Available online at http://sanad.iau.ir/journal/ijim/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol.17, No.1, 2025 Article ID IJIM-1629, 13 pages
Research Article

Evaluating the efficiency score of two-stage network DEA in the presence of shared resources using a multiplier model

Sh.Soofizadeh [1], R.Fallahnejad [2 *], E.Abdali [2]
[1] Department of Basic Sciences, Technical and Vocational University (TVU). Tehran, Iran.
[2] Khorramabad Branch, Islamic Azad University &Department of Mathematics, Khorramabad, Iran.

Submission Date:2021/08/25, Revised Date:2023/09/07, Date of Acceptance:2024/08/05

Abstract
In recent years, several network DEA models have been developed. Of these, Amirteimoori et al. (2016) proposed an additive model to measure the efficiency of the two-stage network DEA model when stages (sub-DMUs) consume shared input sources in both operation stages. However, the method proposed by Amirteimoori et al. (2016) does not obtain efficiency scores for each stage in the network system, as well as, for the whole operation of the system, and just by applying an additive model the efficiency and inefficiency of the DMUs is determined. However, it can be argued that this is a weakness of this method. To overcome this deficiency, we propose an approach for estimating the efficiency score of the two-stage network DEA model in the presence of shared input sources. Numerical examples show the applicability of the approach.

Keywords: Data envelopment analysis, Two-stage network, Shared input sources, Efficiency, Decision-making subunits, Additive model.

* Corresponding author: Email: fallahnejadreza@yahoo.com

1. Introduction

Data envelopment analysis (DEA) for the first time introduced by Charnes et al. (1978), the DEA has become the principal technique for evaluating the efficiency of a set of peer decision-making units (DMUs) that use multiple inputs to produce multiple outputs. Originally, the DEA technique was developed to evaluate the efficiency of a system treated as a black box, without considering its internal structure. When the operations of the component processes in network systems are taken into account, several models for measuring the system and process efficiencies of network systems have been developed [1]. For example, Fare and Grosskopf (1996) developed a network DEA model when intermediate measures exist [2]. Seiford and Zhu (1999) used a two-stage network structure to measure the efficiency of US commercial banks Zhu (2003) and Chen and Zhu (2004) introduced an approach that may conclude that two inefficient stages lead to an overall efficient DMU with the inputs of the first stage and outputs of the second stage [3-5]. Castelli, Pesenti, and Ukovich (2004) discussed DMUs with two-stage for estimating the efficiency of the network [6]. Golany et al. (2006) developed a model for evaluating the efficiency of systems composed of two subsystems arranged in series [7]. Kao and Hwang (2008) proposed the DEA models by considering the series relation between the stages of network systems [8]. Tone and Tsutsui (2009) proposed a slacks-based network DEA model that can deal with intermediate measures [9]. Chen et al. (2009) introduced an additive decomposition approach to the two-stage network systems studied by Kao and Hwang (2008) [10]. Kao (2009) developed a parallel DEA model to measure the efficiency of the two-stage network system which is composed of parallel production units [11]. Kao and Hwang (2010) presented a model for measuring efficiency and indicating the relevance between the efficiency of the system and its subsystem in the network system [12]. Wu (2010) introduced bi-level programming DEA with constrained resources in the network systems [13]. Fukuyama and Weber (2010) considered a slacks-based model for a two-stage process with bad outputs [14]. Cook et al. (2010) and Liu et al. (2016) reviewed DEA models for evaluating the efficiency of network structures [15,16]. Cook et al. (2010) extended the approach to evaluating the efficiency of general network structures [15]. Chen et al. (2009) and Cook et al. (2010) used DMU-specific weights to reflect the "sizes" of the stages within a DMU in the measuring efficiency [10]. Wu et al. (2011) introduced an approach for comparison of stochastic frontier analysis in the network [17]. Cook and Zhu (2014) provided comprehensive coverage of recent research on the modeling of internal structures and two-stage networks using DEA [18]. Lozano (2015) introduced a model for evaluating the efficiency of systems with joint inputs to formulate a parallel-stage network DEA approach that uses a non-oriented, network SBM of efficiency model to evaluate the efficiency of the overall system and subsystems [19]. Avkiran (2015) introduced a model for assessing the efficiency of dynamic network systems in commercial banking with an emphasis on testing robustness [20]. Tavassoli et al. (2015) proposed a network system to measure the technical efficiency of railway transportation services [21]. Yu et al. (2015) proposed a novel fixed cost allocation based on the two-stage network DEA approach to measure the efficiency of the overall system and of the individual processes [22]. Despotis et al. (2016) show that the weighting technique used by Chen et al. (2009) is biased towards the second stage [24]. They presented an approach to estimating unique and unbiased efficiency scores for the individual stages. Ma et al. (2017) proposed an approach for efficiency measurement and decomposition in hybrid two-stage DEA with additional inputs [24]. Kao (2017) proposed an approach for efficiency measurement and frontier projection identification for general two-stage systems in data envelopment analysis [25]. Soofizadeh and Fallahnejad (2022) evaluated the performance of decision-making units in network DEA by using the Nash bargaining game by considering common inputs in the presence of undesirable outputs, which in the first stage removes undesirable outputs from the system and only the desired outputs entered the second stage [26].

Amirteimoori et al. (2016) proposed an additive model to solve the two-stage network DEA model when decision-making subunits use shared input sources in both operation stages. However, the method proposed by Amirteimoori et al. (2016) does not obtain an efficiency score for each stage in the network system, as well as for the whole operation of the system. it can be argued that this is a weakness of this method. The current study shows that using a multiplier model approach for a two-stage network DEA in the presence of shared input sources within stages can obtain not only the efficiency score of the overall process but also the individual processes [27].

Amirteimoori et al. (2016) proposed an additive model to measure the efficiency of the two-stage network DEA model when decision-making subunits use shared input sources in both operation stages. However, the method proposed by Amirteimoori et al. (2016) does not obtain efficiency scores for each stage in the network system, as well as for the whole system. Here, it can be argued that this is a weakness of this method. The current study shows that using a multiplier model approach for a two-stage network DEA in the presence of shared input sources within stages can obtain not only the efficiency score of the overall process but also the individual processes (stages) [27].

