like polygroups that they were studied by Comer [9], also see in [10,19]. Specially, Comer and Davvaz developed the algebraic theory for polygroups. A polygroup is a completely regular, reversible in itself multigroup. According [9], a polygroup is a multi-valued system $\mathcal{M} = < P, \circ, ^{-1} >$, with $e \in P$, $^{-1}: P \to P, \circ : P \times P \to \mathcal{P}^*(P)$, where the following axioms hold for all x, y, z in P:

1. $(x \circ y) \circ z = x \circ (y \circ z),$
2. $e \circ x = x \circ e = x,$
3. $x \in y \circ z$ implies $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x.$

In this definition, $\mathcal{P}^*(P)$ is the set of all the non-empty subsets of P, and if $x \in P$ and A, B are two non-empty subsets of P, then $A \circ x = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b$, and $A \circ x = A \circ \{x\}$, $x \circ A = \{x\} \circ A$.

The following elementary facts about polygroups follow easily from the axioms: $e \in x \circ x^{-1} \bigcap x^{-1} \circ x$, $e^{-1} = e$ and $\left(x^{-1}\right)^{-1} = x$.

If K is a non-empty subset of P, then K is called a subpolygroup of P if $e \in K$ and $< K, \circ, e, ^{-1} >$ is a polygroup. The subpolygroup N of P is said to be normal in P if $a^{-1} \circ N \circ a \subseteq N$, for every $a \in P$. There are several kinds of homomorphisms between polygroups [10]. In this paper, we apply only the notion of strong homomorphisms. Let $< P, \circ, e, ^{-1} >$ and $< P', \circ, e, ^{-1} >$ be two polygroups. A mapping φ from P into P' is said to be a strong homomorphism if $\varphi(e) = e$ and $\varphi(a \circ b) = \varphi(a) * \varphi(b)$ for all $a, b \in P$. A strong homomorphism φ is said to be an isomorphism if φ is one to one and onto.

Definition 2.1. Let $\mathcal{P} =< P, \circ, e, ^{-1} >$ be a polygroup and Ω be a non-empty set. A map $\alpha : P \times \Omega \to \mathcal{P}^*(\Omega)$, where $\alpha(p, \omega) :=^p \omega$ is called a (left) polygroupaction on Ω if the following axioms hold:

1. ${}^{e}\omega = \omega,$
2. ${}^{h}\left({}^{p}\omega\right) = {}^{h \circ p}\omega$, where ${}^{p}A = \bigcup_{a \in A} {}^{p}a$ and ${}^{B}\omega = \bigcup_{b \in B} {}^{b}\omega$ for all $A \subseteq \Omega$ and $B \subseteq P$.
3. $\bigcup_{\omega \in \Omega} {}^{p}\omega = \Omega,$
4. for all $p \in P,\ a \in {}^{p}b \Rightarrow b \in {}^{p^{-1}}a$.

Definition 2.2. [11] A crossed polymodule $\chi = (C, P, \partial, \alpha)$ consists of polygroups $< C, *, e, ^{-1} >$ and $< P, \circ, e, ^{-1} >$ together with a strong homomorphism $\partial : C \to P$ and a (left) action $\alpha : P \times C \to \mathcal{P}^*(C)$ on C, such that the following conditions hold:

1. $\partial\left({}^{p}c\right) = p \circ \partial(c) \circ p^{-1}$ for all $c \in C$ and $p \in P$,

2. ${}^{\partial(c)}c = c * c' * c^{-1}$ for all $c, c' \in C$.

Example 2.1. Every polygroup P has its trivial subpolygroup 1 consisting of just the identity element of P.

This subpolygroup is always a normal subpolygroups. Therefore we have crossed polymodule $\left(1, P\right) = \left(1, P, c_1, \mathrm{id}_c |_1\right)$.

Example 2.2. Every polygroup P contains the whole polygroup P as a normal subpolygroup. Therefore, we have crossed polymodule $(P,P)=(P,P,c,\mathrm{id}_P)$.

Example 2.3. [11] A conjugation crossed polymodule is an inclusion of a normal subpolygroup N of P, with action given by conjugation. In particular, for any polygroup P the identity map $\mathrm{id}_P : P \to P$ is a crossed polymodule with the action of P on itself by conjugation. Indeed, there are two canonical ways in which a polygroup P may be regarded as a crossed polymodule: via the identity map or the inclusion of the trivial subpolygroup.

Example 2.4. [11] If C is a P-polymodule, then there is a well-defined action α of P on C. This together with the zero homomorphism yields a crossed polymodule (C,P,∂,α).

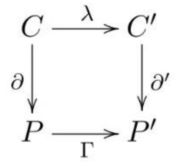
Example 2.5. Let P be a polygroup and $N \trianglelefteq P$ be a normal subpolygroup. Consider the polygroup morphism

$$C_N : P \to \mathrm{Aut}(N)$$
$$p \to (c_p |_N^N \; n \to n^p)$$

Then we have a crossed polymodule

$$(N,P)=(N,P,C_N,\mathrm{id}_P |_N).$$

Definition 2.3. Let $\chi=(C,P,\partial,\alpha)$ and $\chi'=(C',P',\partial',\alpha')$ be two crossed polymodules. A crossed polymodule morphism $f=(\lambda,\Gamma):\chi\to\chi'$ is a tuple of strong homomorphism, such that the diagram



commutes, and $\lambda(p\alpha c)=\Gamma(p)\alpha'\lambda(c)$ for all $p\in P,\;\; c\in C$.

