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Research Article

A survey of generalized derivatives and generalized subdifferential of convex fuzzy functions and their applications

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Abstract
We are interested in proposing the convex fuzzy functions from $\mathbb{R}$ into $\mathbb{E}$ to describe the concepts of left and right generalized derivatives. In this way, several properties of the concepts are discussed. As well, the notions of generalized subdifferential for these functions are studied and calculated. Finally, the properties of the concepts are illustrated and applied through some examples.

Keywords: Convex fuzzy function, Generalized derivatives, Level-wise generalized Hukuhara subdifferential, Generalized subgradient, Generalized subdifferential.

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1. Introduction

At first, in 1967, Hukuhara defined the difference between two intervals in such a way that for each interval A the algebraic difference $A-A=0$ always exists [1]. Based on this definition, Puri and Ralescu defined the Hukuhara difference (H-difference) in 1983. According to this derivative, a differentiable function in its support interval has an increasing length [2]. After that, Bede and Gal introduced the strong generalized derivative using H-difference. Hereupon, a differentiable function in its support interval has a decreasing length [3]. Also, they presented the weak generalized derivative as an extension of their strong generalized derivative. Since the H-difference of fuzzy numbers only exists in limited terms, an extension of it was presented by Stefanini in 2010, which is called the generalized Hukuhara difference [4]. Many times, the gH-difference obtains when the H-difference does not exist, but not always. Therefore, Bede and Stefanini nominated the generalized difference (g-difference), that always exists for the fuzzy numbers defined on $\mathbb{R}$ [5]. But, the g-difference result is not always a fuzzy number. Hence, Gomes and Barros suggested the convex hull of the resulting set [6]. Finally, Bede and Stefanini proposed the concepts of generalized Hukuhara differentiability (gH-differentiability), Level-wise gH-differentiability (L_{gH}-differentiability) and generalized differentiability (g-differentiability) [5].

On the other hand, there are many essential topics in fuzzy optimization, among which fuzzy convex analysis is the most significant. The convexity of fuzzy mapping was presented by Nada and Car in 1992 [7]. The fuzzy mapping convexity and its optimization applications were produced in [8-12]. Quasi-convexity and convexity were explored by Yan-Xu for fuzzy functions [11]. Also, pseudo-convexity was investigated by Syau for multi-variable fuzzy functions [13]. Furukawa was interested in Lipschitz continuity and convexity for fuzzy functions [14]. Various descriptions of several types of convexity for fuzzy mapping can be seen in [8,9,15]. Thence, Noor presented the fuzzy preinvex functions and their properties [16]. The directional derivatives and subdifferential for convex fuzzy mappings from $\mathbb{R}^n$ into $\mathbb{E}$ in terms of H-difference were investigated by Wang and Wu [17] and in this way, for convex fuzzy mappings on n-dimensional scales were established by Wang, Qin, and Agarwal [18].

This paper includes the fuzzy topological concepts, such as g-differentiability of convex fuzzy functions and its properties. Fuzzy differentiability, especially g-differentiability, is one of the important properties of fuzzy functions. Since many functions used in fuzzy convex optimization problems do not have this property, a more comprehensive notion called generalized subdifferentiability (g-subdifferentiability) is defined as a tool for solving such problems. Furthermore, the revolutionary idea behind the g-subdifferential concept, which distinguished it from other notions of fuzzy derivatives is its set-valued property in each α-cut as the non-smoothness indication of a convex fuzzy function around the reference individual point. For this reason, we define and start studying the theoretical aspect of generalized subdifferential (g-subdifferential) in terms of α-cuts (or the set of generalized subgradients (g-subgradients)) for convex fuzzy functions and our aim is not to consider the real applications. It is clear that this research can be widely used in the control systems of electronic and engineering fuzzy optimization. In the following, Level-wise generalized Hukuhara subdifferential (L_{gH}-subdifferential) and g-subdifferential concepts are developed and calculated in terms of α-cuts for convex fuzzy functions. Eventually, we survey the properties of these concepts, one of which is that a convex fuzzy function may not be L_{gH}-differentiable at a point in its domain, but the concepts of L_{gH}-subdifferential and g-subdifferential at this point exist. This expression is the most important advantage of this paper.

Our paper construction is as you can see; First, in Section 2, several fuzzy concepts, g-difference, and g-differentiability are expressed. Then, in Section 3, convex fuzzy functions are studied, and the left and right generalized derivatives of these functions are considered in Section 4. After that, g-subgradients and g-subdifferential of these functions are analyzed and some results are in Section 5. Finally, these concepts are calculated precisely in some practical examples.

2. Literature review

This section contains the fundamental explanations that are the preliminaries of this paper. The fuzzy numbers set denoted by $\mathbb{E}$, which is convex, normal, and upper semi-continuous, that is compactly supported fuzzy sets determined over $\mathbb{R}$. If $x \in \mathbb{E}$; the support of x is specified as $\mathrm{supp}(x) = \{t \in \mathbb{R} : x(t) > 0\}$. If $\alpha \in [0,1]$, the α-cuts of x are determined by $[x]_\alpha = \{t \in \mathbb{R} : x(t) \geq \alpha\}$ and the closure of the support of x is considered for 0-cut (we abbreviated by cl(supp(x)) the closure of the support of x), i.e., $[x]_0 = cl(\mathrm{supp}(x))$.

Moreover, if $\alpha \in [0,1]$ we express $[x]_\alpha$ by $[x_\alpha^-, x_\alpha^+]$, which is closed. If $\lambda \in \mathbb{R}$ and $x, y \in \mathbb{E}$, the scalar multiplication and addition in terms of α-cuts are determined, respectively by $[\lambda \odot x]_\alpha = \lambda [x]_\alpha$ and $[x \oplus y]_\alpha = [x]_\alpha + [y]_\alpha$. Suppose that $a \leq b \leq c \leq d$, a trapezoidal fuzzy number is defined by $x = \langle a,b,c,d \rangle$, which for each $\alpha \in [0,1]$; its α-cuts are $[x]_\alpha = [a + \alpha(b-a), d - \alpha(d-c)]$. Note that, if $b = c$ then $x = \langle a,b,d \rangle$ is a triangular fuzzy number. The H-difference is defined by $x \ominus_H y = z \Leftrightarrow x = y \oplus z$; and its α-cuts are $[x \ominus_H y]_\alpha = [x_\alpha^- - y_\alpha^-, x_\alpha^+ - y_\alpha^+]$, if it exists. It is outstanding that $x \ominus_H x = 0$ for each $x \in \mathbb{E}$, (0 represents the singleton $\{0\}$), but $x - x \neq 0$.

Definition 2.1. Here and subsequently, K_C stands for the set of all closed and bounded intervals in $\mathbb{R}$, i.e., $K_C = \{P \mid P = [p^-, p^+] : p^-, p^+ \in \mathbb{R},\ p^- \leq p^+\}$ [19].

Definition 2.2. If $\mathcal{M} \in \mathbb{R}$, we define $\tilde{\mathcal{M}} \in \mathbb{E}$ as the below form [17]:

$$\tilde{\mathcal{M}} = \begin{cases} 1, & if \quad x = \mathcal{M}, \\ 0, & if \quad x \neq \mathcal{M}. \end{cases}$$

Indeed, $\mathbb{R}$ can be embedded in $\mathbb{E}$.

Definition 2.3. Suppose that $x, y \in \mathbb{E}$ with $[x]_\alpha = [x_\alpha^-, x_\alpha^+]$ and $[y]_\alpha = [y_\alpha^-, y_\alpha^+]$. For each $\alpha \in [0,1]$, we determine the below terms [17]:

1. $x_\alpha^- \le y_\alpha^-, x_\alpha^+ \le y_\alpha^+ \Leftrightarrow x \preceq y$;
2. $x \ne y, x \preceq y \Leftrightarrow x \prec y$.

Definition 2.4. Let $K \subseteq \mathbb{E}$. The fuzzy number $\gamma \in \mathbb{E}$ is the (necessarily unique) infimum of K, if it is a lower bound of K and if for every lower bound δ of K, we have $\delta \preceq \gamma$. This fuzzy number is indicated by $\inf K$, and by $\min K$ when $\inf K \in K$. The supremum of K is $\sup K = -\inf\{-\alpha \mid \alpha \in K\}$. This fuzzy number is denoted by $\max K$ when $\sup K \in K$ [17].

Definition 2.5. The fuzzy function $\varphi : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{E}$ is increasing, if $\varphi(x) \preceq \varphi(y)$ for every $x, y \in \mathbb{I}$ with $x \le y$. Similarly, φ is decreasing, if $\varphi(x) \succeq \varphi(y)$ for every $x, y \in \mathbb{I}$ with $x \le y$ [17].

Proposition 2.1. Suppose that $x \in \mathbb{E}$. It can be determined entirely by $x = (x^-, x^+)$, where the endpoints are the real-valued functions $x^-, x^+ : [0,1] \to \mathbb{R}$ fulfilling the below terms [5]:

1. $x^-(\alpha) = x_\alpha^- \in \mathbb{R}$ is increasing, monotonic, bounded, right-continuous for $\alpha = 0$ and left-continuous for each $\alpha \in (0,1]$;
2. $x^+(\alpha) = x_\alpha^+ \in \mathbb{R}$ is decreasing, monotonic, bounded, right-continuous for $\alpha = 0$ and left-continuous for each $\alpha \in (0,1]$;
3. $x_\alpha^- \le x_\alpha^+$, for each $\alpha \in [0,1]$.

Proposition 2.2. Suppose that $\{C_\alpha : \alpha \in (0,1]\}$ is a collection of real intervals with the below terms [5]:

1. $C_\alpha \in K_C$, for each $\alpha \in (0,1]$;
2. $C_\beta \subseteq C_\alpha$, for each $0 < \alpha < \beta \le 1$, i.e., they are nested;
3. $C_\alpha = \bigcap_{n=1}^{\infty} C_{\alpha_n}$, for any increasing convergence sequence $\alpha_n \in (0,1]$ with $\alpha_n \to \alpha$.

Then a unique fuzzy number x exists with $[x]_0 = cl\left(\bigcup_{\alpha \in (0,1]} C_\alpha\right)$ and $[x]_\alpha = C_\alpha$ for each $\alpha \in [0,1]$. **Lemma 2.1.** Suppose that $\varphi : \mathbb{R} \to \mathbb{E}$ and $t_0 \in \mathbb{R}$ [5]. If

1. $\lim_{t \to t_0} [\varphi(t)]_\alpha = C_\alpha = [C_\alpha^-, C_\alpha^+]$ uniformly w.r.t. $\alpha \in [0,1]$;

2. C_α satisfies the terms in Proposition 2.2 or C_α^-, C_α^+ satisfy the terms in Proposition 2.1;

then $\lim_{t\to t_0}\varphi(t)=C$, where $\left[C\right]_\alpha = C_\alpha = \left[C_\alpha^-, C_\alpha^+\right]$.

Definition 2.6. [5] The gH-difference of $x, y \in \mathbb{E}$, if it exists, is obtained by

$$x \ominus_{gH} y = z \Leftrightarrow \begin{cases} (i) \quad x = y \oplus z, \\ or \\ (ii) \quad y = x \oplus (-1)z. \end{cases}$$

In terms of α -cuts:

$$\left[x \ominus_{gH} y\right]_\alpha = \left[\min\left\{x_\alpha^- - y_\alpha^-, x_\alpha^+ - y_\alpha^+\right\}, \max\left\{x_\alpha^- - y_\alpha^-, x_\alpha^+ - y_\alpha^+\right\}\right].$$

The below cases are possible when $z = x \ominus_{gH} y \in \mathbb{E}$ exists:

$$\text{Case (i)} \begin{cases} z_\alpha^- = x_\alpha^+ - y_\alpha^+ & \text{increasing,} \\ z_\alpha^+ = x_\alpha^- - y_\alpha^- & \text{decreasing,} \\ z_\alpha^- \le z_\alpha^+, & \forall \alpha \in \left[0,1\right]. \end{cases}$$

$$\text{Case (ii)} \begin{cases} z_\alpha^- = x_\alpha^- - y_\alpha^- & \text{increasing,} \\ z_\alpha^+ = x_\alpha^+ - y_\alpha^+ & \text{decreasing,} \\ z_\alpha^- \le z_\alpha^+, & \forall \alpha \in \left[0,1\right]. \end{cases}$$

It is evident that $z \in \mathbb{R}$ if and only if Cases (i) and (ii) are both valid.