The rest of the study is organized as follows: Section 2 provides the additive model for the two-stage network, in which the main work of Amirteimoori et al. (2016) is introduced for a two-stage network with shared inputs. Section 3 presents the multiplier model for estimating the efficiency score of the two-stage network which consider the two-stage network efficiency in the presence of shared input sources. Sections 4 and 5 provide the explanatory examples and the conclusion, respectively.

2. An additive model for a two-stage network with shared input sources within stages

Referring to Amirteimoori et al. (2016), we consider n DMUs for evaluating the efficiency of the two-stage network system, here, assume that each DMU consists of m inputs as x_{ij} $(i=1,...,m)$ and S outputs as $y_{rj}\,(r=1,...,s)$, and we consider $x_{ij}\,(i=1,...,m)$ and y_{rj} $(r=1,...,s)$ as the input and output values of $DMU_j\,(j=1,...,n)$, respectively. In addition, we assume that there are D outputs as $z_{dj}\,(d=1,...,D)$ for each DMU_j at stage 1. Then, we consider these D outputs as the inputs to stage 2 and are called ‘‘intermediate measures”.

Fig. 1 shows a two-stage network system, in which some of the inputs in the two stages consume an input as $k_{tp}\,(t=1,...,T)$, jointly. As seen in Fig. 1, any intermediate measure is divided into two parts $z_j = z_j^{(1)} + z_j^{(2)}$. We assume that for each $DMU_j\,(j=1,...,n)$ there are m inputs as $x_{ij}\,(i=1,...,m)$ to the first stage and D outputs (intermediate measure) as z_{dj} $(d=1,...,D)$ from the first stage. These D outputs (intermediate measure) are flexible so that they can become the final outputs to the first stage and or the inputs to the second stage.

Suppose that only part of variable z_j is consumed by stage 2. Here, it is assumed that Some portion $0 \leq \beta \leq 1$ of this intermediate measure can be used as input to the second stage and the remainder $0 \leq \bar{\beta} = 1 - \beta \leq 1$ is considered as final output from the first stage. According to Amirteimoori et al. (2016), The observed splits of the intermediate measure as z_j are $z_j^{(1)}$ and $z_j^{(2)}$, in which $z_j^{(1)}$ is used as input to the second stage and $z_j^{(2)}$ is the final output from the first stage. Obviously $z_j^{(1)} + z_j^{(2)} = z_j$, and the outputs from the second stage are as y_{rj} $(r = 1,\dots,s)$ and also assume that for each DMU_j, there are T inputs as k_{tp} $(t = 1,\dots,T)$ that should be shared among the two stages in the two-stage network system. Amirteimoori et al. (2016) assumed that Some portion $0 \leq \alpha \leq 1$ of the shared inputs k_p *is* allocated to the first stage and the remainder $0 \leq \bar{\alpha} \leq 1$ is allocated to the second stage with $\alpha + \bar{\alpha} = 1$. The observed inputs as $k_p^{(1)}$ and $k_p^{(2)}$ are allocated to stages 1 and 2, respectively, and we have $k_p^{(1)} + k_p^{(2)} = k_p$.

Amirteimoori et al. (2016) proposed the following additive model:

$$
\begin{aligned}
& E_o^{overall*} = max \left(\sum_{i=1}^{m} s_i^- + \sum_{d=1}^{D} (s_d^{(Z_1\text{-}intermediate)} + s_d^{(Z_2)}) + \sum_{t=1}^{T} (s_t^{(k_1)} + s_t^{(k_2)}) + \sum_{r=1}^{S} s_r^+ \right) \\
& s.t. \qquad\qquad\qquad\qquad\qquad\qquad (1) \\
& \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io} \qquad i = 1,\dots,m \\
& \sum_{j=1}^{n} \lambda_j z_{dj}^{(1)} + s_d^{(Z_1\text{-}intermediate)} = z_{do}^{(1)} \qquad d = 1,\dots,D \\
& \sum_{j=1}^{n} \lambda_j z_{dj}^{(2)} - s_d^{(Z_2)} = z_{do}^{(2)} \qquad d = 1,\dots,D \\
& \sum_{j=1}^{n} \lambda_j k_{tj}^{(1)} + s_t^{(k_1)} = k_{to}^{(1)} \qquad t = 1,\dots,T \\
& \sum_{j=1}^{n} \lambda_j k_{tj}^{(2)} + s_t^{(k_2)} = k_{to}^{(2)} \qquad t = 1,\dots,T \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro} \qquad r = 1,\dots,S \\
& -z_{do}^{(1)} \leq s_d^{(Z_1)} \leq z_{do}^{(1)} \qquad d = 1,\dots,D \\
& s_i^-, s_d^{(Z_2)}, s_t^{(k_1)}, s_t^{(k_2)}, s_r^+, \lambda_j \qquad for\ \ all\ \ i,d,t,r,j. \\
& s_d^{(Z_1\text{-}intermediate)} \qquad free\ \ in\ \ sign.
\end{aligned}
$$

Note that, in Amirteimoori et al. (2016) "released a part of the intermediate measures slacks which are inputs to stage 2, so that these slacks can accept negative values. If there is a set of slacks S_i^{-*} $(i = 1,\dots,m)$, $S_d^{(z_2)*}$ $(d = 1,\dots,D)$, $S_t^{(k_1)*}$ *and* $S_t^{(k_2)*}$ $(t = 1,\dots,T)$, S_r^{+*} $(r = 1,\dots,s)$ and $S_d^{(z_1-intermediate)*}$ $(d = 1,\dots,D)$ that make*s* $S_o^{overall*} = 0$, then DMU_o is called efficient. Otherwise, it is called non-efficient". It should be noted. Here, an important issue is that this model is not used to calculate the efficiency score for such network systems, To overcome this deficiency, in the next section we propose a novel approach for estimating the efficiency score of two-stage network DEA in the presence of shared input sources, and the current study shows that using a multiplier model approach for two-stage network DEA in the presence of shared input sources within stages can obtain not only efficiency score of the whole network system but also the two stages. For this purpose, in the next section, we present a fractional multiplier model in which the shared inputs can be divided between the two stages in the network system parametrically. In the following, we will be able to use a new technique to convert the fractional model into a linear model.

