Available online at http://sanad.iau.ir/journal/ijim/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol, 16, No.1, 2024, Article ID IJIM-1629, 12 pages
Research Article

Science and Research Branch (IAU)

Intuitionistic Fuzzy Multiset Finite Automata: An Algebraic-Based Study

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Submission Date: 2023/03/03, Revised Date: 2023/12/11, Date of Acceptance: 2024/05/18

Abstract

The current study aims to introduce the notions of intuitionistic fuzzy multiset finite automata (IFMFA) concerning a given IFMFA $\mathcal{M}$ with states Q. For a subset T of Q, we present the notion of intuitionistic fuzzy multiset submachine generated by T. Furthermore, the behavior of IFMFA is studied and explicated by using algebraic techniques. Further, it is shown that the union and the intersection of a family of an IFMFSA are IFMFSA, as well. Subsequently, it is proved that if IFMFA $\mathcal{M}$ has a basis, then the cardinality of the basis is unique. Moreover, the language of IFMFA is examined and some theorems are suggested.

Keywords: Multiset, Intuitionistic, Intuitionistic automata, Behavior.

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1. Introduction

Fuzzy automaton was introduced by Wee [1] in 1967 and Santos [2] in 1978. Fuzzy automata offer a constructive useful surrounding for ambiguous and confusing computation and have revealed their importance in solving meaningful problems concerning learning systems, pattern recognition, and database theory. Moreover, the intuitionistic fuzzy sets introduced by Atanassov [3] are highly useful in dealing with vagueness. It was conducted by adding non-membership value, which may express more accurate and flexible information than fuzzy sets. Using the notion of intuitionistic fuzzy sets, Jun [4] proposed the concept of intuitionistic fuzzy finite state machines as a generalization of fuzzy finite state machines. Further in 2015, Shamsizadeh and Zahedi presented the notion of max-min intuitionistic general fuzzy automaton [5-8]. Real-life systems have been modeled via various mathematical notions, such as weighted graphs, weighted automata, labeled transition systems, weighted networks, weighted Petri nets, discrete event systems, etc., depending on the fields of application. For a chosen mathematical notion, it happens frequently that multiple models exist for the observed system. Thus, it is not a surprise that numerous techniques have been developed to determine the minimality of these models. In this respect, several studies have been carried out on the minimization problem of fuzzy finite automaton (see e.g., [9-13]). A multiset, which is a collection of elements where these elements can occur several times, is a generalization of a set [14]. Multiset processing has emerged substantially in different fields of mathematics, computer science, biology, and biochemistry (see e.g., [15-18]). Csuhajvarjú Martin-Vide and Mitrana [19] have proposed multiset automata. Further, Sharma, Tiwari, and Sharan [20] and Tiwari, Vinay, and Dubey [21] have discussed the issues of minimization of fuzzy multiset automata and the algebraic properties of fuzzy multiset finite automata, respectively. Recently, Shamsizadeh et al then studied reduced and irreducible of fuzzy multiset finite automata [22,23]. In this paper, we have discussed the notion of behavior of multiset finite automata and algebraic properties of intuitionistic fuzzy multiset finite automata. In fact, finite multiset automata processing has appeared frequently in various areas of mathematics, Petri nets, membrane computing, biology, and biochemistry. Even, membrane computing has been connected with the theory of mealy multiset automata. The present paper is organized as follows: In Sect. 2, we recall some concepts of multisets, intuitionistics and fuzzy multiset finite automaton (FMFA), and in Sect. 3 we focus on the study of the concepts of intuitionistic fuzzy multiset finite automata (IFMFA) and intuitionistic fuzzy multiset submachine generated by T. Also, we study and explain the behavior of IFMFA by using algebraic techniques. Moreover, we prove that if IFMFA M has a basis, then the cardinality of the basis is unique. Also, we prove that if $\mathcal{M} = (Q, \Sigma, A, B, C)$ is a swap intuitionistic fuzzy multiset finite automata with threshold (a, b) and $\{q_1, q_2, \ldots, q_n\}$ is a basis of $\mathcal{M}$ with threshold (a, b), then $\mathcal{M} = \ll q_1 \gg^{(a,b)} \sqcup \ll \{q_2\} \gg^{(a,b)} \sqcup \ldots \sqcup \ll q_n \gg^{(a,b)}$.

2. Preliminaries

In this section, some concepts and definitions related to multisets and automata are introduced.