Definition 2.7. [5] If $\varphi:(a,b) \to \mathbb{E}$, $t_0 \in (a,b)$, $h \in \mathbb{R}$ and $t_0 + h \in (a,b)$. The gH-derivative of φ at t_0 is determined by

$$\varphi'_{gh}(t_0) = \lim_{h\to 0}\frac{1}{h}\left[\varphi(t_0+h) \ominus_{gH} \varphi(t_0)\right]. \qquad (2.1)$$

φ is said to be gH-differentiable at t_0 if $\varphi'_{gh}(t_0) \in \mathbb{E}$ exists and satisfies (2.1).

Theorem 2.1. [5] If $\varphi:(a,b) \to \mathbb{E}$ and $\left[\varphi(t)\right]_\alpha = \left[\varphi_\alpha^-(t), \varphi_\alpha^+(t)\right]$ such that the real functions $\varphi_\alpha^-(t)$ and $\varphi_\alpha^+(t)$ are uniformly w.r.t. $\alpha \in \left[0,1\right]$ differentiable at $t \in (a,b)$, then for each $\alpha \in \left[0,1\right]$, $\varphi'_{gh}(t)$ exists and in terms of α -cuts, φ'_{gh} is defined by

$$\left[\varphi'_{gh}(t)\right]_\alpha = \left[\min\left\{\left(\varphi_\alpha^-\right)'(t), \left(\varphi_\alpha^+\right)'(t)\right\}, \max\left\{\left(\varphi_\alpha^-\right)'(t), \left(\varphi_\alpha^+\right)'(t)\right\}\right].$$

Definition 2.8. [5] Suppose that $\varphi:[a,b]\to\mathbb{E}$ and $t_0\in(a,b)$. If $\left(\varphi_\alpha^-\right)'(t_0)$ and $\left(\varphi_\alpha^+\right)'(t_0)$ exist, then

1. for all $\alpha\in[0,1]$, φ is said to be [(i)-gH]-differentiable at t_0 if
$$\left[\varphi'_{gh}(t_0)\right]_\alpha=\left[\left(\varphi_\alpha^-\right)'(t_0),\left(\varphi_\alpha^+\right)'(t_0)\right], \quad (2.2)$$
2. for all $\alpha\in[0,1]$, φ is said to be [(ii)-gH]-differentiable at t_0 if
$$\left[\varphi'_{gh}(t_0)\right]_\alpha=\left[\left(\varphi_\alpha^+\right)'(t_0),\left(\varphi_\alpha^-\right)'(t_0)\right]. \quad (2.3)$$

Definition 2.9. [5] Suppose that $\varphi:(a,b)\to\mathbb{E}$, $t_0\in(a,b)$, $h\in\mathbb{R}$ and $t_0+h\in(a,b)$. The L_{gH}-derivative of φ at t_0 is determined as the set of interval-valued gH-derivatives, if they exist, i.e.,
$$\varphi'_{L_{gh}}(t_0)_\alpha=\lim_{h\to 0}\frac{1}{h}\left(\left[\varphi(t_0+h)\right]_\alpha\ominus_{gH}\left[\varphi(t_0)\right]_\alpha\right).$$
φ is said to be L_{gH}-differentiable at t_0, if for each $\alpha\in[0,1]$, $\varphi'_{L_{gh}}(t_0)_\alpha\in K_C$ and the L_{gH}-derivative of φ at t_0 is the collection of intervals $\left\{\varphi'_{L_{gh}}(t_0)_\alpha:\alpha\in[0,1]\right\}$ and represented by $\varphi'_{L_{gh}}(t_0)$.

Definition 2.10. [5] The g-difference of $x,y\in\mathbb{E}$ is determined for any $\alpha\in[0,1]$ in terms of α-cuts by
$$\left[x\ominus_g y\right]_\alpha=\overline{conv}\left(\bigcup_{\beta\geq\alpha}\left([x]_\beta\ominus_{gH}[y]_\beta\right)\right). \quad (2.4)$$

Proposition 2.3. [5] The g-difference of $x,y\in\mathbb{E}$ exists and it is in $\mathbb{E}$. For any $\alpha\in[0,1]$, the g-difference in terms of α-cuts is determined by
$$\left[x\ominus_g y\right]_\alpha=\left[\inf_{\beta\geq\alpha}\min\left\{x_\beta^--y_\beta^-,\ x_\beta^+-y_\beta^+\right\},\sup_{\beta\geq\alpha}\max\left\{x_\beta^--y_\beta^-,\ x_\beta^+-y_\beta^+\right\}\right].$$

Remark 2.1. [5] If $x\ominus_{gH} y\in\mathbb{E}$, then $x\ominus_{gH} y=x\ominus_g y$; particularly $x\ominus_g x=0$.

Proposition 2.4. [5] If $x,y\in\mathbb{E}$, then

1. $0\ominus_g(x\ominus_g y)=y\ominus_g x$;
2. $(x+y)\ominus_g y=x$;
3. $x\ominus_g y=y\ominus_g x=z\Leftrightarrow z=-z$; also, $z=0\Leftrightarrow x=y$.

Proposition 2.5. If $x,y\in\mathbb{E}$, the below terms are available:

1. If $x \succeq y$, then $x \ominus_g y \succeq 0$;
2. If $x \succeq 0$, then $\ominus_g x \preceq 0$;
3. If $x \preceq y$, then $x \ominus_g y \preceq 0$.

Proof. According to Proposition 2.1 in [1], if $x \succeq y$ then $x \ominus_{gH} y \succeq 0$. Furthermore, by Remark 2.1, $x \ominus_g y = x \ominus_{gH} y$, so $x \ominus_g y \succeq 0$. (2) and (3) are proved, similarly. □

Definition 2.11. [5] The Hausdorff distance of $x, y \in \mathbb{E}$ is determined by

$$D(x,y) = \sup_{\alpha \in [0,1]} \left\{ \left\| [x]_\alpha \ominus_{gH} [y]_\alpha \right\|_* \right\},$$

the norm of the interval $[x, y]$ is determined by

$$\left\| [x,y] \right\|_* = \max\left\{ |x|, |y| \right\}.$$

Whereas the gH-difference $[x]_\alpha \ominus_{gH} [y]_\alpha$ always exists, we can say the metric D is well-defined. Consequently, $(\mathbb{E}, D)$ is a complete metric space. This definition is equal to the normal definitions of metric fuzzy numbers spaces, e.g., [20-23].

Proposition 2.6. [5] For any $x, y \in \mathbb{E}$, we obtain $D(x,y) = \sup_{\alpha \in [0,1]} \left\{ \left\| [x]_\alpha \ominus_{gH} [y]_\alpha \right\|_* \right\} = \left\| x \ominus_g y \right\|$, and $\|.\| = D(.,0)$.

Remark 2.2. [5] By Remark 2.1, whenever $x \ominus_{gH} y$ exists, we can conclude

$$D(x,y) = \left\| x \ominus_g y \right\| = \left\| x \ominus_{gH} y \right\|.$$

Definition 2.12. [5] The g-derivative of $\varphi : (a,b) \to \mathbb{E}$ is determined for any $t_0 \in (a,b)$ by

$$\varphi_g'(t_0) = \lim_{h \to 0} \frac{\varphi(t_0 + h) \ominus_g \varphi(t_0)}{h}, \quad (2.5)$$

where $h \in \mathbb{R}$ be such that $t_0 + h \in (a,b)$. φ is said to be g-differentiable at t_0, if $\varphi_g'(t_0) \in \mathbb{E}$ satisfying (2.5) exists.

Theorem 2.2. [5] If $\varphi : (a,b) \to \mathbb{E}$ is uniformly $\mathrm{L_{gH}}$-differentiable at t_0, then $\varphi_g'(t_0)$ exists, and in terms of α-cuts, φ_g' is determined by

$$\left[\varphi_g'(t_0) \right]_\alpha = \overline{conv} \left(\bigcup_{\beta \geq \alpha} \varphi_{L_{gh}}'(t_0)_\beta \right), \quad \forall \alpha \in [0,1]. \quad (2.6)$$

Theorem 2.3. [5] If $\varphi:[a,b]\to\mathbb{E}$, $\left[\varphi(t)\right]_\alpha=\left[\varphi_\alpha^-(t),\varphi_\alpha^+(t)\right]$ such that $\left(\varphi_\alpha^-\right)'$ and $\left(\varphi_\alpha^+\right)'$ exist w.r.t. t, uniformly w.r.t. $\alpha\in[0,1]$, then φ_g' exists and in terms of α-cuts, it is determined by

$$\left[\varphi_g'(t)\right]_\alpha=\left[\inf_{\beta\ge\alpha}\min\left\{\left(\varphi_\beta^-\right)'(t),\left(\varphi_\beta^+\right)'(t)\right\},\sup_{\beta\ge\alpha}\max\left\{\left(\varphi_\beta^-\right)'(t),\left(\varphi_\beta^+\right)'(t)\right\}\right].$$

3. Convex fuzzy functions

Here, we assign the fuzzy function $\varphi:\mathbb{I}\subseteq\mathbb{R}\to\mathbb{E}$, where $\mathbb{I}$ is half-open, open or closed, finite or infinite. For simplicity of notation, we write the interior of $\mathbb{I}$ by $\mathbb{I}^o$. It means that if $t_0\in\mathbb{I}^o$, $h\in\mathbb{R}$ is such that $t_0+h,t_0-h\in\mathbb{I}^o$.

Definition 3.1. [17] φ is convex, if

$$\varphi\left(\lambda t+(1-\lambda)w\right)\preceq\lambda\odot\varphi(t)\oplus(1-\lambda)\odot\varphi(w),\quad\forall t,w\in\mathbb{I},\quad\forall 0\le\lambda\le 1.$$

Definition 3.2 A fuzzy set-valued function $\varphi:\mathbb{I}\subseteq\mathbb{R}\to 2^{\mathbb{E}}$ is convex, if

$$\varphi\left(\lambda t+(1-\lambda)w\right)\subseteq\lambda\odot\varphi(t)\oplus(1-\lambda)\odot\varphi(w),\quad\forall t,w\in\mathbb{I},\quad\forall 0\le\lambda\le 1.$$

Theorem 3.1. [17] If $\left[\varphi(t)\right]_\alpha=\left[\varphi_\alpha^-(t),\varphi_\alpha^+(t)\right]$, then

(i) φ is convex if and only if $\varphi_\alpha^-(t)$ and $\varphi_\alpha^+(t)$ are both convex for each $\alpha\in[0,1]$.

(ii) φ is increasing (decreasing) if and only if $\varphi_\alpha^-(t)$ and $\varphi_\alpha^+(t)$ are both increasing (decreasing) for each $\alpha\in[0,1]$.

In the next sections, we will look more closely at the properties of the convex fuzzy functions. In the beginning, we will state the below lemmas.