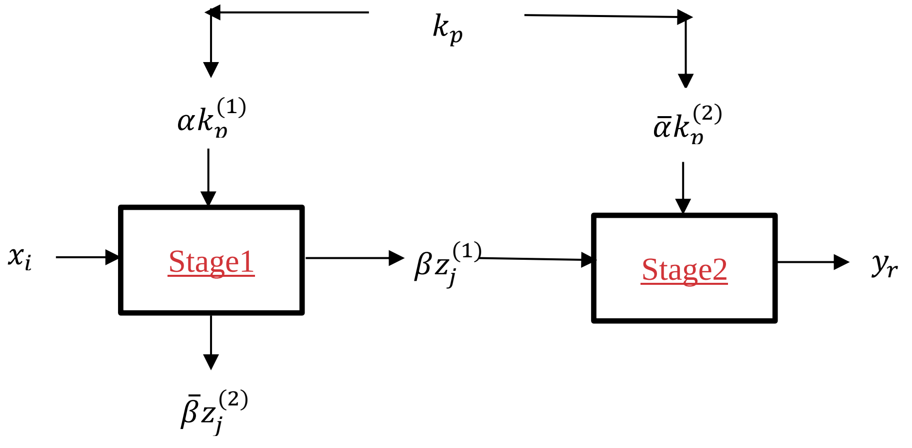

Fig.1.Two-stage network in the presence of shared resources

3. A multiplier model for estimating the efficiency score of a two-stage network with shared inputs

To consider the internal structure of a two-stage network system with shared inputs, this study proposes a two-stage network DEA model in the presence of shared input sources for evaluating the efficiency score of the two-stage network system. Fig. 1. presents the two-stage network system. In the first stage, each $DMU_j\,(j=1,\ldots,n)$ consumes inputs as i=1,..,m x_{ij} to generate intermediate measures (outputs) as $z_{dj}\left(d=1,\ldots,D\right)$. Thus, these intermediate measures are the inputs in the second stage to produce the final outputs as $y_{rj}\,(r=1,\ldots,S)$. Suppose that shared input sources exist among J DMUs, and each DMU_j receives a part of shared input sources such as $\sum_{j=1}^{n} k_j = k$. Because the operator of DMU_j can freely assign the shared input sources between the two stages, the shared input sources are considered as an extra input, so DMU_j allocates some parts α_j of shared input sources to the first stage and the remaining $\left(1-\alpha_j\right) = \bar{\alpha}_j$ to the second stage. This study considers the shared input sources and uses the following model to obtain the efficiency score for DMU_o under the assumption of CRS:

$$E_o^{overall*} = max\left(w_1 \times \frac{\sum_{d=1}^{D} \omega_d z_{do} + \sum_{r=1}^{s_1} u_r^1 y_{ro}^1}{\sum_{i=1}^{m} v_i x_{io} + \alpha_p v_{i+1} k_{po}} + w_2 \times \frac{\sum_{r=1}^{s_2} u_r^2 y_{ro}^2}{\sum_{d=1}^{D} \omega_d z_{do} + \bar{\alpha}_p \omega_{d+1} k_{po}} \right)$$

$$S.t. \qquad (2)$$

$$\frac{\sum_{d=1}^{D} \omega_d z_{dj} + \sum_{r=1}^{s_1} u_r^1 y_{rj}^1}{\sum_{i=1}^{m} v_i x_{ij} + \alpha_j v_{i+1} k_{pj}} \le 1$$

$$\frac{\sum_{r=1}^{s_2} u_r^2 y_{rj}^2}{\sum_{d=1}^{D} \omega_d z_{dj} + \bar{\alpha}_j \omega_{d+1} k_{pj}} \le 1$$

$$v_i, v_{i+1}, \omega_d, \omega_{d+1}, u_r^1, u_r^2 \ge 0$$

Here, y_{ro}^1 *and* y_{ro}^2 are values for outputs of stage 1 (intermediate measures) and outputs of stage 2, respectively, like models of centralized resource allocation v_{i+1} and ω_{d+1} are the same weights for $k_p^{(1)}$ and $k_p^{(2)}$, respectively. (See, e.g., Hakim et al. (2016) and Fang (2013)). We then define the overall efficiency of the network system as: $E_o^{overall} = (w_1 \times E_o^1 + w_2 \times E_o^2)$ where :

$$w_1 = \frac{\sum_{i=1}^{m} v_i x_{io} + \alpha_p v_{i+1} k_{po}}{\sum_{i=1}^{m} v_i x_{io} + \alpha_p v_{i+1} k_{po} + \sum_{d=1}^{D} \omega_d z_{do} + \bar{\alpha}_p \omega_{d+1} k_{po}}$$

$$and \qquad (3)$$

$$w_2 = \frac{\sum_{d=1}^{D} \omega_d z_{do} + \bar{\alpha}_p \omega_{d+1} k_{po}}{\sum_{i=1}^{m} v_i x_{io} + \alpha_p v_{i+1} k_{po} + \sum_{d=1}^{D} \omega_d z_{do} + \bar{\alpha}_p \omega_{d+1} k_{po}}$$

In the above formula, we consider user-specified weights as w_1 and w_2 for stage 1 and stage 2 such that $w_1 + w_2 = 1$. Then we define E_o^1 *and* E_o^2 for measuring the efficiency scores of stage 1 and stage 2, respectively, and can be calculated using the following:

$$E_o^1 = \frac{\sum_{d=1}^{D} \omega_d z_{do} + \sum_{r=1}^{S_1} u_r^1 y_{ro}^1}{\sum_{i=1}^{m} v_i x_{io} + \alpha_p v_{i+1} k_{po}}$$

$$and \qquad (4)$$

$$E_o^2 = \frac{\sum_{r=1}^{S_2} u_r^2 y_{ro}^2}{\sum_{d=1}^{D} \omega_d z_{do} + \bar{\alpha}_p \omega_{d+1} k_{po}}$$