Definition 1. *[14] If Σ is a finite alphabet, then $\alpha: \Sigma \to N$ is a multiset over Σ, where N denotes the set of natural numbers including 0. The α norm of Σ is defined by $|\alpha| = \sum_{a\in\Sigma} \alpha(a)$.*

We shall denote by $\Sigma^{\oplus}$ the set of all multiset over Σ. The multiset $0_\Sigma \in \Sigma^{\oplus}$ is defined by $0_\Sigma(a) = 0$, for every $a \in \Sigma$. For $b \in \Sigma$, we shall denote by $< b >^{(a,b)}$ a singleton multiset and is defined by:

$$< b > (a) = \begin{cases} 1 & \text{if } b = a \\ 0 & \text{otherwise} \end{cases},$$

for every $a \in \Sigma$. For a given set A, let $\bar{A} = \{< a > | a \in A\}$. For two multisets $\alpha, \beta \in \Sigma^{\oplus}$, the operations inclusion $\subseteq$, addition $\oplus$ and difference $\ominus$ are defined as follows:

1. $\alpha \subseteq \beta$ if $\alpha(a) \leq \beta(a)$,
2. $(\alpha \oplus \beta)(a) = \alpha(a) + \beta(a)$,
3. $(\alpha \ominus \beta)(a) = \max(0, \alpha(a) - \beta(a))$,

for every $\alpha \in \Sigma$. Furthermore, $\alpha \subset \beta$ if $\alpha \subseteq \beta$ and $\alpha \neq \beta$. Clearly, $\Sigma^{\oplus}$ is a commutative monoid with identity element 0_Σ with respect to $\oplus$.

Definition 2. *[7] A multiset finite automata (MFA) is a 5-tuple $\mathcal{M} = (Q, \Sigma, \delta, q_0, F)$, where*

1. Q and Σ are nonempty finite sets called the state-set and input-set,
2. $\delta: Q \times \Sigma^{\oplus} \to 2^Q$ is a map called transition map,
3. $q_0 \in Q$ is called the initial state,
4. $F \subseteq Q$ is called the set of final states.

Now, we recall the following concepts of fuzzy multiset finite automaton (FMFA) from [21].

Definition 3. *A fuzzy multiset finite automaton (or FMFA, for short) is a 5-tuple $\mathcal{M} = (Q, \Sigma, \delta, \iota, \tau)$, where*

1. Q and Σ are nonempty finite sets called the state-set and input-set, respectively,
2. $\delta: Q \times \Sigma^{\oplus} \times Q \to [0, 1]$ is a map called fuzzy transition map,
3. $\iota: Q \to [0, 1]$ is a map called the fuzzy set of initial states,
4. $\tau: Q \to [0, 1]$ is a map called the fuzzy set of final states.

A configuration of fuzzy multiset finite automaton $\mathcal{M}$ is a pair (q, β), where q and β denote current state and current multiset, respectively. The transition from configuration (q, β) leads to configuration (p, γ) with membership value $k \in [0, 1]$ if there exists a multiset $\alpha \in \Sigma^{\oplus}$ with $\alpha \subseteq \beta, \delta(q, \alpha, p) = k$ and $\gamma = \beta \ominus \alpha$ and is denoted by $(q, \beta) \overset{k}{\to} (p, \gamma)$. $\overset{k'^*}{\to}$ denote the reflexive and transitive closure of $\overset{k'}{\to}$, i. e., for $(q, \beta), (p, \gamma) \in Q \times \Sigma^{\oplus}$, $(q, \beta) \overset{k'^*}{\to} (p, \gamma)$ if for some $n \geq 0$, there exist $(n + 1)$ states $q_0, \ldots, q_n$ and $(n + 1)$ multisets $\beta_0, \beta_1, \ldots, \beta_n$ such that $p_0 = q, p_n = p, \beta_0 = \beta, \beta_n = \gamma$ and $(p_i, \beta_i) \overset{k_i}{\to} (p_{i+1}, \beta_{i+1})$, for every $i = 0, 1, \ldots, n - 1$, where $k' = k_0 \wedge k_1 \wedge \ldots \wedge k_{n-1}$. Now, we define

$$\mu_M((q, \beta) \to^* (p, \gamma)) = \vee \{\mu_{\mathcal{M}}((q, \beta)) \to^* (r, \beta \ominus \alpha))$$
$$\wedge \mu_{\mathcal{M}}((r, \beta \ominus \alpha)) \to^* (p, \gamma)) | r \in Q, \alpha \in \Sigma^{\oplus}, \alpha \subseteq \beta\},$$

and

$$\mu_{\mathcal{M}}((q, \beta)) \to^* (p, \beta)) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}. \qquad (1)$$