Lemma 3.1. φ is convex, if and only if for each $z,w,t\in\mathbb{I}$ with $z<w<t$, we get

$$\varphi(w)\preceq\frac{t-w}{t-z}\odot\varphi(z)\oplus\frac{w-z}{t-z}\odot\varphi(t).\qquad(3.7)$$

Proof. Suppose that φ is convex and $z,w,t\in\mathbb{I}$ be arbitrary with $z<w<t$. If $\lambda:=\frac{t-w}{t-z}$, then $1-\lambda=\frac{w-z}{t-z}$. So $0\le\lambda\le 1$, and $w=\lambda z+(1-\lambda)t$. By the convexity of φ, we get

$$\varphi(w)=\varphi\left(\lambda z+(1-\lambda)t\right)\preceq\lambda\odot\varphi(z)\oplus(1-\lambda)\odot\varphi(t)=\frac{t-w}{t-z}\odot\varphi(z)\oplus\frac{w-z}{t-z}\odot\varphi(t).$$

Vice versa, assume that (3.7) holds for each $z,w,t\in\mathbb{I}$ with $z<w<t$. Let $z,t\in\mathbb{I}$ be arbitrary with $z<t$ and $0\le\lambda\le 1$, then

$$z<t\underset{\times(1-\lambda)}{\rightarrow}z(1-\lambda)<(1-\lambda)t\underset{+(\lambda z)}{\rightarrow}\lambda z+z(1-\lambda)<\lambda z+(1-\lambda)t\Rightarrow z<\lambda z+(1-\lambda)t.$$

Also, we get

$$z<t\underset{\times(\lambda>0)}{\rightarrow}\lambda z<\lambda t\underset{+(1-\lambda)t}{\rightarrow}\lambda z+(1-\lambda)t<\lambda t+(1-\lambda)t=t.$$

Thus, $z<\lambda z+(1-\lambda)t<t$. By the hypothesis we have

$$\varphi(\lambda z+(1-\lambda)t)\preceq\frac{t-(\lambda z+(1-\lambda)t)}{t-z}\odot\varphi(z)\oplus\frac{\lambda z+(1-\lambda)t-z}{t-z}\odot\varphi(t)=\lambda\odot\varphi(z)\oplus(1-\lambda)\odot\varphi(t).$$

Which justifies the convexity of φ. □

Lemma 3.2. If φ is convex, then for each $z,w,t\in\mathbb{I}$ with $z<w<t$, we get

$$\frac{\varphi(w)\ominus_g\varphi(z)}{w-z}\preceq\frac{\varphi(t)\ominus_g\varphi(z)}{t-z}\preceq\frac{\varphi(t)\ominus_g\varphi(w)}{t-w}. \quad (3.8)$$

Proof. Let $z,w,t\in\mathbb{I}$ be fixed and arbitrary, by setting $x:=\frac{w-z}{t-z}\in(0,1)$. Therefore

$$\varphi(w)=\varphi((w-z)+z)=\varphi\left(\frac{t-z}{t-z}(w-z)+z\right)=\varphi(x(t-z)+z)$$

$$\Rightarrow\varphi(w)=\varphi(xt+(1-x)z). \quad (3.9)$$

By the convexity of φ and (3.9), we get

$$\varphi(w)\preceq x\odot\varphi(t)\oplus(1-x)\odot\varphi(z).$$

Then, we get

$$\varphi(w)\ominus_g\varphi(z)\preceq x\odot[\varphi(t)\ominus_g\varphi(z)]=\frac{w-z}{t-z}\odot[\varphi(t)\ominus_g\varphi(z)],$$

and equivalently

$$\frac{\varphi(w)\ominus_g\varphi(z)}{w-z}\preceq\frac{\varphi(t)\ominus_g\varphi(z)}{t-z}.$$

Also, we can see that

$$\varphi(w)\ominus_g\varphi(z)\preceq(x-1)\odot[\varphi(t)\ominus_g\varphi(z)]=\frac{w-t}{t-z}\odot[\varphi(t)\ominus_g\varphi(z)].$$

The proof is completed by showing that

$$\frac{\varphi(t)\ominus_g\varphi(z)}{t-z}\preceq\frac{\varphi(t)\ominus_g\varphi(w)}{t-w}. \quad \square$$

4. Left and Right generalized derivatives of the convex fuzzy functions

Throughout this part, we assign $\varphi:\mathbb{I}\subseteq\mathbb{R}\to\mathbb{E}$ as a convex fuzzy function and introduce the left and right g-derivatives for it. Also, the basic properties of these concepts can be visualized easily.

Definition 4.1. The left and right gH-derivatives of φ at $t_0\in\mathbb{I}$ are respectively determined as below:

$$\varphi'_{-gh}(t_0)=\lim_{h\to0^+}\frac{\varphi(t_0)\ominus_{gH}\varphi(t_0-h)}{h}, \quad (4.10)$$

$$\varphi'_{+gh}(t_0)=\lim_{h\to0^+}\frac{\varphi(t_0+h)\ominus_{gH}\varphi(t_0)}{h}. \quad (4.11)$$

φ is left and right gH-differentiable at t_0, if $\varphi'_{-gh}(t_0),\varphi'_{+gh}(t_0)\in\mathbb{E}$, respectively satisfying (4.10) and (4.11) exist.

Definition 4.2. If $t_0\in\mathbb{I}^o$ and $\alpha\in[0,1]$. The left $\mathrm{L_{gH}}$-derivative of φ at t_0 is determined as the set of interval-valued left gH-derivatives, if they exist, i.e.,

$$\varphi'_{-L_{gh}}(t_0)_\alpha=\lim_{h\to0^+}\frac{[\varphi(t_0)]_\alpha\ominus_{gH}[\varphi(t_0-h)]_\alpha}{h}. \quad (4.12)$$

φ is left $\mathrm{L_{gH}}$-differentiable at t_0, if for each $\alpha\in[0,1]$, $\varphi'_{-L_{gh}}(t_0)_\alpha\in K_C$. The left $\mathrm{L_{gH}}$-derivative of φ at t_0 is determined by the collection of intervals $\left\{\varphi'_{-L_{gh}}(t_0)_\alpha:\alpha\in[0,1]\right\}$ and denoted by $\varphi'_{-L_{gh}}(t_0)$. Similarly, the right $\mathrm{L_{gH}}$-derivative of φ at t_0 is determined by $\left\{\varphi'_{+L_{gh}}(t_0)_\alpha:\alpha\in[0,1]\right\}$ and denoted by $\varphi'_{+L_{gh}}(t_0)$, where

$$\varphi'_{+L_{gh}}(t_0)_\alpha=\lim_{h\to0^+}\frac{[\varphi(t_0+h)]_\alpha\ominus_{gH}[\varphi(t_0)]_\alpha}{h}. \quad (4.13)$$

Definition 4.3. The left and right g-derivatives of φ at $t_0\in\mathbb{I}^o$ are respectively defined as below:

$$\varphi'_{-g}(t_0)=\lim_{h\to0^+}\frac{\varphi(t_0)\ominus_g\varphi(t_0-h)}{h}, \quad (4.14)$$

$$\varphi'_{+g}(t_0)=\lim_{h\to0^+}\frac{\varphi(t_0+h)\ominus_g\varphi(t_0)}{h}. \quad (4.15)$$

φ is left and right g-differentiable at t_0, if $\varphi'_{-g}(t_0), \varphi'_{+g}(t_0) \in \mathbb{E}$ respectively satisfying (4.14) and (4.15) exist.

Definition 4.4. For any $t_0 \in \mathbb{I}^o$ and $\alpha \in [0,1]$, consider

$$S_{-L_{gh}}(t_0,h)_\alpha = \frac{[\varphi(t_0)]_\alpha \ominus_{gH} [\varphi(t_0-h)]_\alpha}{h},$$

and

$$S_{+L_{gh}}(t_0,h)_\alpha = \frac{[\varphi(t_0+h)]_\alpha \ominus_{gH} [\varphi(t_0)]_\alpha}{h}.$$

Note that, for each $\alpha \in [0,1]$, $S_{-L_{gh}}(t_0,h)_\alpha, S_{+L_{gh}}(t_0,h)_\alpha \in K_C$. The left and right L_{gH}-quotients of φ at t_0 are determined by the collections of intervals $\left\{S_{-L_{gh}}(t_0,h)_\alpha : \alpha \in [0,1]\right\}$ and $\left\{S_{+L_{gh}}(t_0,h)_\alpha : \alpha \in [0,1]\right\}$, respectively. These functions are considered as the functions of h, and denoted by $S_{-L_{gh}}(t_0,h)$ and $S_{+L_{gh}}(t_0,h)$, respectively.

Definition 4.5. The left and right g-quotients of φ at $t_0 \in \mathbb{I}^o$ are considered as the functions of h and respectively determined as below:

$$S_{-g}(t_0,h) = \frac{\varphi(t_0) \ominus_g \varphi(t_0-h)}{h},$$
$$S_{+g}(t_0,h) = \frac{\varphi(t_0+h) \ominus_g \varphi(t_0)}{h}.$$

Theorem 4.1. If φ is uniformly left and right L_{gH}-differentiable at $t_0 \in \mathbb{I}^o$, then φ is left and right g-differentiable at t_0. Also, uniformly w.r.t. $\alpha \in [0,1]$, the collection of intervals

$$\left[S_{-g}(t_0,h)\right]_\alpha = \overline{conv}\left(\bigcup_{\beta \ge \alpha} \frac{[\varphi(t_0)]_\beta \ominus_{gH} [\varphi(t_0-h)]_\beta}{h}\right),$$
$$\left[S_{+g}(t_0,h)\right]_\alpha = \overline{conv}\left(\bigcup_{\beta \ge \alpha} \frac{[\varphi(t_0+h)]_\beta \ominus_{gH} [\varphi(t_0)]_\beta}{h}\right),$$

respectively, converges to $\left[\varphi'_{-g}(t_0)\right]_\alpha$ and $\left[\varphi'_{+g}(t_0)\right]_\alpha$ as $h \to 0^+$.

Proof. Suppose that $t_0 \in \mathbb{I}^o$ and $\alpha \in [0,1]$ denote the intervals

$$S_{-L_{gh}}(t_0,h)_\alpha = \frac{[\varphi(t_0)]_\alpha \ominus_{gH} [\varphi(t_0-h)]_\alpha}{h},\ b_\alpha = \lim_{h\to 0^+} S_{-L_{gh}}(t_0,h)_\alpha = \varphi'_{-L_{gh}}(t_0)_\alpha,$$

$$\left[S_{-g}(t_0,h)\right]_\alpha = \overline{conv}\left(\bigcup_{\beta\geq\alpha} S_{-L_{gh}}(t_0,h)_\beta\right) = \frac{1}{h}\left(\left[\varphi(t_0)\right]_\alpha \ominus_g \left[\varphi(t_0-h)\right]_\alpha\right),$$
$$B_\alpha = \overline{conv}\left(\bigcup_{\beta\geq\alpha} b_\beta\right).$$

Suppose that $S_{-g}(t_0,h)$ and B are the fuzzy numbers that contain the collection of intervals $\left\{\left[S_{-g}(t_0,h)\right]_\alpha : \alpha\in[0,1]\right\}$ and $\left\{B_\alpha : \alpha\in[0,1]\right\}$ as α -cuts, respectively. In fact, the α -cuts $\left\{\left[S_{-g}(t_0,h)\right]_\alpha : \alpha\in[0,1]\right\}$ and $\left\{B_\alpha : \alpha\in[0,1]\right\}$ satisfy the terms in Proposition 2.1.

We will prove the existence of $\lim_{h\to 0^+} S_{-g}(t_0,h) = B,$ and the left g-derivative of φ at t_0, accordingly, we will prove $\varphi'_{-g}(t_0) = B$.