3. Categories and polyfunctors

In this section we introduce the concept of categories and polyfunctors in the polygroups. Let $\mathcal{P},\mathcal{Q},\mathcal{R}$ and $\mathcal{K}$ be categories of polygroups, so a category is given by

$$\mathfrak{P}=(\mathrm{Mor}(\mathcal{P}),\mathrm{Ob}(\mathcal{P}),(s,i,t),\bullet)$$

where $\mathrm{Mor}(\mathcal{P})$ is the set of morphisms, $\mathrm{Ob}(\mathcal{P})$ is the set of objects, (polygroups). $s:\mathrm{Mor}(\mathcal{P})\to\mathrm{Ob}(\mathcal{P})$ is the source map, $i:\mathrm{Ob}(\mathcal{P})\to\mathrm{Mor}(\mathcal{P})$ is the map sending an object to its identity morphism, $t:\mathrm{Mor}(\mathcal{P})\to\mathrm{Ob}(\mathcal{P})$ is the target map, and $(\bullet)$ is the composition of morphisms.

If $P_1 \xrightarrow{u} P_2 \xrightarrow{v} P_3$ is in $\mathcal{P}$, then we implicitly suppose given objects P_1, P_2 and $P_3 \in \mathrm{Ob}(\mathcal{P})$ and morphisms $u, v \in \mathrm{Mor}(\mathcal{P})$ with $us = P_1, ut = P_2$ and $vs = P_2, vf = P_3$.

Definition 3.1. A morphism $P_1 \xrightarrow{u} P_2$ from $\mathrm{Mor}(\mathcal{P})$ is called a strong isomorphism, If there exists a morphism $v \in \mathrm{Mor}(\mathcal{P})$ such that $u \bullet v = \mathrm{id}_{P_1}$ and $v \bullet u = \mathrm{id}_{P_2}$ hold, we write $v = u^{-1}$, and we call u^{-1} the inverse of u.

If $P_1, P_2 \in \mathrm{Ob}(\mathcal{P})$, then we write $(P_1, P_2)_{\mathcal{P}} = \{a \in \mathrm{Mor}(\mathcal{P}) \mid \ as = P_1, at = P_2\}$ for the set morphisms from P_1 to P_2.

Definition 3.2. A polyfunctor from $\mathcal{P}$ to $\mathcal{Q}$ is given by $\mathcal{F}\colon(\mathrm{Mor}(\mathcal{F}), \mathrm{Ob}(\mathcal{F}))$, where $\mathrm{Ob}(\mathcal{F}): \mathrm{Ob}(\mathcal{P}) \to \mathrm{Ob}(\mathcal{Q})$ and $\mathrm{Mor}(\mathcal{F}): \mathrm{Mor}(\mathcal{P}) \to \mathrm{Mor}(\mathcal{Q})$. A polyfunctor must satisfy these conditions:

$$us\mathrm{Ob}(\mathcal{F}) = u\mathrm{Mor}(\mathcal{F})s,$$
$$ut\mathrm{Ob}(\mathcal{F}) = u\mathrm{Mor}(\mathcal{F})t,$$
$$P_1 i\mathrm{Mor}(\mathcal{F}) = P_1\mathrm{Ob}(\mathcal{F})i,$$

for $u \in \mathrm{Mor}(\mathcal{P})$, $P_1 \in \mathrm{Ob}(\mathcal{P})$,

$$(u \bullet v)\mathrm{Mor}(\mathcal{F}) = u\mathrm{Mor}(\mathcal{F}) \bullet v\mathrm{Mor}(\mathcal{F}),$$

and for $P_1 \xrightarrow{u} P_2 \xrightarrow{v} P_3$ in $\mathcal{P}$.

Also, for $P_1 \in \mathrm{Ob}(\mathcal{P})$, we write $P_1\mathcal{F} = P_1\mathrm{Ob}(\mathcal{F}) \in \mathrm{Ob}(\mathcal{Q})$, and for $u \in \mathrm{Mor}(\mathcal{P})$, $u\mathcal{F} = u\mathrm{Mor}(\mathcal{Q}) \in \mathrm{Mor}(\mathcal{Q})$.

If $\mathcal{F}: \mathcal{P} \to \mathcal{Q}$ and $\mathcal{G}\colon \mathcal{Q} \to \mathcal{R}$ are polyfunctors, then we write $(\mathcal{F} * \mathcal{G}): \mathcal{P} \to \mathcal{R}$ for the composite of $\mathcal{F}$ and $\mathcal{G}$, also if unambiguous, we write $\mathcal{F}\mathcal{G} = \mathcal{F} * \mathcal{G}$ for brevity.