In the study, we assume that $v_{i+1} = \omega_{d+1} = \mu_p$, because shared input sources are the same type of inputs, as well as α_j *and* $\bar{\alpha}_j$ are parameters to allocate shared input sources between stage 1 and stage 2, respectively. Thus, under CRS, Model (2) becomes:

$$E_o^{overall*} = max\left(\frac{\sum_{d=1}^{D}\omega_d z_{do} + \sum_{r=1}^{s_1} u_r^1 y_{ro}^1 + \sum_{r=1}^{s_2} u_r^2 y_{ro}^2}{\sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D}\omega_d z_{do} + \mu_p k_{po}}\right)$$

$$s.t. \qquad (5)$$

$$\frac{\sum_{d=1}^{D}\omega_d z_{dj} + \sum_{r=1}^{s_1} u_r^1 y_{rj}^1}{\sum_{i=1}^{m} v_i x_{ij} + \alpha_j \mu_p k_{pj}} \le 1$$

$$\frac{\sum_{r=1}^{s_2} u_r^2 y_{rj}^2}{\sum_{d=1}^{D}\omega_d z_{dj} + \bar{\alpha}_j \mu_p k_{pj}} \le 1$$

$$v_i, \omega_d, u_r^1, u_r^2, \mu_p \ge 0$$

To transform Model (2) into a linear programming problem, we apply the Charnes–Cooper transformation introduced by Charnes and Cooper (1962), thus:

$$E_o^{overall*} = max \sum_{d=1}^{D}\omega_d z_{do} + \sum_{r=1}^{s_1} u_r^1 y_{ro}^1 + \sum_{r=1}^{s_2} u_r^2 y_{ro}^2$$

$$s.t. \qquad (6)$$

$$\sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D}\omega_d z_{do} + \mu_p k_{po} = 1$$

$$\sum_{d=1}^{D}\omega_d z_{dj} + \sum_{r=1}^{s_1} u_r^1 y_{rj}^1 - \left(\sum_{i=1}^{m} v_i x_{ij} + \alpha_j \mu_p k_{pj}\right) \le 0$$

$$\sum_{r=1}^{s_2} u_r^2 y_{rj}^2 - \left(\sum_{d=1}^{D}\omega_d z_{dj} + \bar{\alpha}_j \mu_p k_{pj}\right) \le 0$$

$$v_i, \omega_d, u_r^1, u_r^2, \mu_p \ge 0$$

Here, α_j and $\bar{\alpha}_j = (1 - \alpha_j)$ are considered as parameters in estimating efficiency scores for all DMU_j. In the next section using numerical examples show the applicability of the approach.

4. Explanatory examples

In this section, using an example from Amirteimoori et al. (2016) (Data on 35 sample branches is selected and derived from operations during the last six months of 2008. We use six variables from the data set as inputs and outputs. Stage 1 uses funds from customers (x_1) and several checking accounts (x_2). The outputs from this stage are deposits (z_1). Some portion of these deposits is distributed among the customers in stage 2 and the reminder should be transferred to the central bank. The outputs from stage 2 are the number of transactions (y_1), loans (y_2), and profits (y_3). The process is depicted in Fig.2. Both stages consume the operating costs, k_1 is the operational costs consumed by stage 1 and k_2 is the costs consumed by stage 2).

we look at the problem of estimating efficiency scores using the proposed multiplier models. Table 1 shows a listing of the normalized data set. The results from the proposed model (4) and (6) are reported in Table 2. Where the values observed under columns of v_1^* and v_2^* are optimum weights for x_1 and x_2, and u_1^*, u_2^* and u_3^* are for y_1, y_2 and y_3. and also $\mu_1^* \; and \; \mu_2^*$ are for $k_1 \; and \; k_2$ (shared inputs), and $\omega_1^* and \; \omega_2^*$ are for $z_1 \; and \; z_2$. $E_1^* \; and \; E_2^*$ are the values of efficiency scores of stage 1 and stage 2, respectively, finally, $E^{overall*}$ is the value of efficiency scores of the whole operation of the system. As can be seen, in table2 only DMU30 has an efficiency score of 1.0000, because both stages 1 and 2 also have an efficiency score of 1.0000, while some of DMUs have only an efficiency score of 1.0000 at stage1 and or stage2, however, this does not guarantee the efficiency score of the overall system.

According to table1, we can say that the values of $\alpha_j \; and \; \bar{\alpha}_j$ already allocated to $k_1 \; and \; k_2$, here, we do not consider values for $\alpha_j \; and \; \bar{\alpha}_j$ in estimating efficiency scores of stage 1, stage 2, and the overall system of 35 DMUs. While in the next example, we will assign values of $\alpha_j \; and \; \bar{\alpha}_j$ for shared inputs between two stages, and then estimate efficiency scores of stage 1, stage 2, and the overall system. Table3 shows a listing of the data set, here, it is assumed that $\bar{\alpha}_j = \alpha_j$ =0.5, and the results from the proposed model (4) and (6) are reported in Table4, here, the difference between this example and the preceding example is that in the formula for $\alpha_j and \; \bar{\alpha}_j$ values are considered and then estimating efficiency scores of stage1, stage2, and the overall system. In addition, we can investigate the results of the performance evaluation of the network system for different values of $\alpha_j and \; \bar{\alpha}_j$.

In Tables 3 and 4, we can see that an efficiency score has been obtained for each unit under evaluation. However, according to the results of Amirteimoori et al. (2016), we saw that his model was not able to provide an efficiency score for units and sub-units, and only determined efficiency and non-efficiency.

the method proposed by Amirteimoori et al. (2016) does not obtain efficiency scores for each stage in the network system, as well as, for the whole operation of the system, and just by applying an additive model the efficiency and inefficiency of the DMUs is determined. However, it can be argued that this is a weakness of this method. To overcome this deficiency, we propose an approach for estimating the efficiency score of the two-stage network DEA model in the presence of shared input sources. Numerical examples show the applicability of the approach. And this issue can be an innovation for this research.