Let us consider the intervals $\left[S_{-g}(t_0,h)\right]_\alpha = \left[\, S^-_{-g}(t_0,h)_\alpha, S^+_{-g}(t_0,h)_\alpha\,\right]$ and $B_\alpha = \left[B^-_\alpha, B^+_\alpha\right]$ such that

$$S^-_{-g}(t_0,h)_\alpha = \inf_{\beta\geq\alpha} S^-_{-L_{gh}}(t_0,h)_\beta,\ S^+_{-g}(t_0,h)_\alpha = \sup_{\beta\geq\alpha} S^+_{-L_{gh}}(t_0,h)_\beta, B^-_\alpha = \inf_{\beta\geq\alpha} b^-_\beta,\ B^+_\alpha = \sup_{\beta\geq\alpha} b^+_\beta,$$

and from the assumption uniformly w.r.t. α, $S_{-L_{gh}}(t_0,h)_\alpha$ converges to $\varphi'_{-L_{gh}}(t_0)_\alpha$ as $h\to 0^+$. So, for each $\varepsilon>0,\ \delta_\varepsilon>0$ exists, where

$$0<h<\delta_\varepsilon \Rightarrow b^-_\alpha - \frac{\varepsilon}{4} < S^-_{-L_{gh}}(t_0,h)_\alpha < b^-_\alpha + \frac{\varepsilon}{4},\quad \forall\alpha\in[0,1],$$
$$0<h<\delta_\varepsilon \Rightarrow b^+_\alpha - \frac{\varepsilon}{4} < S^+_{-L_{gh}}(t_0,h)_\alpha < b^+_\alpha + \frac{\varepsilon}{4},\quad \forall\alpha\in[0,1].$$

According to the infimum and supremum definitions, for a given $\varepsilon>0$ and for each α and h, there exist $\lambda_i\geq 0,\ i=1,2,3,4,$ such that

$$S^-_{-g}(t_0,h)_\alpha > S^-_{-L_{gh}}(t_0,h)_{\lambda_1} - \frac{\varepsilon}{4},\ B^-_\alpha > b^-_{\lambda_2} - \frac{\varepsilon}{4},$$
$$S^+_{-g}(t_0,h)_\alpha < S^+_{-L_{gh}}(t_0,h)_{\lambda_3} + \frac{\varepsilon}{4},\ B^+_\alpha < b^+_{\lambda_4} + \frac{\varepsilon}{4}.$$

Consequently, for each $\varepsilon>0$ and $\alpha\in[0,1]$, $\delta_\varepsilon>0$ exists, such that if $0<h<\delta_\varepsilon$, then

$$S^-_{-g}(t_0,h)_\alpha > S^-_{-L_{gh}}(t_0,h)_{\lambda_1} - \frac{\varepsilon}{4} > b^-_{\lambda_1} - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} \geq B^-_\alpha - \frac{\varepsilon}{2},$$

$$B^-_\alpha > b^-_{\lambda_2} - \frac{\varepsilon}{2} > S^-_{-L_{gh}}(t_0,h)_{\lambda_2} - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} \geq S^-_{-g}(t_0,h)_\alpha - \frac{\varepsilon}{2},$$

$$S^+_{-g}(t_0,h)_\alpha < S^+_{-L_{gh}}(t_0,h)_{\lambda_3} + \frac{\varepsilon}{4} < b^+_{\lambda_3} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \le B^+_\alpha + \frac{\varepsilon}{2},$$

$$B^+_\alpha < b^+_{\lambda_4} + \frac{\varepsilon}{4} < S^+_{-L_{gh}}(t_0,h)_{\lambda_4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \le S^+_{-g}(t_0,h)_\alpha + \frac{\varepsilon}{2}.$$

Finally, for the same values of h, as $\varepsilon \to 0$ and passing to the limit as $h \to 0^+$, we obtain

$$\lim_{h\to 0^+} \left\| S_{-g}(t_0,h) \ominus_g B \right\| = \lim_{h\to 0^+} \sup_{\alpha\in[0,1]} \left\| \left[S_{-g}(t_0,h) \right]_\alpha \ominus_{gH} B_\alpha \right\|_*$$
$$= \lim_{h\to 0^+} \sup_{\alpha\in[0,1]} \max \left\{ \left| S^-_{-g}(t_0,h)_\alpha - B^-_\alpha \right|, \left| S^+_{-g}(t_0,h)_\alpha - B^+_\alpha \right| \right\} = 0,$$

i.e.,

$$\lim_{h\to 0^+} D\left(\frac{\varphi(t_0) \ominus_g \varphi(t_0 - h)}{h}, \varphi'_{-g}(t_0) \right) = \lim_{h\to 0^+} \sup_{\alpha\in[0,1]} \left\| \left[S_{-g}(t_0,h) \right]_\alpha \ominus_{gH} B_\alpha \right\|_*$$
$$= \lim_{h\to 0^+} \sup_{\alpha\in[0,1]} \max \left\{ \left| S^-_{-g}(t_0,h)_\alpha - B^-_\alpha \right|, \left| S^+_{-g}(t_0,h)_\alpha - B^+_\alpha \right| \right\} = 0.$$

We conclude that for each $\alpha \in [0,1]$, $\varphi'_{-g}(t_0)$ exists. Also, the existence of $\varphi'_{+g}(t_0)$ can be shown with a similar argument. □

Example 4.1. Suppose that the convex function $\varphi : (-1,1) \to \mathbb{E}$ is determined by α-cuts for each $\alpha \in [0,1]$ as follows:

$$[\varphi(t)]_\alpha = \left[\varphi^-_\alpha(t), \varphi^+_\alpha(t) \right] = \left[(-1+\alpha)(1+|t|), (1-\alpha)(1+|t|) \right].$$

For a given $t \in (-1,1)$ and for each $\alpha \in [0,1]$, $\varphi^-_\alpha(t)$ and $\varphi^+_\alpha(t)$ satisfy the terms in Proposition 2.1. Thus, a unique fuzzy number $\varphi(t) \in \mathbb{E}$ exists with these α-cuts.

Obviously, for each $\alpha \in [0,1]$, $\left(\varphi^-_\alpha\right)'(0)$ and $\left(\varphi^+_\alpha\right)'(0)$ do not exist. On the other hand

$$\left[S_{-g}(0,h) \right]_\alpha = \overline{conv}\left(\bigcup_{\beta\ge\alpha} \frac{[\varphi(0)]_\beta \ominus_{gH} [\varphi(0-h)]_\beta}{h} \right)$$
$$= \frac{1}{h}\left[\inf \min_{\beta\ge\alpha} \left\{ (-1+\beta)|h|, (1-\beta)|h| \right\}, \sup \max_{\beta\ge\alpha} \left\{ (-1+\beta)|h|, (1-\beta)|h| \right\} \right]$$
$$= \frac{1}{h}\left[(-1+\alpha)|h|, (1-\alpha)|h| \right].$$

Note that, for each $\alpha \in [0,1]$, $\left[S_{-g}(0,h) \right]_\alpha$ converges to $\left[\varphi'_{-g}(0) \right]_\alpha = [-1+\alpha, 1-\alpha]$, as

$h \to 0^+$ and $\left[\varphi'_{-g}(0)\right]_\alpha$ satisfy the terms in Proposition 2.1. So φ is left g-differentiable at $t=0$. In the same way, we prove the right g-differentiability of φ at $t=0$.

Lemma 4.1 The left and right g-quotients in Definition 4.5 exist on $\mathbb{I}^o$ and they are decreasing and increasing on $\mathbb{I}^o$, respectively.

Proof. By the existence of $\varphi(t) \ominus_g \varphi(t-h)$, we get

$$S_{-g}(t,h) = \frac{\varphi(t) \ominus_g \varphi(t-h)}{h} \in \mathbb{E}.$$

We consider the fuzzy number $S_{-g}(t,h)$ and denote

$$S^-_{-g}(t,h) = \frac{1}{h}\left[\varphi(t) \ominus_g \varphi(t-h)\right]^-, \quad S^+_{-g}(t,h) = \frac{1}{h}\left[\varphi(t) \ominus_g \varphi(t-h)\right]^+.$$

By Proposition 2.3, for any $h>0$ and for each $\alpha \in [0,1]$

$$\begin{aligned} S^-_{-g}(t,h)_\alpha &= \inf_{\beta \ge \alpha} \min\left\{\frac{\varphi^-_\beta(t)-\varphi^-_\beta(t-h)}{h}, \frac{\varphi^+_\beta(t)-\varphi^+_\beta(t-h)}{h}\right\} \\ &\le \sup_{\beta \ge \alpha} \max\left\{\frac{\varphi^-_\beta(t)-\varphi^-_\beta(t-h)}{h}, \frac{\varphi^+_\beta(t)-\varphi^+_\beta(t-h)}{h}\right\} = S^+_{-g}(t,h)_\alpha, \end{aligned}$$

and

$$\left[S_{-g}(t,h)\right]_\alpha = \frac{1}{h}\left[\varphi(t) \ominus_g \varphi(t-h)\right]_\alpha$$
$$= \frac{1}{h}\left[\inf_{\beta\ge\alpha} \min\left\{\varphi^-_\beta(t)-\varphi^-_\beta(t-h), \varphi^+_\beta(t)-\varphi^+_\beta(t-h)\right\}, \sup_{\beta\ge\alpha} \max\left\{\varphi^-_\beta(t)-\varphi^-_\beta(t-h), \varphi^+_\beta(t)-\varphi^+_\beta(t-h)\right\}\right].$$

Since φ is convex, by part (i) in Theorem 3.1, we conclude that for any fixed t, uniformly for $\alpha \in [0,1]$, $\varphi^-_\alpha(t)$ and $\varphi^+_\alpha(t)$ are convex. Also, they are both right continuous at 0 and left continuous for $\alpha \in [0,1]$, then the quotients of h are as below:

$$S^-_-(t,h)_\alpha = \frac{\varphi^-_\alpha(t)-\varphi^-_\alpha(t-h)}{h}, \quad S^+_-(t,h)_\alpha = \frac{\varphi^+_\alpha(t)-\varphi^+_\alpha(t-h)}{h},$$

which are both right continuous at 0 and left continuous for $\alpha \in (0,1]$. Moreover,

$$\inf_{\beta\ge\alpha} \min\left\{\frac{\varphi^-_\beta(t)-\varphi^-_\beta(t-h)}{h}, \frac{\varphi^+_\beta(t)-\varphi^+_\beta(t-h)}{h}\right\},$$

and

$$\sup_{\beta\geq\alpha}\max\left\{\frac{\varphi_\beta^-(t)-\varphi_\beta^-(t-h)}{h},\frac{\varphi_\beta^+(t)-\varphi_\beta^+(t-h)}{h}\right\}.$$

satisfy the same properties. Therefore,

$$\inf_{\beta\geq\alpha}\min\left\{S_-^-(t,h)_\beta,S_-^+(t,h)_\beta\right\},$$

$$\sup_{\beta\geq\alpha}\max\left\{S_-^-(t,h)_\beta,S_-^+(t,h)_\beta\right\},$$

w.r.t. $\alpha\in[0,1]$ are respectively, increasing and decreasing; So, by Proposition 2.1, they determine a fuzzy number. As a consequence, for any $t\in\mathbb{I}^o$ the α-cuts $\left[S_{-g}(t,h)\right]_\alpha$ define a fuzzy number, i.e., $S_{-g}(t,h)\in\mathbb{E}$.