Definition 3.3. A polyfunctor $\mathcal{F}: \mathcal{P} \to \mathcal{Q}$ is called an isopolyfunctor from $\mathcal{P}$ to $\mathcal{Q}$, if there exists a polyfunctor $\mathcal{G}: \mathcal{Q} \to \mathcal{P}$ such that $\mathcal{F}\mathcal{G} = \mathrm{id}_{\mathcal{P}}$ and $\mathcal{G}\mathcal{F} = \mathrm{id}_{\mathcal{Q}}$, and we write $\mathcal{F}^{-1} = \mathcal{G}$. Also, if $\mathcal{P} = \mathcal{Q}$ then an isopolyfunctor $\mathcal{F}: \mathcal{P} \to \mathcal{Q}$ is called an autopolyfunctor.

We denote the set of autopolyfunctors from $\mathcal{P}$ to $\mathcal{P}$ by $\mathrm{Aut}(\mathcal{P})$, that is

$$\{\mathcal{F} \mid \mathcal{F}\colon \mathcal{P} \to \mathcal{P}, \mathcal{F} \text{ is an autopolyfunctor}\}.$$

Also, if $\mathcal{F} \in \mathrm{Aut}(\mathcal{P})$, we write $\left(\mathcal{P} \xrightarrow[\sim]{\mathcal{F}} \mathcal{P}\right) = \left(\mathcal{P} \xrightarrow{\mathcal{F}} \mathcal{P}\right)$ for $\mathcal{F}, \mathcal{G} \in \mathrm{Aut}(\mathcal{P})$, and $\mathcal{F}^{\mathcal{G}} = \mathcal{G}^{-1}\mathcal{F}\mathcal{G}$.

Let $\mathcal{F}: \mathcal{P} \to \mathcal{Q}$ is a polyfunctor, $\mathcal{P}' \subseteq \mathcal{P}$ and $\mathcal{Q}' \subseteq \mathcal{Q}$, also, suppose given a polyfunctor $\mathcal{F}': \mathcal{P}' \to \mathcal{Q}'$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P}' & \xrightarrow{\mathcal{F}'} & \mathcal{Q}' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ \mathcal{P} & \xrightarrow[\mathcal{F}]{} & \mathcal{Q} \end{array}$$

In other words, for $P' \in \mathrm{Ob}(\mathcal{P}')$ and $u \in \mathrm{Mor}(\mathcal{P}')$, we have $P'\mathcal{F}' = P'\mathcal{F}$ and $u\mathcal{F}' = u\mathcal{F}$, then we write $\mathcal{F}|_{\mathcal{P}'}^{\mathcal{Q}'} = \mathcal{F}' : \mathcal{P}' \to \mathcal{Q}'$.

Also, if $\mathcal{P}'$ is a subcategory of $\mathcal{P}$, then we write $J_{\mathcal{P}',\mathcal{P}} : \mathcal{P}' \to \mathcal{P}$, for embedding polyfunctor from $\mathcal{P}'$ to $\mathcal{P}$. We often abbreviate $J = J_{\mathcal{P}',\mathcal{P}} : \mathcal{P}' \to \mathcal{P}$. In the following, we write $[\mathcal{P},\mathcal{Q}]$ for the category of polyfunctors from $\mathcal{P}$ to $\mathcal{Q}$. The set of objects $\mathrm{Ob}([\mathcal{P},\mathcal{Q}])$ of this category consists of the polyfuncrors from $\mathcal{P}$ to $\mathcal{Q}$. The set of morphisms $\mathrm{Mor}([\mathcal{P},\mathcal{Q}])$ consists of the transformations between such polyfunctors.

Definition 3.4. Let $F, G \in \mathrm{Ob}([\mathcal{P},\mathcal{Q}])$ be polyfunctors from $\mathcal{P}$ to $\mathcal{Q}$. A transformation $(F \xrightarrow{a} G) \in \mathrm{Mor}([\mathcal{P},\mathcal{Q}])$ from F to G is a tuple of morphisms $\left(P_1F \xrightarrow{P_1a} P_1G\right)_{P_1 \in \mathrm{Ob}(\mathcal{P})}$

with the property that the following diagram is commutative for $P_1 \xrightarrow{u} P_2 \in \mathrm{Mor}(\mathcal{P})$

$$\begin{array}{ccc} P_1F & \xrightarrow{P_1a} & P_1G \\ {\scriptstyle uF}\downarrow & & \downarrow{\scriptstyle uG} \\ P_2F & \xrightarrow[P_2a]{} & P_2G \end{array}$$

Certainly, for more explanation, sometimes we write

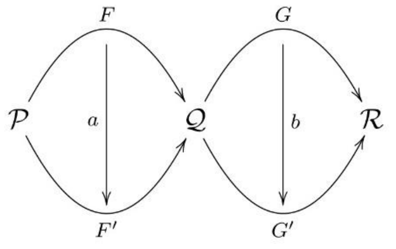
$$a = (P_1F \xrightarrow{P_1a} P_1G)_{P_1 \in Ob(\mathcal{P})} = \begin{array}{c} P_1 \\ \downarrow{\scriptstyle u} \\ P_2 \end{array} \longrightarrow \begin{array}{ccc} P_1F & \xrightarrow{P_1a} & P_1G \\ {\scriptstyle uF}\downarrow & & \downarrow{\scriptstyle uG} \\ P_2F & \xrightarrow{P_2a} & P_2G \end{array}$$

for the transformation from F to G.