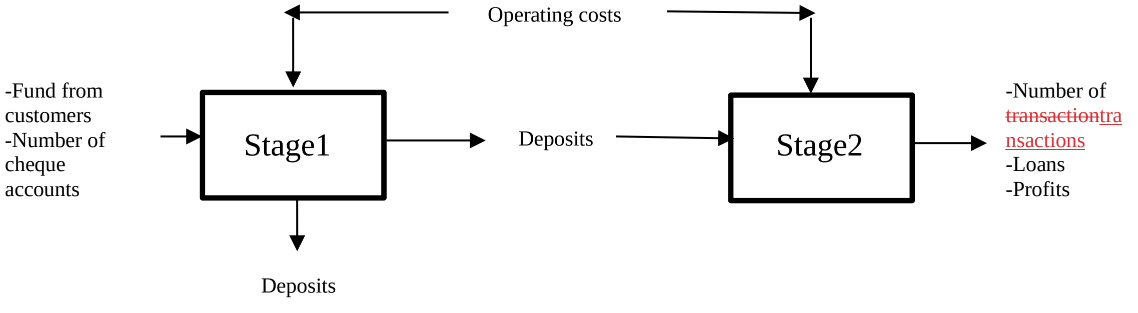

Fig.2.performance analysis in Iranian bank branches

Table 1. Bank branch data.

DMU	x_1	x_2	k_1	k_2	z_1	z_2	y_1	y_2	y_3
DMU01	0.7377	0.7795	0.6799	0.5366	0.1211	0.9490	0.5785	0.2910	0.6382
DMU02	0.2819	0.7362	0.5736	0.7241	0.3013	0.9567	0.5372	0.2855	0.8088
DMU03	0.7416	0.7953	0.5497	0.4098	0.8528	0.7180	0.2310	0.7615	0.6317
DMU04	0.5123	0.5827	0.6123	0.7634	0.5460	0.5816	0.7136	0.4790	0.3737
DMU05	0.2735	0.5630	0.1897	0.3265	0.1461	0.1201	0.7009	0.1396	0.5197
DMU06	0.2983	0.5551	0.1562	0.2965	0.4118	0.3786	0.6592	0.5507	0.7997
DMU07	0.5050	0.6811	0.4070	0.5478	0.9346	0.7263	0.7416	0.7529	0.4645
DMU08	0.4341	0.6614	0.7248	0.4332	0.7659	0.8144	0.7286	0.6348	0.3443
DMU09	0.4222	0.6417	0.4139	0.4243	0.6303	0.9059	1.0000	0.8777	0.4424
DMU10	0.3326	0.6260	0.5830	0.8802	0.4286	0.4645	0.7687	0.6104	0.6765
DMU11	0.3211	0.6181	0.8043	0.5062	0.9065	0.4077	0.6328	0.3313	0.8929
DMU12	0.7398	0.5433	0.7014	1.0000	0.7069	0.5980	0.6505	0.4223	0.4513
DMU13	0.4243	0.5787	0.3121	0.3239	0.2023	0.6520	0.3825	0.5140	0.9089
DMU14	0.4295	0.6575	0.5301	0.3783	0.9435	0.9377	0.4231	0.9647	1.0000
DMU15	0.4895	0.6929	0.6888	0.5375	0.8149	0.2932	0.4284	0.2197	0.8847
DMU16	0.4732	0.4646	0.7427	0.1512	0.5964	0.3763	0.3742	0.3721	0.7421
DMU17	0.3457	0.7795	0.8508	0.4074	0.5175	0.3682	0.3800	0.3767	0.4193
DMU18	0.5410	0.8543	0.6418	0.7915	0.6812	0.7006	0.4671	0.7551	0.6595
DMU19	0.8630	1.0000	0.4691	0.7669	0.7006	0.8078	0.3200	0.7184	0.7622
DMU20	0.2767	0.8386	0.4870	0.9283	1.0000	0.9434	0.4696	1.0000	0.6070
DMU21	0.3577	0.7953	0.3917	0.2506	0.1065	0.9475	0.4573	0.6064	0.6842
DMU22	0.7439	0.6732	0.5646	0.4216	0.6339	0.5905	0.3385	0.4084	0.6033
DMU23	1.0000	0.7362	0.6764	0.2273	0.6177	0.7994	0.7311	0.5586	0.6003
DMU24	0.3713	0.7835	0.3824	0.2334	0.6846	0.5957	0.4573	0.7724	0.4607
DMU25	0.6205	0.5472	0.4678	0.4592	0.7197	0.4940	0.4252	0.6258	0.6868
DMU26	0.2630	0.6220	0.6529	0.5147	0.9998	0.4172	0.5186	0.4733	0.5678
DMU27	0.2842	0.5866	1.0000	0.2553	0.9607	0.5185	0.5889	0.4193	0.4388
DMU28	0.2406	0.6811	0.8692	0.4270	0.9288	0.6205	0.4769	0.5734	0.3122
DMU29	0.7693	0.4882	0.5804	0.8785	0.8270	0.7154	0.4401	0.3831	0.4151
DMU30	0.3097	0.7795	0.3039	0.1590	0.5203	1.0000	0.5576	0.3568	0.2210
DMU31	0.7377	0.7913	0.5425	0.6870	0.7011	0.9131	0.4746	0.8947	0.3249
DMU32	0.2819	0.8307	0.3221	0.4639	0.5551	0.6878	0.4837	0.6106	0.4944
DMU33	0.7416	0.8071	0.6488	0.2009	0.7917	0.5013	0.4500	0.3567	0.2004
DMU34	0.5123	0.5787	0.6103	0.7603	0.9550	0.6231	0.5442	0.4784	0.1712
DMU35	0.2735	0.5433	0.8357	0.9853	0.9644	0.7293	0.7488	0.6547	0.2230

Table2. Efficiency scores for stage 1, stage 2, and the overall system using models (4) and (6).