Similarly, we obtain $S_{+g}(t,h)\in\mathbb{E}$. For $t\in\mathbb{I}^o$ and $h>0$ consider

$$S_{-g}(t,h)=\frac{\varphi(t)\ominus_g\varphi(t-h)}{h}.$$

Let $0<h'<h$. We use the second inequality in (3.8) with $z=t-h$ and $w=t-h'$ to obtain $S_{-g}(t,h)\preceq S_{-g}(t,h')$. Therefore, $S_{-g}(t,h)$ is decreasing w.r.t. h. Similarly, it can be seen that $S_{+g}(t,h)$ is increasing w.r.t. h. □

Theorem 4.2. If φ is uniformly left and right $\mathrm{L_{gH}}$-differentiable on $\mathbb{I}^o$, then left and right g-derivatives of φ exist on $\mathbb{I}^o$. Also, for any $t\in\mathbb{I}^o$

$$\varphi'_{-g}(t)=\sup_{h>0}S_{-g}(t,h)\ ,\ \varphi'_{+g}(t)=\inf_{h>0}S_{+g}(t,h).$$

Proof. Consider $t\in\mathbb{I}^o$. By Lemma 4.1, $S_{-g}(t,h)$ is decreasing on $(0,\delta)\subset\mathbb{I}^o$. If $\delta_1\in(0,\delta)$, then $S_{-g}(t,h)$ is decreasing on $(0,\delta_1]$. Consequently, a fuzzy number $\tilde{\mathcal{M}}$ exists, such that for each $h\in(0,\delta_1]$, $S_{-g}(t,h)\preceq\tilde{\mathcal{M}}$, i.e., the collection $\left\{S_{-g}(t,h):h\in(0,\delta_1]\right\}$ is decreasing and bounded from above on $\mathbb{I}^o$. So, a subsequence $h_n>0$ exists, such that $h_n\to 0^+$ as $n\to\infty$, and

$$\lim_{n\to\infty}S_{-g}(t,h_n)=\lim_{n\to\infty}\frac{\varphi(t)\ominus_g\varphi(t-h_n)}{h_n}$$

$$=\sup_{h>0}\frac{\varphi(t)\ominus_g\varphi(t-h)}{h}=\sup_{h>0}S_{-g}(t,h).$$

By Theorem 2.1 in [24], Theorem 3.1, and Lemma 4.1, $S \subset [0,1]$ exists such that $mS = 0$ (for abbreviation mS denotes the Lebesgue measure of S) and

$$\lim_{n\to\infty}\left[S_{-g}(t,h_n)\right]_\alpha = \left[\lim_{n\to\infty} S^-_{-g}(t,h_n)_\alpha, \lim_{n\to\infty} S^+_{-g}(t,h_n)_\alpha\right]$$
$$= \left[\sup_{h>0} S^-_{-g}(t,h)_\alpha, \sup_{h>0} S^+_{-g}(t,h)_\alpha\right] = \sup_{h>0}\left[S_{-g}(t,h)\right]_\alpha,$$

holds for any $\alpha \in [0,1] \setminus S$. Let $t \in \mathbb{I}^o$ and take a subsequence $h_n > 0$ such that $h_n \to 0^+$ as $n \to \infty$. Now consider the intervals

$$S_{-L_{gh}}(t,h_n)_\alpha = \frac{1}{h_n}\left(\left[\varphi(t)\right]_\alpha \ominus_{gH} \left[\varphi(t-h_n)\right]_\alpha\right),$$

$$b_\alpha = \lim_{n\to\infty} S_{-L_{gh}}(t,h_n)_\alpha = \sup_{h>0} S_{-L_{gh}}(t,h)_\alpha = \varphi'_{-L_{gh}}(t)_\alpha,$$

and

$$S_{-g}(t,h_n)_\alpha = \overline{conv}\left(\bigcup_{\beta\geq\alpha} S_{-L_{gh}}(t,h_n)_\beta\right) = \frac{1}{h_n}\left(\left[\varphi(t)\right]_\alpha \ominus_g \left[\varphi(t-h_n)\right]_\alpha\right),$$

$$B_\alpha = \overline{conv}\left(\bigcup_{\beta\geq\alpha} b_\beta\right).$$

Let $S_{-g}(t,h_n)$ and B be the fuzzy numbers with the collection of intervals $\left\{S_{-g}(t,h_n)_\alpha : \alpha \in [0,1]\right\}$ and $\left\{B_\alpha : \alpha \in [0,1]\right\}$ as α -cuts, respectively. In fact, the α -cuts $\left\{S_{-g}(t,h_n)_\alpha : \alpha \in [0,1]\right\}$ and $\left\{B_\alpha : \alpha \in [0,1]\right\}$ satisfy the terms in Proposition 2.1. Consider the subsequence $h_n > 0$ such that $h_n \to 0^+$ as $n \to \infty$, we will prove that $\lim_{n\to\infty} S_{-g}(t,h_n) = \sup_{h>0} S_{-g}(t,h) = B,$ exists.

So, the left g-derivative of φ at t exists on $\mathbb{I}^o$ and equals to B . By denoting the intervals $S_{-g}(t,h_n)_\alpha = \left[S^-_{-g}(t,h_n)_\alpha, S^+_{-g}(t,h_n)_\alpha\right]$ and $B_\alpha = \left[B^-_\alpha, B^+_\alpha\right]$, we get

$$S^-_{-g}(t,h_n)_\alpha = \inf_{\beta\geq\alpha} S^-_{-L_{gh}}(t,h_n)_\beta,\ S^+_{-g}(t,h_n)_\alpha = \sup_{\beta\geq\alpha} S^+_{-L_{gh}}(t,h_n)_\beta,$$

$$B^-_\alpha = \inf_{\beta\geq\alpha} b^-_\beta,\ B^+_\alpha = \inf_{\beta\geq\alpha} b^+_\beta,$$

with the same argument as the corresponding part of Theorem 4.1. Let us consider $\alpha \in [0,1] \setminus S$, then

$$\begin{aligned}\left\|S_{-g}(t,h_n)\ominus_g B\right\| &= \sup_{\alpha\in[0,1]}\left\|\left[S_{-g}(t,h_n)\right]_\alpha \ominus_{gH} B_\alpha\right\|_* \\ &= \sup_{\alpha\in[0,1]}\left\|\left[S^-_{-g}(t,h_n)_\alpha, S^+_{-g}(t,h_n)_\alpha\right]\ominus_{gH}\left[\sup_{h>0} S^-_{-g}(t,h)_\alpha, \sup_{h>0} S^+_{-g}(t,h)_\alpha\right]\right\|_* \\ &= \sup_{\alpha\in[0,1]}\left\|\left[S^-_{-g}(t,h_n)_\alpha, S^+_{-g}(t,h_n)_\alpha\right]\ominus_{gH}\left[\lim_{n\to\infty} S^-_{-g}(t,h_n)_\alpha, \lim_{n\to\infty} S^+_{-g}(t,h_n)_\alpha\right]\right\|_* \\ &= \sup_{\alpha\in[0,1]}\max\left(\left|\lim_{n\to\infty} S^-_{-g}(t,h_n)_\alpha - S^-_{-g}(t,h_n)_\alpha\right|, \left|\lim_{n\to\infty} S^+_{-g}(t,h_n)_\alpha - S^+_{-g}(t,h_n)_\alpha\right|\right),\end{aligned}$$

passing to the limit of a subsequence $h_n > 0$ such that $h_n \to 0^+$ as $n \to \infty$, we get

$$\lim_{n\to\infty}\sup_{\alpha\in[0,1]}\left\|\left[S_{-g}(t,h_n)\right]_\alpha \ominus_{gH} B_\alpha\right\|_* = 0,$$

almost everywhere holds for $\alpha \in [0,1]$. It follows that

$$\varphi'_{-g}(t) = \lim_{n\to\infty} S_{-g}(t,h_n) = \sup_{h>0} S_{-g}(t,h) = B.$$

Therefore, $\varphi'_{-g}(t) = \sup_{h>0} S_{+g}(t,h).$

Similarly, since $S_{+g}(t,h)$ is increasing and bounded from below on $\mathbb{I}^o$, there exists a subsequence $h_n > 0$ such that $h_n \to 0^+$ as $n \to \infty$ and we obtain

$$\lim_{n\to\infty} S_{+g}(t,h_n) = \lim_{n\to\infty}\frac{\varphi(t+h_n)\ominus_g \varphi(t)}{h_n} = \inf_{h>0}\frac{\varphi(t+h)\ominus_g \varphi(t)}{h} = \inf_{h>0} S_{+g}(t,h),$$

and consequently, $\varphi'_{+g}(t) = \inf_{h>0} S_{+g}(t,h).$ Similarly, $\varphi'_{+g}(t)$ exists on $\mathbb{I}^o$. □

Theorem 4.3. If $[\varphi(t)]_\alpha = \left[\varphi^-_\alpha(t), \varphi^+_\alpha(t)\right]$, such that $\varphi^-_\alpha(t)$ and $\varphi^+_\alpha(t)$ are convex, left and right differentiable w.r.t. t, uniformly for $\alpha \in [0,1]$, then φ is left and right g-differentiable at t and we have

$$\left[\varphi'_{-g}(t)\right]_\alpha = \left[\inf_{\beta\ge\alpha}\min\left\{\left(\varphi^-_\beta\right)'_-(t), \left(\varphi^+_\beta\right)'_-(t)\right\}, \sup_{\beta\ge\alpha}\max\left\{\left(\varphi^-_\beta\right)'_-(t), \left(\varphi^+_\beta\right)'_-(t)\right\}\right],$$

$$\left[\varphi'_{+g}(t)\right]_\alpha = \left[\inf_{\beta\ge\alpha}\min\left\{\left(\varphi^-_\beta\right)'_+(t), \left(\varphi^+_\beta\right)'_+(t)\right\}, \sup_{\beta\ge\alpha}\max\left\{\left(\varphi^-_\beta\right)'_+(t), \left(\varphi^+_\beta\right)'_+(t)\right\}\right].$$

Proof. By Proposition 2.3, we obtain

$$\left[S_{-g}\left(t,h\right)\right]_\alpha = \frac{1}{h}\left[\varphi(t)\ominus_g \varphi(t-h)\right]_\alpha$$
$$=\frac{1}{h}\left[\inf_{\beta\geq\alpha}\min\left\{\varphi_\beta^-(t)-\varphi_\beta^-(t-h),\varphi_\beta^+(t)-\varphi_\beta^+(t-h)\right\},\sup_{\beta\geq\alpha}\max\left\{\varphi_\beta^-(t)-\varphi_\beta^-(t-h),\varphi_\beta^+(t)-\varphi_\beta^+(t-h)\right\}\right]$$
$$=\frac{1}{h}\left[\inf_{\beta\geq\alpha}\min\left\{S_-^-\left(t,h\right)_\beta,S_-^+\left(t,h\right)_\beta\right\},\sup_{\beta\geq\alpha}\max\left\{S_-^-\left(t,h\right)_\beta,S_-^+\left(t,h\right)_\beta\right\}\right].$$

Since the functions $\varphi_\beta^-(t)$ and $\varphi_\beta^+(t)$ are left differentiable at t, uniformly for $\beta\in\left[0,1\right]$, we conclude that

$$\lim_{h\to0^+}\left[S_{-g}\left(t,h\right)\right]_\alpha = \lim_{h\to0^+}\frac{1}{h}\left[\varphi(t)\ominus_g \varphi(t-h)\right]_\alpha$$
$$=\left[\inf_{\beta\geq\alpha}\min\left\{\left(\varphi_\beta^-\right)'_-(t),\left(\varphi_\beta^+\right)'_-(t)\right\},\sup_{\beta\geq\alpha}\max\left\{\left(\varphi_\beta^-\right)'_-(t),\left(\varphi_\beta^+\right)'_-(t)\right\}\right],$$

for any $\alpha\in\left[0,1\right]$. Moreover, the quotients $S_-^-\left(t,h\right)_\alpha$ and $S_-^+\left(t,h\right)_\alpha$ are decreasing and bounded from above. Thus, there exists a subsequence $h_n>0$ such that $h_n\to0^+$ as $n\to\infty$, we have

$$\lim_{n\to\infty}S_-^-\left(t,h_n\right)_\alpha=\lim_{n\to\infty}\frac{\varphi_\alpha^-(t)-\varphi_\alpha^-(t-h_n)}{h_n}=\sup_{h>0}\frac{\varphi_\alpha^-(t)-\varphi_\alpha^-(t-h)}{h}=\sup_{h>0}S_-^-\left(t,h\right)_\alpha,$$

$$\lim_{n\to\infty}S_-^+\left(t,h_n\right)_\alpha=\lim_{n\to\infty}\frac{\varphi_\alpha^+(t)-\varphi_\alpha^+(t-h_n)}{h_n}=\sup_{h>0}\frac{\varphi_\alpha^+(t)-\varphi_\alpha^+(t-h)}{h}=\sup_{h>0}S_-^+\left(t,h\right)_\alpha.$$

The proof completes as the same part of Theorem 34 in [4]. Similarly, the right g-differentiability of φ can be proved. □

Corollary 4.1. If $\varphi'_{+g}(t),\varphi'_{-g}(t)\in\mathbb{E}$ exist on $\mathbb{I}^o$, then a subsequence $h_n>0$ exists as $n\to\infty$, such that

$$\varphi'_{-g}(t)=\lim_{n\to\infty}\frac{\varphi(t)\ominus_g\varphi(t-h_n)}{h_n}=\sup_{h>0}\frac{\varphi(t)\ominus_g\varphi(t-h)}{h}=\sup_{h>0}S_{-g}\left(t,h\right),$$

$$\varphi'_{+g}(t)=\lim_{n\to\infty}\frac{\varphi(t+h_n)\ominus_g\varphi(t)}{h_n}=\inf_{h>0}\frac{\varphi(t+h)\ominus_g\varphi(t)}{h}=\inf_{h>0}S_{+g}\left(t,h\right).$$

Theorem 4.4. The below terms are valid:

(i) The functions φ'_{+g} and φ'_{-g} are increasing on $\mathbb{I}^o$.