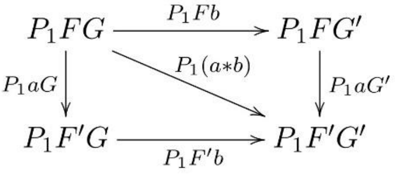
Definition 3.5. If $(F \xrightarrow{a} F') \in \mathrm{Mor}([\mathcal{P},\mathcal{Q}])$ and $(G \xrightarrow{b} G') \in \mathrm{Mor}([\mathcal{P},\mathcal{R}])$ are transformations, then their horizontal composite, is defined by

$$a * b = \left(P_1FG \xrightarrow{P_1(a*b)} P_1F'G'\right)_{P_1 \in \mathrm{Ob}(\mathcal{P})}$$

$$(aG) \bullet (F'b) = (Fb) \bullet (aG')$$



Pay attention for $P_1 \in \text{Ob}(\mathcal{P})$, the following diagram is commutative:



Proposition 3.1. Horizontal composition $*$ is associative.

Proof. If

$$F \xrightarrow{a} F' \in \text{Mor}\left([\mathcal{P},\mathcal{Q}]\right),$$
$$G \xrightarrow{b} G' \in \text{Mor}\left([\mathcal{Q},\mathcal{R}]\right),$$

and $(H \xrightarrow{c} H') \in \text{Mor}\left([\mathcal{R},\mathcal{K}]\right)$, then

$$\begin{aligned}(a*b)*c &= (a*b)H \bullet (F'G')c \\ &= (aG \bullet F'b)H \bullet (F'G')c \\ &= (aGH) \bullet (F'bH \bullet F'G'c) \\ &= (aGH) \bullet F'(bH \bullet G'c) \\ &= a(GH) \bullet F'(c*b) \\ &= a*(b*c).\end{aligned}$$

Definition 3.6. $(F \xrightarrow{a} G) \in \text{Mor}\left([\mathcal{P},\mathcal{Q}]\right)$ is an isotransformation, if for $P_1a \in \text{Mor}(\mathcal{P})$ is an isomorphism, for $P_1 \in \text{Ob}(\mathcal{P})$.

Definition 3.7. If $(F \xrightarrow{a} F'), (F' \xrightarrow{b} F'') \in \text{Mor}\left([\mathcal{P},\mathcal{Q}]\right)$ are transformations, then thier vertical composite is

$$a*b = \left(P_1F \xrightarrow{(P_1a)(P_1b)} P_1F''\right)_{P_1\in\text{Ob}(\mathcal{P})}$$

Proposition 3.2. Vertical composition $\bullet$ is associative**.**

Proof. If $(E \xrightarrow{a} F)$, $(F \xrightarrow{b} G)$, and $(G \xrightarrow{c} H)$ are in $\text{Mor}\left([\mathcal{P},\mathcal{Q}]\right)$, then for $P_1 \in \text{Ob}(\mathcal{P})$, we have

$$\begin{aligned}P_1\big((a\bullet b)\bullet c\big)&=\big(P_1(a\bullet b)\big)\bullet(P_1c)\\&=\big((P_1a)\bullet(P_1b)\big)\bullet(P_1c)\\&=(P_1a)\bullet\big((P_1b)\bullet(P_1c)\big)\bullet(P_1c)\\&=(P_1a)\bullet\big(P_1(b\bullet c)\big)\\&=P_1\big(a\bullet(b\bullet c)\big).\end{aligned}$$

4. Symmetric crossed polymodules

In group theory, we have for each category χ, a symmetric crossed module $S_\chi=\left(M_\chi,G_\chi,\gamma_\chi,f_\chi\right)$, where (M,G,γ,f) is a crossed module, and G_χ consists of the autofunctors of χ and M_χ consists of the isotransformations from the identity id_χ to some autofunctor of χ. In this section, we introduce the concept of the symmetric crossed module to crossed polymodules.

Theorem 4.1. Suppose $\chi=\left(\mathrm{Mor}(\chi),\mathrm{Ob}(\chi),(s,i,t),\bullet\right)$ be a category of polygroups. Also, consider the set $Q_\chi=\mathrm{Aut}(\chi)=\left\{F \mid \quad F:\chi\to\chi, F \text{ is an autopolyfunctor}\right\}$, together with the composition of functors $(*)$, and $P_{Q_\chi}=Q_\chi\bigcup\{H\}$ such that $H\notin Q_\chi$. Then P_{Q_χ} is polygroup by the appropriate hyperoperations.