DMU_j	v_1^*	v_2^*	μ_1^*	μ_2^*	ω_1^*	ω_2^*	u_1^*	u_2^*	u_3^*	E_1^*	E_2^*	$Overall*$
DMU01	0.0050	1.1843	0.1617	0.0655	0.8734	0.0050	0.0050	0.0050	0.0050	0.8448	0.4147	0.8370
DMU02	2.1447	0.5015	0.0050	0.0050	0.9954	0.0050	0.0050	0.0050	0.0762	1.0000	0.3294	0.9834
DMU03	0.0050	0.4880	0.7138	1.5326	0.4306	0.0050	0.3274	0.0050	0.1149	0.6925	0.6157	0.6608
DMU04	0.0495	1.6224	0.0050	0.0655	1.1567	0.0050	0.0050	0.0050	0.0050	0.6947	0.2824	0.6833
DMU05	0.9065	0.4338	0.1654	2.7000	0.3214	0.5909	0.0050	0.1487	0.3517	0.1772	1.0000	0.5821
DMU06	0.0050	0.7817	1.3569	2.4255	0.6919	0.4092	0.0050	0.2037	0.2405	0.6668	0.9493	0.7964
DMU07	0.0050	1.1198	1.0427	0.0655	1.0302	0.0050	0.0050	0.0050	0.0050	0.7704	0.4332	0.7627
DMU08	0.0443	1.4488	0.0050	0.0780	1.0325	0.0084	0.0050	0.0050	0.0050	0.8626	0.5324	0.8558
DMU09	0.0050	1.1723	1.0916	0.0780	1.0787	0.0084	0.0050	0.0050	0.0050	1.0000	0.7627	0.9953
DMU10	1.0185	0.9404	0.0050	0.0050	0.9657	0.0247	0.0050	0.0050	0.1607	0.5570	0.3575	0.5428
DMU11	0.3416	0.4172	0.0050	1.9027	0.2801	0.2820	0.0050	0.1811	0.1642	0.7118	0.5422	0.6049
DMU12	0.0519	1.6999	0.0050	0.0655	1.2121	0.0050	0.0050	0.0050	0.0050	0.7558	0.2100	0.7360
DMU13	0.0050	0.8663	1.2500	1.6141	0.7637	0.0050	0.0050	0.3187	0.1982	0.7702	0.9755	0.8321
DMU14	0.0441	1.4431	0.0050	0.1383	1.0285	0.0050	0.0186	0.0062	0.0050	1.0000	0.8531	0.9955
DMU15	0.0050	0.4573	0.1699	1.3158	0.0050	0.0050	0.0050	0.2845	0.3295	0.7145	0.4097	0.5249
DMU16	0.0050	1.0691	0.0050	3.3353	0.2545	0.0050	0.0050	0.6675	0.4140	0.6841	1.0000	0.8418
DMU17	0.5739	0.5610	0.0050	1.2751	0.4585	0.3251	0.0474	0.0050	0.1979	0.4253	0.3962	0.4147
DMU18	0.3087	0.8696	0.1893	0.0655	0.7996	0.0050	0.0050	0.0050	0.0050	0.5807	0.3208	0.5730
DMU19	0.0050	0.7403	0.8530	0.0050	0.6686	0.0050	0.0050	0.0050	0.0762	0.6282	0.1628	0.6025

DMU20	3.4665	0.0050	0.0050	0.0655	1.0172	0.0050	0.0050	0.0050	0.0050	1.0000	0.2933	0.9750
DMU21	0.4527	1.0049	0.1070	0.1387	0.9372	0.0081	0.0174	0.0053	0.0050	0.9047	1.0000	0.9064
DMU22	0.0050	1.3747	0.1908	0.0655	1.0151	0.0050	0.0050	0.0050	0.0050	0.6130	0.3977	0.6093
DMU23	0.0050	0.7646	0.0977	3.4845	0.5610	0.4256	0.0050	0.1419	0.0050	0.7514	1.0000	0.8506
DMU24	0.4135	0.4692	0.4417	2.3856	0.4728	0.0581	0.4732	0.0050	0.1694	0.6566	1.0000	0.7920
DMU25	0.0050	1.6859	0.2383	0.0655	1.2467	0.0050	0.0050	0.0050	0.0050	0.6313	0.4662	0.6282
DMU26	0.8213	0.8023	0.0050	0.0050	0.6563	0.0530	0.0050	0.0050	0.2821	0.7756	0.1153	0.5885
DMU27	0.6783	0.6628	0.0050	1.5013	0.5420	0.3834	0.0564	0.0050	0.2335	0.8651	0.6050	0.7569
DMU28	3.8525	0.0768	0.0050	0.0655	1.1942	0.0050	0.0050	0.0050	0.0050	0.7598	0.3657	0.7524
DMU29	0.0050	1.9701	0.0050	0.0655	1.3460	0.0050	0.0050	0.0050	0.0050	1.0000	0.1882	0.9733
DMU30	1.2888	0.2434	0.1406	4.8695	0.6077	0.6938	0.0050	0.0050	0.0050	1.0000	1.0000	1.0000
DMU31	0.4431	0.9826	0.1045	0.0655	0.9164	0.0050	0.0050	0.0050	0.0050	0.8627	0.3258	0.8487
DMU32	0.0050	0.0154	6.0072	0.0655	0.9238	0.0050	0.0050	0.0050	0.0050	0.6498	0.4421	0.6461
DMU33	0.5332	0.5213	0.0050	2.3936	0.4260	0.2954	0.2985	0.0050	0.1840	0.5852	0.6226	0.5996
DMU34	0.9658	1.0918	0.0050	0.0050	1.0884	0.0153	0.0050	0.0050	0.1202	0.8977	0.0990	0.8045
DMU35	0.0528	1.7304	0.0050	0.0655	1.2339	0.0050	0.0050	0.0050	0.0050	0.9395	0.2193	0.9128

Table3. The data set (Khodakarami et al. 2015)