(ii) If $c,d\in\mathbb{I}^o$ and $c<d$. Then

$$\varphi'_{-g}(c)=\sup_{h>0}S_{-g}\left(c,h\right)\preceq\inf_{h>0}S_{+g}\left(c,h\right)=\varphi'_{+g}(c),$$

$$\varphi'_{+g}(c)=\inf_{h>0} S_{+g}\left(c,h\right) \preceq \sup_{h>0} S_{-g}\left(d,h\right)=\varphi'_{-g}(d).$$

Proof. The proof of (i) and (ii), are simultaneous. Suppose that $c,d \in \mathbb{I}^o$ such that $c<d$, then $a,b \in \mathbb{I}$ exist, where $c,d \in [a,b] \subset \mathbb{I}$. Let $h>0$ be sufficiently small and $0<h<\frac{d-c}{2}$, then $c-h<c<c+h<d-h<d<d+h$. By Lemma 3.2, we obtain

$$\frac{\varphi(c)\ominus_g \varphi(c-h)}{h} \preceq \frac{\varphi(c+h)\ominus_g \varphi(c)}{h} \preceq \frac{\varphi(d)\ominus_g \varphi(d-h)}{h} \preceq \frac{\varphi(d+h)\ominus_g \varphi(d)}{h}.$$

By the g-quotients, the above inequality is as below:

$$S_{-g}\left(c,h\right) \preceq S_{+g}\left(c,h\right) \preceq S_{-g}\left(d,h\right) \preceq S_{+g}\left(d,h\right), \qquad (4.16)$$

from (4.16), passing to the limit as $h \to 0^+$

$$\varphi'_{-g}(c) \preceq \varphi'_{+g}(c) \preceq \varphi'_{-g}(d) \preceq \varphi'_{+g}(d), \qquad (4.17)$$

$$\Rightarrow \begin{cases} \varphi'_{-g}(c) \preceq \varphi'_{+g}(c), \\ \varphi'_{+g}(c) \preceq \varphi'_{-g}(d). \end{cases} \qquad (4.18)$$

By (4.17), we have

$$\varphi'_{-g}(c) \preceq \varphi'_{-g}(d),\ \varphi'_{+g}(c) \preceq \varphi'_{+g}(d).$$

Hence, φ'_{-g} and φ'_{+g} are increasing. Finally, by (4.18), we have

$$\varphi'_{-g}(c)=\sup_{h>0} S_{-g}\left(c,h\right) \preceq \inf_{h>0} S_{+g}\left(c,h\right)=\varphi'_{+g}(c),$$

$$\varphi'_{+g}(c)=\inf_{h>0} S_{+g}\left(c,h\right) \preceq \sup_{h>0} S_{-g}\left(d,h\right)=\varphi'_{-g}(d). \qquad \square$$

5. Generalized subgradients and generalized subdifferential of the convex fuzzy functions

First, in this section, we assign $\varphi:\mathbb{I}\subseteq\mathbb{R}\to\mathbb{E}$ as a convex fuzzy function and study the concepts of L_{gH}-subgradients, L_{gH}-subdifferential, g-subgradients, and g-subdifferential for it. Then we illustrate how to calculate them. Eventually, we explain the properties of these concepts and use them in some examples for the functions lack g-differentiability property.

Definition 5.1 If $\varphi'_{-L_{gh}}(t_0)_\alpha, \varphi'_{+L_{gh}}(t_0)_\alpha \in K_C$ as the set of intervals of left and right gH-derivatives of φ at t_0 exist and for each $\alpha\in[0,1]$, satisfy the following inequalities

$$\begin{cases}\left(\left[\varphi(t)\right]_{\alpha}\ominus_{gH}\left[\varphi(t_0)\right]_{\alpha}\right)\geq\left(\varphi'_{-L_{gh}}(t_0)_{\alpha}\right)\left(t-t_0\right), & \forall t\in\mathbb{I},\\ and & \\ \left(\left[\varphi(t)\right]_{\alpha}\ominus_{gH}\left[\varphi(t_0)\right]_{\alpha}\right)\geq\left(\varphi'_{+L_{gh}}(t_0)_{\alpha}\right)\left(t-t_0\right), & \forall t\in\mathbb{I},\end{cases}\tag{5.19}$$

then we say that $\varphi'_{-L_{gh}}(t_0)_{\alpha}$ and $\varphi'_{+L_{gh}}(t_0)_{\alpha}$ are L_{gH}-subgradients of φ at t_0. For any $\alpha\in[0,1]$, the $\alpha-L_{gH}$-subdifferential of φ at t_0 is the set of all L_{gH}-subgradients of φ at t_0 and marked by $\partial_{L_{gh}}\varphi(t_0)_{\alpha}$.

Note that, φ is $\alpha-L_{gH}$-subdifferentiable at t_0, if for each $\alpha\in[0,1]$, $\partial_{L_{gh}}\varphi(t_0)_{\alpha}\neq\varnothing$.

Definition 5.2. If $t_0\in\mathbb{I}^o$ and a fuzzy number $u_g\in\mathbb{E}$ satisfies the following inequality

$$\varphi(t)\ominus_g\varphi(t_0)\succeq u_g\odot(t-t_0),\quad\forall t\in\mathbb{I},$$

then u_g is said to be a g-subgradient of φ at t_0. The g-subdifferential of φ at t_0 is the set of all g-subgradients of φ at t_0 and marked by $\partial_g\varphi(t_0)$, i.e.,

$$\partial_g\varphi(t_0)=\left\{u_g\in\mathbb{E}:\varphi(t)\ominus_g\varphi(t_0)\succeq u_g\odot(t-t_0),\quad\forall t\in\mathbb{I}\right\}.$$

φ is called g-subdifferentiable at t_0, if $\partial_g\varphi(t_0)\neq\varnothing$. The fuzzy elements of $\partial_g\varphi(t_0)$ are the g-subgradients of φ at t_0.

Definition 5.3. If $t_0\in\mathbb{I}^o$ and $\varphi'_{-g}(t_0),\varphi'_{+g}(t_0)\in\mathbb{E}$ satisfy the following inequalities

$$\begin{cases}\varphi(t)\ominus_g\varphi(t_0)\succeq\varphi'_{-g}(t_0)\odot(t-t_0), & \forall t\in\mathbb{I},\\ and & \\ \varphi(t)\ominus_g\varphi(t_0)\succeq\varphi'_{+g}(t_0)\odot(t-t_0), & \forall t\in\mathbb{I},\end{cases}\tag{5.20}$$

then $\varphi'_{-g}(t_0)$ and $\varphi'_{+g}(t_0)$ are said to be g-subgradients of φ at t_0.

Theorem 5.1. If for any $t_0\in\mathbb{I}^o$, φ is uniformly left and right L_{gH}-differentiable at t_0, then $\varphi'_{-g}(t_0),\varphi'_{+g}(t_0)\in\partial_g\varphi(t_0)$ exist on $\mathbb{I}^o$ and for each $\alpha\in[0,1]$

$$\begin{cases}\overline{conv}\left(\bigcup_{\beta\geq\alpha}\left(\left[\varphi(t)\right]_{\beta}\ominus_{gH}\left[\varphi(t_0)\right]_{\beta}\right)\right)\geq\overline{conv}\left(\bigcup_{\beta\geq\alpha}\varphi'_{-L_{gh}}(t_0)_{\beta}\right)(t-t_0), & \forall t\in\mathbb{I},\\ and & \\ \overline{conv}\left(\bigcup_{\beta\geq\alpha}\left(\left[\varphi(t)\right]_{\beta}\ominus_{gH}\left[\varphi(t_0)\right]_{\beta}\right)\right)\geq\overline{conv}\left(\bigcup_{\beta\geq\alpha}\varphi'_{+L_{gh}}(t_0)_{\beta}\right)(t-t_0), & \forall t\in\mathbb{I}.\end{cases}\tag{5.21}$$

Proof. Let $t_0 \in \mathbb{I}^o$ be arbitrary, from the assumption and Theorem 4.2, for each $\alpha,\beta \in [0,1]$, $\beta \geq \alpha,$ we get

$$\varphi'_{-L_{gh}}(t_0)_\beta = \sup_{t<t_0}\left(\frac{[\varphi(t)]_\beta \ominus_{gH} [\varphi(t_0)]_\beta}{t-t_0}\right) \Rightarrow \varphi'_{-L_{gh}}(t_0)_\beta \geq \frac{[\varphi(t)]_\beta \ominus_{gH} [\varphi(t_0)]_\beta}{t-t_0}, \quad \forall t < t_0.$$

Since $\varphi'_{-L_{gh}}(t_0)_\beta$ for each $\beta \geq \alpha$ is convex and closed, then

$$\overline{conv}\left(\bigcup_{\beta\geq\alpha}\varphi'_{-L_{gh}}(t_0)_\beta\right) \geq \overline{conv}\left(\bigcup_{\beta\geq\alpha}\left(\frac{[\varphi(t)]_\beta \ominus_{gH} [\varphi(t_0)]_\beta}{t-t_0}\right)\right), \quad \forall t < t_0$$

$$\Rightarrow \overline{conv}\left(\bigcup_{\beta\geq\alpha}\left([\varphi(t)]_\beta \ominus_{gH} [\varphi(t_0)]_\beta\right)\right) \geq \overline{conv}\left(\bigcup_{\beta\geq\alpha}\varphi'_{-L_{gh}}(t_0)_\beta\right)(t-t_0), \quad \forall t < t_0. \quad (5.22)$$

Note that, for each $\alpha,\beta \in [0,1]$, $\beta \geq \alpha$, we obtain

$$\varphi'_{+L_{gh}}(t_0)_\beta = \inf_{t>t_0}\left(\frac{[\varphi(t)]_\beta \ominus_{gH} [\varphi(t_0)]_\beta}{t-t_0}\right) \Rightarrow \varphi'_{+L_{gh}}(t_0)_\beta \leq \frac{[\varphi(t)]_\beta \ominus_{gH} [\varphi(t_0)]_\beta}{t-t_0}, \quad \forall t > t_0.$$

Since $\varphi'_{+L_{gh}}(t_0)_\beta$, for each $\beta \geq \alpha$ is convex and closed, then

$$\overline{conv}\left(\bigcup_{\beta\geq\alpha}\varphi'_{+L_{gh}}(t_0)_\beta\right) \leq \overline{conv}\left(\bigcup_{\beta\geq\alpha}\left(\frac{[\varphi(t)]_\beta \ominus_{gH} [\varphi(t_0)]_\beta}{t-t_0}\right)\right), \quad \forall t > t_0$$

$$\Rightarrow \overline{conv}\left(\bigcup_{\beta\geq\alpha}\left([\varphi(t)]_\beta \ominus_{gH} [\varphi(t_0)]_\beta\right)\right) \geq \overline{conv}\left(\bigcup_{\beta\geq\alpha}\varphi'_{+L_{gh}}(t_0)_\beta\right)(t-t_0), \quad \forall t > t_0. \quad (5.23)$$