Proof. The composition of functors is associative, and therefore, the multiplication in Q_χ is associative. If $F,G\in Q_\chi$, then $F*G\in Q_\chi$. Also $F*\mathrm{id}_\chi=F$ and, $\mathrm{id}_\chi*F=F$. Hence $1_{Q_\chi}=\mathrm{id}_\chi$. We have $F*F^{-1}=\mathrm{id}_\chi$ and $F^{-1}*F=\mathrm{id}_\chi$. So, the inverse for F is F^{-1}. Hence $\left(Q_\chi,*\right)$ is a group. Now, if $H\notin Q_\chi$, then we define on $P_{Q_\chi}=Q_\chi\bigcup\{H\}$, the hyperoperations as follows:

1. $H\otimes H=\mathrm{id}_\chi$,
2. $\mathrm{id}_\chi\otimes F=F\otimes\mathrm{id}_\chi=F$, for all $F\in P_{Q_\chi}$,
3. $H\otimes F=F\otimes H=F$, for all $F\in P_{Q_\chi}-\left\{\mathrm{id}_\chi,H\right\}$,
4. $F\otimes G=F*G$, for all $(F,G)\in Q_\chi^2$ such that $G\neq F^{-1}$,
5. $F\otimes F^{-1}=\left\{\mathrm{id}_\chi,H\right\}$, for all $F\in P_{Q_\chi}-\left\{\mathrm{id}_\chi,H\right\}$.

We show that $\left(P_{Q_\chi},\otimes,\mathrm{id}_\chi,{}^{-1}\right)$ is a polygroup. If $\{F,G,K\}\bigcup\left\{\mathrm{id}_\chi,H\right\}=\varnothing$, then we have two following cases:

(i) $F \neq G^{-1} \neq K$ and $F \neq K^{-1}$. In this case

$$\begin{aligned}(F \circ G) \circ K &= (F * G) * K \\ &= F * (G * K) \\ &= F \otimes (G \otimes K)\end{aligned}$$

(ii) There exists $\{u, v\} \subseteq \{F, G, K\}$ such that $u = v^{-1}$. Without losing generality, suppose that $F = u$ and $G = v$. So $(F \otimes G) \otimes K = \{\mathrm{id}_\chi, H\} \otimes K$. Hence $\{\mathrm{id}_\chi, H\} \otimes K = K$ .Therefore, if $G = K^{-1}$, then $F \otimes (G \otimes K) = F \otimes \{\mathrm{id}_\chi, H\} = F = G^{-1} = K$, and if $G \neq K^{-1}$, we have

$$\begin{aligned}F \otimes (G \otimes K) &= F \otimes (G * K) \\ &= F * (G * K) \\ &= (F * F^{-1}) * K \\ &= \mathrm{id}_\chi * K = K\end{aligned}$$

On the other hand $\otimes$ is the associative.

Now, if $\{F, G, K\} \cup \{\mathrm{id}_\chi, H\} \neq \varnothing$, and let $\mathrm{id}_\chi \in \{F, G, K\}$, then the associativity condition holds. Suppose that $\{F, G, K\} \cup \{\mathrm{id}_\chi, H\} = \{H\}$. Without losing generality, let $F = H$, in this case we have

$$(F \otimes G) \otimes K = F \otimes (G \otimes K) = \begin{cases} H & G = H, K = H \\ K & G = H, K \neq H \\ G & G \neq H, K = H \\ G * K & G \neq K^{-1}, G \neq H \neq K \\ \{\mathrm{id}_\chi, H\} & G = K^{-1}, G \neq H \neq K \end{cases}$$

Therefore, according to the structure of $\otimes$ we conclude that id_χ is the identity element of P_{Q_χ}, and the other conditions for being polygroup hold too.

Proposition 4.1. If Q_χ is a group, then $\dfrac{P_{Q_\chi}}{\beta_P^*} \cong Q_\chi$.

Proof. It is straightforward.

Theorem 4.2. Suppose $\chi = \left(\mathrm{Mor}(\chi), \mathrm{Ob}(\chi), (s,i,t), \bullet\right)$ be a category of polygroups. Consider the set $P_\chi = \left\{(\mathrm{id}_\chi \xrightarrow{a} F) : F \in \mathrm{Aut}(\chi)\right\}$, where a a is an isotransformation, and on P_χ, we define a multiplication by $\left(\mathrm{id}_\chi \xrightarrow[\sim]{a} F\right) * \left(\mathrm{id}_\chi \xrightarrow[\sim]{b} G\right) = \left(\mathrm{id}_\chi \xrightarrow[\sim]{a \neq b} FG\right) = a \bullet (Fb) = b \bullet (aG),$

then $P_{P_\chi} = P_\chi \bigcup \{H\}$ such that $H \notin P_\chi$, by the appropriate hyperoperation is polygroup.

Proof. We have that multiplication $(*)$ is the horizontal composition of transformations. Hence in particular, $(*)$ is associative.