DMU_j	x_1	x_2	k	z_1	z_2	y_1	y_2	y_3
DMU01	2982	0.2	117	8	145	158	5	4760
DMU02	2684	0.5	101	6	135	191	5	3240
DMU03	3753	0.15	84	11	213	217	9	4850
DMU04	2961	0.1	121	9	152	295	13	4190
DMU05	2789	0.35	116	5	139	337	7	4710
DMU06	2951	0.6	135	14	91	263	8	4510
DMU07	2856	0.2	174	8	153	338	13	4930
DMU08	2654	0.45	132	11	175	194	11	4350
DMU09	2921	0.2	110	7	97	172	4	4130
DMU10	2723	0.7	98	10	64	387	3	3860
DMU11	3975	0.5	164	11	142	419	6	5157
DMU12	1855	0.65	135	7	118	476	9	4230
DMU13	4186	0.3	139	13	164	117	10	5970
DMU14	2774	0.2	112	7	143	218	6	3370
DMU15	2657	0.45	176	9	115	176	5	4670
DMU16	3852	0.5	161	12	178	197	12	5110
DMU17	3758	0.1	95	8	126	423	9	4840
DMU18	3984	0.3	153	15	114	259	12	5710
DMU19	3656	0.55	76	11	89	110	9	4380
DMU20	2814	0.6	241	7	135	73	6	3850
DMU21	3881	0.4	135	39	84	198	5	5650
DMU22	3175	0.1	92	6	124	331	6	4140
DMU23	746	0.5	168	7	97	578	8	4470
DMU24	2667	0.2	114	8	119	114	5	3750
DMU25	2894	0.65	139	11	142	135	9	4180
DMU26	3651	0.5	175	9	136	238	7	4460
DMU27	1956	0.1	131	13	157	194	12	4290

Table4. Efficiency scores for stage 1, stage 2, and the overall system using models (4) and (6).

DMU_j	v_1^*	v_2^*	μ_1^*	μ_2^*	ω_1^*	ω_2^*	u_1^*	u_2^*	u_3^*	E_1^*	E_2^*	E^o	α_j	$\bar{\alpha}_j$
DMU01	0.52	1.00	0.69	0.23	0.69	1.00	0.73	0.27	0.73	0.32	1.00	0.52	0.50	0.50
DMU02	0.10	0.86	0.92	0.73	0.92	0.11	0.19	0.60	0.19	0.66	0.98	0.61	0.50	0.50
DMU03	0.82	0.45	0.45	0.78	0.73	0.19	0.92	0.73	0.73	1.00	0.32	0.49	0.50	0.50
DMU04	0.35	0.69	0.19	0.27	0.00	0.69	0.19	0.78	0.92	0.66	0.82	0.45	0.50	0.50
DMU05	0.12	0.82	1.00	0.26	0.13	0.35	0.52	0.99	0.34	0.45	0.69	0.50	0.50	0.50
DMU06	0.23	0.69	0.25	0.80	0.82	0.19	0.73	0.49	0.92	1.00	0.71	0.86	0.50	0.50
DMU07	0.99	0.11	0.92	1.00	1.00	0.23	0.78	0.80	0.19	0.70	1.00	0.86	0.50	0.50
DMU08	0.92	0.73	0.73	0.73	0.39	0.99	0.19	1.00	0.69	0.23	1.00	0.76	0.50	0.50
DMU09	0.39	0.86	0.82	0.52	0.69	0.73	1.00	0.73	0.69	0.73	0.20	0.35	0.50	0.50

DMU10	0.73	0.69	0.99	0.19	0.82	0.90	1.00	0.69	0.23	0.45	0.86	0.78	0.50	0.50
DMU11	0.50	0.35	0.92	1.00	0.99	0.73	0.45	0.35	1.00	0.56	0.84	0.41	0.50	0.50
DMU12	0.99	0.27	0.19	0.65	0.66	0.15	0.19	1.00	0.63	0.65	0.80	0.30	0.50	0.50
DMU13	036	0.92	0.78	0.99	0.37	0.82	0.52	0.25	0.86	1.00	0.10	0.23	0.50	0.50
DMU14	0.66	0.14	0.38	0.73	0.45	1.00	0.73	0.80	0.25	0.92	0.40	0.58	0.50	0.50
DMU15	0.99	1.00	0.45	0.52	0.80	0.35	0.45	0.25	0.25	0.36	0.99	0.92	0.50	0.50
DMU16	0.45	0.78	0.24	0.73	0.25	0.39	0.23	0.16	0.66	0.82	0.45	0.35	0.50	0.50
DMU17	0.69	0.99	0.39	0.17	0.35	0.99	0.80	0.73	0.27	1.00	0.30	0.66	0.50	0.50
DMU18	1.00	0.86	0.99	0.45	0.91	0.50	0.67	0.11	1.00	0.64	0.99	0.82	0.50	0.50
DMU19	1.00	0.45	0.92	0.25	0.35	0.24	0.45	0.27	0.99	0.99	.080	0.89	0.50	0.50
DMU20	0.39	0.19	0.19	0.82	0.73	1.00	0.27	0.73	0.27	0.52	1.00	0.86	0.50	0.50
DMU21	0.80	0.35	0.69	0.36	0.20	0.96	0.52	0.35	0.11	0.64	0.97	0.90	0.50	0.50
DMU22	0.21	0.41	0.18	1.00	1.00	0.82	0.99	0.27	0.99	0.85	0.35	0.45	0.50	0.50
DMU23	0.19	0.25	0.52	0.25	0.39	0.25	0.25	0.73	0.25	1.00	1.00	1.00	0.50	0.50
DMU24	0.66	0.86	0.85	0.82	0.30	0.45	0.21	0.66	0.25	0.67	0.90	0.67	0.50	0.50
DMU25	0.46	0.45	0.23	0.23	0.45	0.25	0.27	0.25	0.25	0.76	0.69	0.73	0.50	0.50
DMU26	0.66	0.96	0.32	0.55	0.99	0.23	0.22	1.00	0.99	0.78	0.90	0.65	0.50	0.50
DMU27	1.00	056	0.39	1.00	0.25	0.73	0.27	0.73	0.25	0.80	1.00	0.92	0.50	0.50

5. Conclusions

In recent years, several network DEA models have been developed. Of these, Amirteimoori et al. (2016) (Additive models for network data envelopment analysis in the presence of shared resources. Transportation Research Part D, Transport and Environment, 411-424.) proposed an additive model to measure the efficiency of two-stage network DEA model when sub-DMUs consume shared input sources in both operation stages. However, the method proposed by Amirteimoori et al. (2016) does not obtain efficiency scores for each stage in the network system, as well as, for the whole operation of the system, and just by applying an additive model the efficiency and inefficiency of the DMUs is determined. However, it can be argued that this is a weakness of this method. To overcome this deficiency, we propose an approach for estimating the efficiency score of the two-stage network DEA model in the presence of shared input sources. The results of numerical examples identify the property of the new model. This study fills in the gap of previous studies in the presence of shared input sources among all sub-DMUs. The model that we propose builds on the CRS assumption in which the shared inputs can be divided between the two stages in the network system parametrically. In the following, we will be able to use a new technique to convert the fractional model into a linear model.