According to part (ii) of Theorem 4.4, for each $\alpha,\beta \in [0,1]$, such that $\beta \geq \alpha$, we get

$$\varphi'_{-L_{gh}}(t_0)_\beta \leq \varphi'_{+L_{gh}}(t_0)_\beta \Rightarrow \overline{conv}\left(\bigcup_{\beta\geq\alpha}\varphi'_{-L_{gh}}(t_0)_\beta\right) \leq \overline{conv}\left(\bigcup_{\beta\geq\alpha}\varphi'_{+L_{gh}}(t_0)_\beta\right),$$

then

$$\overline{conv}\left(\bigcup_{\beta\geq\alpha}\varphi'_{-L_{gh}}(t_0)_\beta\right)(t-t_0) \leq \overline{conv}\left(\bigcup_{\beta\geq\alpha}\varphi'_{+L_{gh}}(t_0)_\beta\right)(t-t_0), \quad \forall t > t_0. \quad (5.24)$$

According to (5.22) and (5.24), we get

$$\overline{conv}\left(\bigcup_{\beta\geq\alpha}\left([\varphi(t)]_\beta \ominus_{gH} [\varphi(t_0)]_\beta\right)\right) \geq \overline{conv}\left(\bigcup_{\beta\geq\alpha}\varphi'_{+L_{gh}}(t_0)_\beta\right)(t-t_0) \geq \overline{conv}\left(\bigcup_{\beta\geq\alpha}\varphi'_{-L_{gh}}(t_0)_\beta\right)(t-t_0), \quad \forall t > t_0.$$

In the following

$$\overline{conv}\left(\bigcup_{\beta\geq\alpha}\left(\left[\varphi(t)\right]_{\beta}\ominus_{gH}\left[\varphi(t_0)\right]_{\beta}\right)\right)\geq\overline{conv}\left(\bigcup_{\beta\geq\alpha}\varphi'_{-L_{gh}}(t_0)_{\beta}\right)(t-t_0),\quad\forall t>t_0.\qquad(5.25)$$

Then by (5.22) and (5.25), for each $\alpha\in[0,1]$ we obtain the first inequality of (5.21) for all $t\in\mathbb{I}$. Similarly, we can prove the second inequality. □

Definition 5.4. If φ is uniformly left and right $\mathrm{L_{gH}}$-differentiable at $t_0\in\mathbb{I}^o$, then for any $\alpha\in[0,1]$ the set of all $\left[u_g\right]_{\alpha}$, satisfy inequalities of (5.21) at t_0 for any $t\in\mathbb{I}$ is the α-cut of g-subdifferential of φ at t_0 and denoted by $\partial_g\varphi(t_0)_{\alpha}$.

Note that, φ for each $\alpha\in[0,1]$ is said to be g-subdifferentiable at t_0 if for each $\alpha\in[0,1]$, $\partial_g\varphi(t_0)_{\alpha}\neq\varnothing$.

Remark 5.1. If φ is g-differentiable at t_0 on $\mathbb{I}$, then $\partial_g\varphi(t_0)_{\alpha}=\left\{\left[\varphi'_g(t_0)\right]_{\alpha}\right\},\ \forall\alpha\in[0,1]$. In the following lemma, we express the fundamental properties of the g-subdifferential in terms of $\alpha-$cuts.

Lemma 5.1. If $\varphi:\mathbb{R}\to\mathbb{E}$ is convex and uniformly $\mathrm{L_{gH}}$-subdifferentiable at t_0, then for each $\alpha\in[0,1]$, $\partial_g\varphi(t_0)_{\alpha}$ is closed and convex.

Proof. At first, we will show that $\partial_g\varphi(t_0)_{\alpha}$, for each $\alpha\in[0,1]$ is closed. For $\alpha\in[0,1]$, let $\left[u_g\right]_{\alpha}\subseteq cl\left(\partial_g\varphi(t_0)_{\alpha}\right)$, according to Proposition 2.2, there exists a sequence of intervals $\left[u_g\right]_{\alpha_n}\subseteq\partial_g\varphi(t_0)_{\alpha}$ such that $\left[u_g\right]_{\alpha_n}\to\left[u_g\right]_{\alpha}$, i.e., $\left[u_g\right]_{\alpha}=\bigcap_{n=1}^{\infty}\left[u_g\right]_{\alpha_n}$ for each increasing sequences $\alpha_n\to\alpha$, converging to $\alpha\in[0,1]$. Since, φ is uniformly $\mathrm{L_{gH}}$-subdifferentiable at t_0, for each $\alpha,\beta\in[0,1]$, $\beta\geq\alpha$, a sequence of intervals $u_{\beta_n-L_{gh}}\subseteq\partial_{L_{gh}}\varphi(t_0)_{\beta}$ exists such that $u_{\beta-L_{gh}}=\bigcap_{n=1}^{\infty}u_{\beta_n-L_{gh}}$ for each increasing sequences $\beta_n\geq\alpha_n$, where $\beta_n\to\beta$ and $\alpha_n\to\alpha$. Therefore, by Theorem 5.1 for each $\alpha_n\in[0,1]$, $n\geq1$ we have

$$\overline{conv}\left(\left[\varphi(t)\right]_{\beta}\ominus_{gH}\left[\varphi(t_0)\right]_{\beta}\right)\geq\overline{conv}\left(\bigcup_{\alpha_n\in[0,1]}\left[u_g\right]_{\alpha_n}\right)(t-t_0),$$

$$\overline{conv}\left(\bigcup_{\beta\geq\alpha}\left(\left[\varphi(t)\right]_{\beta}\ominus_{gH}\left[\varphi(t_0)\right]_{\beta}\right)\right)\geq\overline{conv}\left(\bigcup_{\beta_n\geq\alpha_n}u_{\beta_n-L_{gh}}\right)(t-t_0),\quad\forall t\in\mathbb{R},$$

then, by taking the limit from the above inequalities as $n\to\infty$, we obtain

$$\overline{conv}\left(\bigcup_{\beta\geq\alpha}\left(\left[\varphi(t)\right]_{\beta}\ominus_{gH}\left[\varphi(t_0)\right]_{\beta}\right)\right)\geq\overline{conv}\left(\bigcup_{\alpha\in[0,1]}\bigcap_{n=1}^{\infty}\left[u_g\right]_{\alpha_n}\right)(t-t_0)$$

$$=\overline{conv}\left(\bigcup_{\beta\geq\alpha}\bigcap_{n=1}^{\infty}u_{\beta_n-L_{gh}}\right)(t-t_0)=\overline{conv}\left(\bigcup_{\beta\geq\alpha}u_{\beta-L_{gh}}\right)(t-t_0),\quad\forall t\in\mathbb{R}.$$

It follows that by theorem 35 in [4], for each $\alpha\in[0,1]$,

$$\overline{conv}\left(\bigcup_{\beta\geq\alpha}\left(\left[\varphi(t)\right]_{\beta}\ominus_{gH}\left[\varphi(t_0)\right]_{\beta}\right)\right)\geq\left[u_g\right]_{\alpha}(t-t_0),\quad\forall t\in\mathbb{R}.$$

Finally, according to Definition 5.4, we have $\left[u_g\right]_{\alpha}\subseteq\partial_g\varphi(t_0)_{\alpha}$. We conclude that $\partial_g\varphi(t_0)_{\alpha}$, for each $\alpha\in[0,1]$ is closed.

Now for the proof of convexity, let $\left[u_{1g}\right]_{\alpha},\left[u_{2g}\right]_{\alpha}\subseteq\partial_g\varphi(t_0)_{\alpha}$, and $\lambda\in[0,1]$, from the assumption for each $\alpha,\beta\in[0,1]$, $\beta\geq\alpha$, there exist $u_{1\beta-L_{gh}},u_{2\beta-L_{gh}}\subseteq\partial_{L_{gh}}\varphi(t_0)_{\beta}$ such that for each $\alpha\in[0,1]$,

$$\lambda\ \overline{conv}\left(\bigcup_{\beta\geq\alpha}\left(\left[\varphi(t)\right]_{\beta}\ominus_{gH}\left[\varphi(t_0)\right]_{\beta}\right)\right)\geq\lambda\ \overline{conv}\left(\bigcup_{\beta\geq\alpha}u_{1\beta-L_{gh}}\right)(t-t_0),\quad\forall t\in\mathbb{R}.\qquad(5.26)$$

and

$$(1-\lambda)\ \overline{conv}\left(\bigcup_{\beta\geq\alpha}\left(\left[\varphi(t)\right]_{\beta}\ominus_{gH}\left[\varphi(t_0)\right]_{\beta}\right)\right)\geq(1-\lambda)\ \overline{conv}\left(\bigcup_{\beta\geq\alpha}u_{2\beta-L_{gh}}\right)(t-t_0),\quad\forall t\in\mathbb{R}.\ (5.27)$$

By adding expressions (5.26) and (5.27), we have

$$\overline{conv}\left(\bigcup_{\beta\geq\alpha}\left(\left[\varphi(t)\right]_{\beta}\ominus_{gH}\left[\varphi(t_0)\right]_{\beta}\right)\right)\geq\left(\lambda\ \overline{conv}\left(\bigcup_{\beta\geq\alpha}u_{1\beta-L_{gh}}\right)+(1-\lambda)\ \overline{conv}\left(\bigcup_{\beta\geq\alpha}u_{2\beta-L_{gh}}\right)\right)(t-t_0),\quad\forall t\in\mathbb{R}.$$

Therefore by Theorem 5.1, for each $\alpha\in[0,1]$,

$$\left(\lambda\ \overline{conv}\left(\bigcup_{\beta\geq\alpha}u_{1\beta-L_{gh}}\right)+(1-\lambda)\ \overline{conv}\left(\bigcup_{\beta\geq\alpha}u_{2\beta-L_{gh}}\right)\right)\subseteq\partial_g\varphi(t_0)_{\alpha}$$

$$\Rightarrow\left(\lambda\ \left[u_{1g}\right]_{\alpha}+(1-\lambda)\ \left[u_{2g}\right]_{\alpha}\right)\subseteq\partial_g\varphi(t_0)_{\alpha}.$$

Hence, for each $\alpha\in[0,1]$, $\partial_g\varphi(t_0)_{\alpha}$ is convex. □

Proposition 5.1. If $\varphi:(a,b)\to\mathbb{E}$ is convex and uniformly L_{gH}-subdifferentiable at $t_0\in(a,b)$, then φ is g-subdifferentiable at t_0 and for each $\alpha\in[0,1]$ we have

$$\partial_g \varphi(t_0)_\alpha = \overline{conv}\left(\bigcup_{\beta\geq\alpha}\left\{\varphi'_{-L_{gh}}(t_0)_\beta \cap \varphi'_{+L_{gh}}(t_0)_\beta\right\}\right).$$

Proof. According to Theorem 5.1, for each $\alpha \in [0,1]$,

$$\overline{conv}\left(\bigcup_{\beta\geq\alpha}\left([\varphi(t)]_\beta \ominus_{gH} [\varphi(t_0)]_\beta\right)\right) \geq \overline{conv}\left(\bigcup_{\beta\geq\alpha}\varphi'_{-L_{gh}}(t_0)_\beta\right)(t-t_0), \quad \forall t\in(a,b), \tag{5.28}$$

$$\overline{conv}\left(\bigcup_{\beta\geq\alpha}\left([\varphi(t)]_\beta \ominus_{gH} [\varphi(t_0)]_\beta\right)\right) \geq \overline{conv}\left(\bigcup_{\beta\geq\alpha}\varphi'_{+L_{gh}}(t_0)_\beta\right)(t-t_0), \quad \forall t\in(a,b). \quad (5.29).$$

Then, we get $\varphi'_{+L_{gh}}(t_0)_\beta \subseteq \partial_{L_{gh}}\varphi(t_0)_\beta$ and $\varphi'_{-L_{gh}}(t_0)_\beta \subseteq \partial_{L_{gh}}\varphi(t_0)_\beta,\ \forall \beta\geq\alpha$, so

$$\left\{\varphi'_{-L_{gh}}(t_0)_\beta \cap \varphi'_{+L_{gh}}(t_0)_\beta\right\} \subseteq \partial_{L_{gh}}\varphi(t_0)_\beta,\ \forall \beta\geq\alpha.$$