If $\left(\mathrm{id}_\chi \xrightarrow[\sim]{a} F\right), \left(\mathrm{id}_\chi \xrightarrow[\sim]{b} G\right) \in P_\chi$, then

$$a * b = a \bullet Fb = b \circ aG \quad : \ \mathrm{id}_\chi \to FG.$$

Hence, $a * b$ is an isotransformation to an autofunctor FG. Therefore, $a * b \in P_\chi$. But, we have

$$a * \mathrm{id}_{\mathrm{id}_\chi} = \left(\mathrm{id}_\chi \xrightarrow[\sim]{a} F\right) * \left(\mathrm{id}_\chi \xrightarrow{\mathrm{id}_{\mathrm{id}_\chi}} \mathrm{id}_\chi\right)$$
$$= \mathrm{id}_{\mathrm{id}_\chi} \bullet \left(a\, \mathrm{id}_\chi\right) = a,$$
$$\mathrm{id}_{\mathrm{id}_\chi} * a = \left(\mathrm{id}_\chi \xrightarrow{\mathrm{id}_{\mathrm{id}_\chi}} \mathrm{id}_\chi\right) * \left(\mathrm{id}_\chi \xrightarrow[\sim]{a} F\right)$$
$$= a \bullet \left(\mathrm{id}_{\mathrm{id}_\chi} F\right) = a \bullet \mathrm{id}_F = a,$$

hence, $1_{P_\chi} = \mathrm{id}_{\chi \mathrm{id}_\chi}$. Now, we have

$$\left(a^*\right)^{-1} = a^{-1} F^{-1} : \mathrm{id}_\chi \to F^{-1}$$

so $\left(a^*\right)^{-1}$ is an isotransformation, where F^{-1} is an isofunctor. Hence, $\left(a^*\right)^{-1} \in P_\chi$.

But,

$$a * \left(a^*\right)^{-1} = \left(\mathrm{id}_\chi \xrightarrow[\sim]{a} F\right) * \left(\mathrm{id}_\chi \xrightarrow{a^{-1}F^{-1}} F^{-1}\right)$$
$$= \left(a^{-1}F^{-1}\right) \bullet \left(aF^{-1}\right) = \left(a^{-1} \bullet a\right) F^{-1} = \mathrm{id}_F F^{-1} = \mathrm{id}_{FF^{-1}} = \mathrm{id}_{\mathrm{id}_\chi} = 1_{P_\chi},$$
$$\left(a^*\right)^{-1} * a = \left(\mathrm{id}_\chi \xrightarrow[\sim]{a^{-1}F^{-1}} F^{-1}\right) * \left(\mathrm{id}_\chi \xrightarrow{a} F\right)$$
$$= a \bullet \left(a^{-1} F^{-1} F\right) = a \bullet a^{-1} = \mathrm{id}_{\mathrm{id}_\chi} = 1_{P_\chi},$$

hence, $\left(a^{*}\right)^{-1}$ is the inverse of a.

Now, if $K \notin P_{\chi}$, then we define on $P_{P_{\chi}} = P_{\chi} \bigcup \{K\}$ the hyperoprations as follows:

1. $K \otimes K = \mathrm{id}_{\chi}$,
2. $\mathrm{id}_{\chi} \otimes F = F \otimes \mathrm{id}_{\chi} = F$, for all $F \in P_{P_{\chi}}$,
3. $K \otimes F = F \otimes K = F$, for all $F \in P_{P_{\chi}} - \{\mathrm{id}_{\chi}, K\}$,
4. $F \otimes G = F * G$, for all $(F,G) \in P_{\chi}^{2}$ such that $G \neq F^{-1}$,
5. $F \otimes F^{-1} = \{\mathrm{id}_{\chi}, K\}$, for all $F \in P_{P_{\chi}} - \{\mathrm{id}_{\chi}, K\}$,

hence $\left(P_{P_{\chi}}, \otimes, \mathrm{id}_{\chi}, {}^{-1}\right)$ is a polygroup.

Theorem 4.3. Suppose $\chi = \left(\mathrm{Mor}(\chi), \mathrm{Ob}(\chi), (s,i,t), \bullet\right)$ be a category of polygroups, and suppose given functors $F, G : \chi \to \chi$. Let given transformations $\left(\mathrm{id}_{\chi} \xrightarrow{a} F\right)$ and $\left(\mathrm{id}_{\chi} \xrightarrow{b} F\right)$ such that $a * b = b * a = \mathrm{id}_{\mathrm{id}_{\chi}}$ holds, then

1. We have $F, G \in \mathrm{Aut}(\chi)$, i.e., the functors F and G are autofunctors, and we have $G = F^{-1}$.
2. The transformations a and b are isotransformations.

Proof. It is straightforward.

Theorem 4.4. Let $\chi = \left(\mathrm{Mor}(\chi), \mathrm{Ob}(\chi), (s,i,t), \bullet\right)$ be a category of polygroups, and $V = (P, Q, \alpha, f)$ be a crossed polymodule. Also $Q_{\chi} = \mathrm{Aut}(\chi)$,

$$P_{\chi} = \left\{\left(\mathrm{id}_{\chi} \xrightarrow{a} F\right) \mid \quad F \in \mathrm{Aut}(\chi)\right\},$$

where, a is an isotransformation. Then we have a polyaction of Q_{χ} on P_{χ}, given by the polygroup morphism

$$\alpha_{\chi} : Q_{\chi} \to \mathrm{Aut}(P_{\chi})$$
$$Q \to \left(\mathrm{id}_{\chi} \xrightarrow[\sim]{a} F\right) \to \left(\mathrm{id}_{\chi} \xrightarrow[\sim]{Q^{-1} a Q} Q^{-1} F Q\right),$$

and a polygroup morphism

$$f_{\chi} : P_{\chi} \to Q_{\chi}$$
$$\left(\mathrm{id}_{\chi} \xrightarrow[\sim]{a} F\right) \to F.$$