Future research in line with this method, could abandon the CRS assumption and adopt the VRS assumption to investigate the issue of shared input sources among all sub-DMUs.

References

[1] Charnes, A., Cooper, W.W., Rhodes, E., 1978. Measuring the efficiency of decision making units. Eur. J. Oper. Res. 2 (6), 429–444.

[2] Färe, R., and Grosskopf, S., 1996. Productivity and intermediate products: a frontier approach. Economic Letters, 50(1), 65-70.

[3] Seiford, L.M., Zhu, J., (1999). Profitability and marketability of the top 55 U.S. commercial banks. Manage. Sci. 45 (9), 1270–1288.

[4] Zhu, J., (2003). Quantitative Models for Performance Evaluation and Benchmarking: Data Envelopment Analysis with Spreadsheets and Dea Excel Solver. Kluwer Academic.

[5] Chen, Y., and Zhu, J., 2004. Measuring information technology's indirect impact on firm performance. Information Technology & Management Journal, 5(1-2), 9-22.

[6] Castelli, L., Pesenti, R. and Ukovich, W., 2004, DEA-like models for the efficiency evaluation of hierarchically structured DMUs. European Journal of Operational Research, 154(2), 465–476.

[7] Golany, B., Hackman, S.T., and Passy, U., 2006. An efficiency measurement framework for multi-stage production systems. Ann. Oper. Res. 145 (1), 51–68.

[8] Kao, C., and Hwang, S.-N., 2008. Efficiency decomposition in two-stage data envelopment analysis: an application to non-life insurance companies in Taiwan. Eur. J. Oper. Res. 185 (1), 418–429.

[9] Tone, K., Tsutsui, M., (2009). Network DEA: a slacks-based measure approach. Eur. J. Oper. Res. 197 (1), 243–252.

[10] Chen, Y., Cook, W.D., Li, N., & Zhu, J., 2009. Additive efficiency decomposition in two-stage DEA. European Journal of Operational Research, 196(3), 1170-1176.

[11] Kao, C., 2009. Efficiency measurement for parallel production systems. Eur. J. Oper. Res. 196 (3), 1107–1112.

[12] Kao, C., Hwang, S.-N., 2010. Efficiency measurement for network systems: IT impact on firm performance. Decis. Support Syst. 48 (3), 437–446.

[13] Wu, D.D., (2010). Bi-Level programming data envelopment analysis with constrained resource. Eur. J. Oper. Res. 207 (2), 856–864.

[14] Fukuyama, H. and Weber, W.L., 2010. A Slacks-based Inefficiency Measure for a Two-stage System with Bad Outputs. Omega-The International Journal of Management Science, 38(5), 398-409.

[15] Cook, W.D., Zhu, J., Bi, G., and Yang, F., 2010. Network DEA: Additive efficiency decomposition. Eur. J. Oper. Res, 207(2), 1122-1129.

[16] Liu, J.S., Lu, L.Y.Y., and Lu, W.M., 2016. Research fronts in data envelopment analysis. Omega-The International Journal of Management Science, 41, 3-15.

[17] Wu, D.D., Zhou, Z., Birge, J.R., (2011). Estimation of potential gains from mergers in multiple periods: a comparison of stochastic frontier analysis and data envelopment analysis. Ann. Oper. Res. 186 (1), 357–381.

[18] Cook, W.D., and Zhu, J. (Eds.), 2014. Data Envelopment Analysis - A Handbook of Modeling. Internal Structure and Network. Springer, New York.

[19] Lozano, S., (2015). A joint-inputs network DEA approach to production and pollution-generating technologies. Expert Syst. Appl. 42 (21), 7960–7968.

[20] Avkiran, N.K., 2015. An illustration of dynamic network DEA in commercial banking including robustness tests. Omega 55, 141–150.

[21] Tavassoli, M., Faramarzi, G.R., Farzipoor Saen, R., (2015). A joint measurement of efficiency and effectiveness using network data envelopment analysis approach in the presence of shared input. OPSEARCH 52 (3), 490–504.

[22] Yu, M.M., Chen, L.H., Hsiao, B. (2015). A fixed cost allocation based on the two-stage network data envelopment approach. Journal of business research. 1817-1822.

[23] Despotis, D.K., Koronakos, G., and Sotiros, D., 2016. Composition versus decomposition in two-stage network DEA: A reverse approach. Journal of Productivity Analysis, 45(1).

[24] Ma, J., Qi, L., Deng, L., (2017) Eifficiency measurement and decomposition in hybrid two-stage DEA with additional inputs. Expert Syst. Appl. 348-357.

[25] Kao, C., 2017.Eifficiency measurement and frontier projection identification for general two-stage system in data envelopment analysis. Eur. J. Oper. Res. 679-689.

[26] Soofizadeh, S., Fallahnejad, R. (2022). 'A bargaining game model for performance evaluation in network DEA considering shared inputs in the presence of undesirable outputs', Journal of Mathematical Modeling, 10(2), pp. 227-245. doi: 10.22124/jmm.2021.19170.1643.

[27] Amirteimoori, A. Despotis, D, K. Kordrostami, S. and Azizi, H., 2016. Additive models for network data envelopment analysis in the presence of shared resources. Transportation Research Part D, Transport and Environment. 411-424.