Since $\left\{\varphi'_{-L_{gh}}(t_0)_\beta \cap \varphi'_{+L_{gh}}(t_0)_\beta\right\}$ for each $\beta\geq\alpha$ is convex and closed, then

$$\overline{conv}\left(\bigcup_{\beta\geq\alpha}\left\{\varphi'_{-L_{gh}}(t_0)_\beta \cap \varphi'_{+L_{gh}}(t_0)_\beta\right\}\right) \subseteq \partial_g\varphi(t_0)_\alpha.$$

Now let for any $u_{\alpha-L_{gh}} \in K_C$, also for $\alpha\in[0,1]$ be fixed and for any $\beta\geq\alpha$, such that

$$u_{\beta-L_{gh}} \subseteq \left\{\varphi'_{-L_{gh}}(t_0)_\beta \cap \varphi'_{+L_{gh}}(t_0)_\beta\right\} \Rightarrow \overline{conv}\left(\bigcup_{\beta\geq\alpha}u_{\beta-L_{gh}}\right)\left[u_g\right]_\alpha \subseteq \overline{conv}\left(\bigcup_{\beta\geq\alpha}\left\{\varphi'_{-L_{gh}}(t_0)_\beta \cap \varphi'_{+L_{gh}}(t_0)_\beta\right\}\right).$$

According to Definition 5.4, for each $\alpha\in[0,1]$,

$$\overline{conv}\left(\bigcup_{\beta\geq\alpha}\left([\varphi(t)]_\beta \ominus_{gH} [\varphi(t_0)]_\beta\right)\right) \geq \overline{conv}\left(\bigcup_{\beta\geq\alpha}u_{\beta-L_{gh}}\right)(t-t_0), \quad \forall t\in(a,b).$$

If $t-t_0>0$, by (5.28), we get

$$\overline{conv}\left(\bigcup_{\beta\geq\alpha}\left([\varphi(t)]_\beta \ominus_{gH} [\varphi(t_0)]_\beta\right)\right) \geq \overline{conv}\left(\bigcup_{\beta\geq\alpha}u_{\beta-L_{gh}}\right)(t-t_0). \tag{5.30}$$

If $t-t_0<0$, by (5.29), we get

$$\overline{conv}\left(\bigcup_{\beta\geq\alpha}\left([\varphi(t)]_\beta \ominus_{gH} [\varphi(t_0)]_\beta\right)\right) \geq \overline{conv}\left(\bigcup_{\beta\geq\alpha}u_{\beta-L_{gh}}\right)(t-t_0). \tag{5.31}$$

Therefore, according to Definition 5.4 for each $\alpha\in[0,1]$ inequalities of (5.30) and (5.31) at t_0 for any $\mathrm{t}\in(a,b)$ are hold and

$$\overline{conv}\left(\bigcup_{\beta\geq\alpha}\left(\left[\varphi(t)\right]_{\beta}\ominus_{gH}\left[\varphi(t_0)\right]_{\beta}\right)\right)\geq\overline{conv}\left(\bigcup_{\beta\geq\alpha}u_{\beta-L_{gh}}\right)(t-t_0),\quad\forall t\in(a,b)$$

$$\Rightarrow\overline{conv}\left(\bigcup_{\beta\geq\alpha}u_{\beta-L_{gh}}\right)=\left[u_g\right]_{\alpha}\subseteq\partial_g\varphi(t_0)_{\alpha}.$$

Hence

$$\overline{conv}\left(\bigcup_{\beta\geq\alpha}\left\{\varphi'_{-L_{gh}}(t_0)_{\beta}\cap\varphi'_{+L_{gh}}(t_0)_{\beta}\right\}\right)\subseteq\partial_g\varphi(t_0)_{\alpha}.$$

In other words

$$\overline{conv}\left(\bigcup_{\beta\geq\alpha}\left\{\varphi'_{-L_{gh}}(t_0)_{\beta}\cap\varphi'_{+L_{gh}}(t_0)_{\beta}\right\}\right)\supseteq\partial_g\varphi(t_0)_{\alpha}.$$

We conclude that for each $\alpha\in[0,1]$ we have

$$\partial_g\varphi(t_0)_{\alpha}=\overline{conv}\left(\bigcup_{\beta\geq\alpha}\left\{\varphi'_{-L_{gh}}(t_0)_{\beta}\cap\varphi'_{+L_{gh}}(t_0)_{\beta}\right\}\right).\qquad\square$$

Below, we give two practical examples, that show how to calculate the g-subdifferential in terms of α -cuts for convex fuzzy functions.

Example 5.1. Suppose that $\varphi:\mathbb{R}\to\mathbb{E}$ is determined as below:

$$\varphi(t)=\langle -1,0,1\rangle\odot|t|,$$

such that for each $\alpha\in[0,1]$, its α -cuts are determined by

$$\left[\varphi(t)\right]_{\alpha}=\left[(\alpha-1)|t|,(1-\alpha)|t|\right].$$

Obviously, $\left(\varphi_{\alpha}^{-}\right)'(t_0)$ and $\left(\varphi_{\alpha}^{+}\right)'(t_0)$ do not exist, thus by [21], φ is not L_{gH} -differentiable at $t_0=0$. But φ is L_{gH}-subdifferentiable, and then it is g-subdifferentiable at $t_0=0$. Then according to Definition 5.4 and Proposition 5.1, for each $\alpha\in[0,1]$ we get

$$\partial_g\varphi(0)_{\alpha}=\left\{\left[u_g\right]_{\alpha}\in K_C:\overline{conv}\left(\bigcup_{\beta\geq\alpha}\left(\left[\varphi(t)\right]_{\beta}\ominus_{gH}\left[\varphi(0)\right]_{\beta}\right)\right)\geq\overline{conv}\left(\bigcup_{\beta\geq\alpha}u_{\beta-L_{gh}}\right)(t-0),\ \forall t\in\mathbb{R}\right\}$$

$$=\left\{\left[u_g\right]_{\alpha}\in K_C:\overline{conv}\left(\bigcup_{\beta\geq\alpha}\left[\beta-1,1-\beta\right]|t|\right)\geq\overline{conv}\left(\bigcup_{\beta\geq\alpha}u_{\beta-L_{gh}}\right)t,\ \forall t\in\mathbb{R}\right\}.$$

If $t>0$, we obtain

$$\partial_g \varphi(0)_\alpha = \left\{ \left[u_g\right]_\alpha \in K_C : \overline{conv}\left(\bigcup_{\beta \ge \alpha} \left[\beta - 1, 1 - \beta\right] \right) \ge \left[u_g\right]_\alpha \right\}. \quad (5.32)$$

If $t < 0$, we get

$$\partial_g \varphi(0)_\alpha = \left\{ \left[u_g\right]_\alpha \in K_C : \overline{conv}\left(\bigcup_{\beta \ge \alpha} \left[\beta - 1, 1 - \beta\right](-t) \right) \ge \overline{conv}\left(\bigcup_{\beta \ge \alpha} u_{\beta - L_{gh}} \right) t, \ \forall t \in \mathbb{R} \right\}$$

$$= \left\{ \left[u_g\right]_\alpha \in K_C : \overline{conv}\left(\bigcup_{\beta \ge \alpha} \left[\beta - 1, 1 - \beta\right] \right) \le \left[u_g\right]_\alpha \right\}. \quad (5.33)$$

Then according to Definition 5.4 and Proposition 5.1, $\left[u_g\right]_\alpha$ for each $\alpha \in \left[0,1\right]$ satisfies inequalities (5.32) and (5.33) at $t_0 = 0$. for any $t \in \mathbb{R}$, then we get

$$\partial_g \varphi(0)_\alpha = \left\{ \left[u_g\right]_\alpha \in K_C : \left[u_g\right]_\alpha \in \overline{conv}\left(\bigcup_{\beta \ge \alpha} \left[\beta - 1, 1 - \beta\right] \right) \right\}.$$

$\varphi_\alpha^-(t)$ and $\varphi_\alpha^+(t_0)$, for any $t \ne 0$, $\alpha \in \left[0,1\right]$ are both differentiable and

$$\partial \varphi_\alpha^-(t) = \left\{ \left(\varphi_\alpha^-\right)'(t) \right\} = \left\{ \left(\alpha - 1\right)\frac{t}{|t|} \right\}, \ \ \partial \varphi_\alpha^+(t) = \left\{ \left(\varphi_\alpha^+\right)'(t) \right\} = \left\{ \left(1 - \alpha\right)\frac{t}{|t|} \right\}.$$

Example 5.2. Let $\varphi : \mathbb{R} \to \mathbb{E}$ be convex and defined as follows

$$\varphi(t) = \left\langle 1, 2, 3 \right\rangle \odot 2t,$$

such that for each $\alpha \in \left[0,1\right]$, its α -cuts are defined by

$$\left[\varphi(t)\right]_\alpha = \left[\left(1+\alpha\right)2t, \left(3-\alpha\right)2t\right].$$

Obviously, $\varphi_\alpha^-(t)$ and $\varphi_\alpha^+(t_0)$ are differentiable in $\mathbb{R}$. According to Definition 5.4 and Proposition 5.1, for each $\alpha \in \left[0,1\right]$, we get

$$\partial_g \varphi(t_0)_\alpha = \left\{ \left[u_g\right]_\alpha \in K_C : \overline{conv}\left(\bigcup_{\beta \ge \alpha} \left(\left[\varphi(t)\right]_\beta \ominus_{gH} \left[\varphi(t_0)\right]_\beta \right) \right) \ge \overline{conv}\left(\bigcup_{\beta \ge \alpha} u_{\beta - L_{gh}} \right)(t - t_0), \ \forall t \in \mathbb{R} \right\}$$

$$= \left\{ \left[u_g\right]_\alpha \in K_C : \overline{conv}\left(\bigcup_{\beta \ge \alpha} \left(\left(\left[1+\beta, 3-\beta\right]2t\right) \ominus_{gH} \left(\left[1+\beta, 3-\beta\right]2t_0\right) \right) \right) \ge \overline{conv}\left(\bigcup_{\beta \ge \alpha} u_{\beta - L_{gh}} \right)(t - t_0), \ \forall t \in \mathbb{R} \right\}.$$

If $t > t_0$, for each $\alpha \in \left[0,1\right]$ we have

$$\left\{ \left[u_g\right]_\alpha \in K_C : \left[u_g\right]_\alpha \le 2\left[1+\alpha, 3-\alpha\right] \right\}. \quad (5.34)$$

If $t < t_0$, for each $\alpha \in [0,1]$ we have

$$\left\{ \left[u_g \right]_\alpha \in K_C : \left[u_g \right]_\alpha \geq 2[\alpha - 3, -\alpha - 1] \right\}. \qquad (5.35)$$

Therefore, according to Definition 5.4 and Proposition 5.1, $\left[u_g \right]_\alpha$ for each $\alpha \in [0,1]$ satisfies inequalities of (5.34) and (5.35) at t_0 for any $t \in \mathbb{R}$. Also, according to Remark 5.1, for each $\alpha \in [0,1]$ we have

$$\partial_g \varphi(t_0)_\alpha = \left\{ \left[\varphi'_g(t_0) \right]_\alpha \right\} = \left\{ \left[u_g \right]_\alpha \in K_C : \left[u_g \right]_\alpha \in \left(2[1+\alpha, 3-\alpha] \bigcap 2[\alpha - 3, -\alpha - 1] \right) \right\}.$$

6. Conclusion

The concepts of g-difference and g-differentiability are so operational, that we utilized them to define the left and right g-derivatives for the convex fuzzy functions. Moreover, the concepts of L_{gH}-subdifferential and g-subdifferential with their properties, and how to calculate the g-subdifferential for convex fuzzy functions in terms of α-cuts and their applications are explained with several practical examples. It is suggested to do some research, based on considering g-subdifferential convex fuzzy mapping.

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