Then $\left(P_{\chi}, Q_{\chi}, \alpha_{\chi}, f_{\chi}\right)$ is a crossed polymodule, (Symmetric Crossed Polymodule on χ), and we write

$$S_{\chi} = \left(P_{\chi}, Q_{\chi}, \alpha_{\chi}, f_{\chi}\right).$$

Also, we write, for $\left(\mathrm{id}_{\chi} \xrightarrow[\sim]{a} F\right) \in P_{\chi}$, and $Q \in Q_{\chi}$,

$$\begin{aligned} a^{Q} &= (a)\left(Q\alpha_{\chi}\right) \\ &= Q^{-1}aQ : \mathrm{id}_{\chi} \to F^{Q} \\ &= Q^{-1}FQ \end{aligned}$$

for the polyaction of Q on a.

Proof. α_{χ} is well-defined, because according to the assumption $Q \in Q_{\chi}$ and $\mathrm{id}_{\chi} \xrightarrow[\sim]{a} F$, $\mathrm{id}_{\chi} \xrightarrow[\sim]{b} H \in P_{\chi}$, we have

$$Q^{-1}aQ : \mathrm{id}_{\chi} \to F^{Q} = Q^{-1}FQ$$

where $Q^{-1}FQ$ is an aut of unctor of χ, and $Q^{-1}aQ$ is an isotransformation. Hence $Q^{-1}aQ \in P_{\chi}$. But,

$$\begin{aligned} Q^{-1}(a * b)Q &= Q^{-1}(a \bullet Fb)Q \\ &= Q^{-1}aQ \bullet Q^{-1}FbQ \\ &= Q^{-1}aQ \bullet (Q^{-1}FQ)(Q^{-1}bQ) \\ &= (Q^{-1}aQ) * (Q^{-1}bQ). \end{aligned}$$

Also,

$$\left(Q^{-1}\right)^{-1}\left(Q^{-1}aQ\right)Q^{-1} = QQ^{-1}aQQ^{-1} = a,$$

$$Q^{-1}\left(\left(Q^{-1}\right)^{-1} aQ^{-1}\right) = Q^{-1}QaQ^{-1}Q = a.$$

Now, we show that α_{χ} is a polygroup morphism. If $Q, H \in Q_{\chi}$ and $\left(\mathrm{id}_{\chi} \xrightarrow[\sim]{a} F\right) \in P_{\chi}$, then

$$
\begin{aligned}
(a)\left((QH)_{\alpha_\chi}\right) &= \left(\mathrm{id}_\chi \xrightarrow[\sim]{(QH)^{-1}a(QH)} (QH)^{-1}F(QH)\right) \\
&= \left(\mathrm{id}_\chi \xrightarrow[\sim]{H^{-1}Q^{-1}aQH} H^{-1}Q^{-1}FQH\right) \\
&= \left(\mathrm{id}_\chi \xrightarrow[\sim]{Q^{-1}aQ} Q^{-1}FQ\right)\left(H\alpha_\chi\right) \\
&= \left(\mathrm{id}_\chi \xrightarrow[\sim]{a} F\right)\left(Q\alpha_\chi\right)\left(H\alpha_\chi\right) \\
&= (a)\left(Q\alpha_\chi\right)\left(H\alpha_\chi\right).
\end{aligned}
$$

But f_χ is a polygroup morphism, since if $\left(\mathrm{id}_\chi \xrightarrow[\sim]{a} F\right),\left(\mathrm{id}_\chi \xrightarrow[\sim]{b} Q\right) \in P_\chi$, then

$$
\begin{aligned}
(a*b)\, f_\chi &= \left(\mathrm{id}_\chi \xrightarrow[\sim]{a*b} FQ\right) f_\chi = FQ \\
&= \left(\mathrm{id}_\chi \xrightarrow[\sim]{a} F\right) f_\chi * \left(\mathrm{id}_\chi \xrightarrow[\sim]{b} Q\right) f_\chi \\
&= \left(af_\chi\right) * \left(bf_\chi\right).
\end{aligned}
$$

For two conditions of crossed polymodule, if $\left(\mathrm{id}_\chi \xrightarrow[\sim]{a} F\right) \in P_\chi$ and $Q \in Q_\chi$, then

$$
(a^Q)\, f_\chi = \left(\mathrm{id}_\chi \xrightarrow[\sim]{Q^{-1}aQ} Q^{-1}FQ\right) f_\chi = Q^{-1}FQ = F^Q = \left(af_\chi\right)^Q .
$$

Also suppose given $\left(\mathrm{id}_\chi \xrightarrow[\sim]{a} F\right),\left(\mathrm{id}_\chi \xrightarrow[\sim]{b} Q\right) \in P_\chi$, then

$$
\begin{aligned}
a^b &= b^{-1} * a * b \\
&= \left(\mathrm{id}_\chi \xrightarrow[\sim]{b^{-1}Q^{-1}} Q^{-1}\right) * \left(\mathrm{id}_\chi \xrightarrow[\sim]{a} F\right) * \left(\mathrm{id}_\chi \xrightarrow[\sim]{b} Q\right) \\
&= \left(\left(b^{-1}Q^{-1}\right) * \left(Q^{-1}a\right)\right) * \left(\mathrm{id}_\chi \xrightarrow[\sim]{b} Q\right) \\
&= b \bullet \left(\left(b^{-1}Q^{-1}\right) \bullet \left(Q^{-1}a\right)\right) Q \\
&= b \bullet \left(b^{-1}Q^{-1}Q\right) \bullet \left(Q^{-1}aQ\right) \\
&= b \bullet b^{-1} \bullet \left(Q^{-1}aQ\right) = Q^{-1}aQ = a^Q = a^{bf_\chi}
\end{aligned}
$$

Definition 4.1. $\left(P_\chi, Q_\chi, \alpha_\chi, f_\chi\right)$ is a crossed polymodule, and called the Symmetric Crossed Polymodule on χ and we write

$$
S_\chi := \left(P_\chi, Q_\chi, \alpha_\chi, f_\chi\right).
$$

5. Conclusion

The paper presented a thorough definition of polyfunctors and transformations for polygroups.We also, introduced the notion of symmetric crossed modules to symmetric crossed polymodules. The study's findings expanded the classical results of crossed modules to encompass crossed polymodules of polygroups. In further studies, it will be interesting and useful to investigate the functors for crossed squares and also obtain the properties of symmetric crossed squares and study them. Then findings extended the classical results of crossed squares to crossed polysquares of polygroups.

References

[1] Whitehead, J. H. C. (1949), “Combinatorial homotopy II”, Bull. Amer. Math. Soc., Vol. 55, pp. 453-496.

[2] Alp, M. (2005), “Actor of crossed modules of algebroids”, proc.16th Int. Conf. Jangjeon Math. soc., Vol. 16, pp. 6-15.

[3] Alp, M. (2008), “Pullback crossed modules of algebroids”, Iranian J. Sci. Tech.,Transaction A, Vol. 32 No. A3, pp. 145-181.

[4] Dehghanizadeh, M. A. and Davvaz, B. (2018), “On central automorphisms of crossed modules”, Carpathian Math. Publ., Vol. 10 No. 2, pp. 288-295

[5] Dehghanizadeh, M. A. and Davvaz, B. (2019), “On the nilpotent crossed modules”, in: International Conference on Recent Achievements in Mathematical Science. Yazd, Iran, pp. 51-52.

[6] Dehghanizadeh, M. A. and Davvaz, B. (2019), “On the Representation and Characters of Cat^1-Groups and Crossed modules”, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., Vol. 68 No. 1, pp. 70-86.

[7] Dehghanizadeh, M. A. and Davvaz, B. (2021), “n-Complete Crossed Modules and Wreath Products of Groups”, Journal of New Results in Science, Vol. 10 No. 1, pp. 38-45.

[8] Dehghanizadeh, M. A. and Davvaz, B. (2019), “On the Solvable and Nilpotent Crossed Modules”, in: 11th Iranian Group Theory Conference, Yazd, Iran, pp. 28-29.

[9] Comer, S. D. (1984), “Polygroups derived from cogroups”, J. Algebra, Vol. 89, pp. 397-405.

[10] Davvaz, B. (2013), “Polygroup theory and related systems”, World Sci. Publ.

[11] Alp, M. and Davvaz, B. (2015), “On Crossed Polymodules and Fundamental Relations”, U.P.B. Sci., Bull., Series A, Vol. 77 No. 2, pp. 129-140.

[12] Arvasi, Z. (1997), “Crossed squares and 2-crossed modules of commutative algebras”, Theory and Applications of Categories, Vol. 3 No 7, pp. 160-181.

[13] Arvasi, Z. and Porter, T. (1998), “Freeness conditions for 2-crossed modules of commutative algebras”, Applied Categorical Structures, Vol. 6, pp. 455-471.

[14] Brown, R. and Gilbert, N. D. (1989), “Algebraic models of 3-types and autophism structures for crossed modules”, Proc. London Math. Soc., Vol. 59 No. 3, pp.51-73.

[15] Brown, R. and Loday, J. -L. (1987), “Van Kampen theorems for diagrams of spaces”, Topology, Vol. 26, pp. 311-334.

[16] Brown, R. and Mosa, G. H. (1988), "Double categories, R-categories and crossed modules", U. C. N. W maths preprint, Vol. 88 No. 11, pp.1-18.

[17] Dehghanizadeh, M. A., Davvaz, B. and Alp, M. (2018), "On crossed polysquares and fundamental relations", Sigma J. Eng. Nat., Vol. 9 N0. 1, pp. 1-16.

[18] Dehghanizadeh, M. A., Davvaz, B. and Alp, M. (2022), "On crossed polysquare version of homotopy kernels", Journal of Mathematical Extension, Vol. 16 No. 3(7), pp. 1-37.

[19] Davvaz, B. (2000), "On Polygroups and Permutation Polygroups", Math., Balkanica (N. S.), Vol. 14 No. 1-2, pp. 41-58.