Definition 4. *[3] Let E be a (crisp) fixed set and let A be a given subset of E. An intuitionistic fuzzy set (IFS) A^+ in E is an object of the following form*

$$A^+ = \{< x, \mu_A(x), \nu_A(x) > | x \in E\},$$

where the functions $\mu_A: E \to [0,1]$ and $\nu_A: E \to [0,1]$ define the value of membership and the value of non-membership of the element $x \in E$ to the set A, respectively and for every $x \in E$, $0 \le \mu_A(x) + \nu_A(x) \le 1$. Obviously, every ordinary fuzzy set $\{(x, \mu_A(x)) | x \in E\}$ has an intuitionistic form $\{< x, \mu_A(x), 1 - \mu_A(x) > | x \in E\}$. If $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, then $\pi_A(x)$ is the value of non-determinacy (uncertainty) of the membership of element $x \in E$ to the set A. In the case of ordinary fuzzy sets, where $\nu_A(x) = 1 - \mu_A(x)$, we have $\pi_A(x) = 0$, for every $x \in E$.

3. Intuitionistic fuzzy multiset finite automata

In this section, the notion of intuitionistic fuzzy multiset finite automata (IFMFA) is presented. Moreover, the behavior of IFMFA is studied and explained by using algebraic techniques.

Definition 5. *An intuitionistic fuzzy multiset finite automata (IFMFA) $\mathcal{M}$ is defined as: $\mathcal{M} = (Q, \Sigma, A, B, C)$, where*

1. Q is a set of states,
2. Σ is a non-empty set of input alphabet,
3. $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy set, where $\mu_A, \nu_A: Q \times \Sigma^{\oplus} \times Q \to [0, 1]$ and μ_A, ν_A are called intuitionistic fuzzy transition relation of states,
4. $B: (\mu_B, \nu_B)$ is an intuitionistic fuzzy set, $\mu_B, \nu_B: Q \to [0, 1]$ and μ_B, ν_B are called intuitionistic initial function,
5. $C: (\mu_C, \nu_C)$ is an intuitionistic fuzzy set, $\mu_C, \nu_C: Q \to [0, 1]$ and μ_C, ν_C are called intuitionistic output function.

The instruction $\mu_A(p_1, \theta \oplus \sigma, p_2)(\nu_A(p_1, \theta \oplus \sigma, p_2))$ stands for the current state p_1, with inputting the multiset θ or σ being scanned, can go to the state p_2, that is, the selection of intuitionistic fuzzy mutiset relation between applicable instructions (p_1, θ, p_2) and (p_1, σ, p_2) possibly being scanned by different parts of the inputting multiset is non-deterministic. A configuration of an IFMFA is described by $(p, \theta) \in (Q, \Sigma^{\oplus})$. The IFMFA is non-deterministic, so there may be several transition relations that are possible in a given configuration. Thus, for an IFMFA $\mathcal{M}$ and its two configurations (p_1, θ) and (p_2, σ), we define a move from (p_1, θ) to (p_2, σ) with degree of membership and nonmembership $u, v \in [0, 1]$, written as $(p_1, \theta) \xrightarrow{(u,v)} (p_2, \sigma)$ or $\mu_{\mathcal{M}}(p_1, \theta) \to (p_2, \sigma)) = u$ and $\nu_{\mathcal{M}}(p_1, \theta) \to (p_2, \sigma)) = v$. , if there exists a multiset $\omega \in \Sigma^{\oplus}$ with $\omega \subseteq \theta$ such that $\mu_A(p_1, \omega, p_2) = u$ and $\nu_A(p_1, \omega, p_2) = v$. We use $\to^*$ to denote reflexive and transitive closure of $\to$. Let $(p_1, \theta), (p_2, \sigma) \in (Q, \Sigma^{\oplus})$, we have $(p_1, \theta) \xrightarrow{(l,k)^*} (p_2, \sigma)$ if there exists $n + 1 (n \ge 1)$ configurations $(p_1, \theta), (q_1, \theta_1), \ldots, (q_{n-1}, \theta_{n-1}), (p_2, \sigma)$, such that $(p_1, \theta) \xrightarrow{(l_1,k_1)} (q_1, \theta_1) \xrightarrow{(l_2,k_2)} \ldots \xrightarrow{(l_{n-1},k_{n-1})} (q_{n-1}, \theta_{n-1}) \xrightarrow{(l_n,k_n)} (p_2, \sigma), l_i, k_i \in [0, 1]$, then $l = l_1 \wedge l_2 \wedge \ldots \wedge l_{n-1} \vee l_n$ and $k = k_1 \vee k_2 \vee \ldots \vee k_n$. Naturally, the membership degrees and nonmembership degrees of a configuration (p_1, θ), which derive another configuration (p_2, σ), are expressed as follows:

$$\mu_M((p_1,\theta) \rightarrow^* (p_2,\sigma)) = \vee \{\mu_{\mathcal{M}}((p_1,\theta)) \rightarrow^* (s,\theta \ominus \omega)) \wedge \mu_{\mathcal{M}}((s,\theta \ominus \omega)) \rightarrow^* (p_2,\sigma)) \,|\, s \in Q, \omega \in \Sigma^{\oplus}, \omega \subseteq \theta\},$$

$$\nu_M((p_1,\theta) \rightarrow^* (p_2,\sigma)) = \wedge \{\nu_{\mathcal{M}}((p_1,\theta)) \rightarrow^* (s,\theta \ominus \omega)) \vee \nu_{\mathcal{M}}((s,\theta \ominus \omega)) \rightarrow^* (p_2,\sigma)) \,|\, s \in Q, \omega \in \Sigma^{\oplus}, \omega \subseteq \theta\},$$

and

$$\mu_{\mathcal{M}}((p_1,\theta)) \rightarrow^* (p_2,\theta)) = \begin{cases} 1 & \text{if } p_1 = p_2 \\ 0 & \text{if } p_1 \neq p_2 \end{cases}, \qquad (2)$$

$$\nu_{\mathcal{M}}((p_1,\theta)) \rightarrow^* (p_2,\theta)) = \begin{cases} 0 & \text{if } p_1 = p_2 \\ 1 & \text{if } p_1 \neq p_2 \end{cases}. \qquad (3)$$

Definition 6. *Let $\mathcal{M} = (Q,\Sigma,A,B,C)$ be an IFMFA. Let $p,q \in Q$. Then p is called an immediate successor of q with threshold (a,b) if there exists $\alpha \in \Sigma$ such that $\mu_{\mathcal{M}}((q,\alpha) \rightarrow (p,0_\Sigma)) \geq a$ and $\nu_{\mathcal{M}}((q,\alpha) \rightarrow (p,0_\Sigma)) \leq b$. Also, we say that p is a successor of q with threshold (a,b) if there exists $\beta \in \Sigma^{\oplus}$ such that $\mu_{\mathcal{M}}((q,\beta) \rightarrow^* (p,0_\Sigma)) \geq a$ and $\nu_{\mathcal{M}}((q,\beta) \rightarrow (p,0_\Sigma)) \leq b$, where $a,b \in [0,1]$ and $0 \leq a+b \leq 1$. We say that p is a strong successor of q if and only if p is a successor of q with threshold $(1,0)$. If p is a strong successor of q, then p is a successor of q with threshold arbitrary (a,b), where $0 \leq a+b \leq 1$ and $a,b \in [0,1]$.*

Accordingly, let $a,b \in [0,1]$ and $0 \leq a+b \leq 1$.

Theorem 1. *Let $\mathcal{M} = (Q,\Sigma,A,B,C)$ be a IFMFA and $p,q,r \in Q$. Then the following holds:*
1. q is a strong successor of q,
2. if p is a successor of q with threshold (a_1,b_1) and r is a successor of p with threshold (a_2,b_2), then r is a successor of q with threshold (a,b), where $a = a_1 \wedge a_2, b = b_1 \vee b_2$.

Proof. 1. By definitions $\mu_{\mathcal{M}}$ and $\nu_{\mathcal{M}}$, we have $\mu_{\mathcal{M}}((q,0_\Sigma) \longrightarrow (q,0_\Sigma)) = 1 \geq 1$ and $\nu_{\mathcal{M}}((q,0_\Sigma) \longrightarrow (q,0_\Sigma)) = 0 \leq 0$. Then q is a strong successor of q. 2. Since p is a successor of q with threshold (a_1,b_1), then there exists $\alpha \in \Sigma^{\oplus}$ such that $\mu_{\mathcal{M}}((q,\alpha) \longrightarrow^* (p,0_\Sigma)) \geq a_1$ and $\nu_{\mathcal{M}}((q,\alpha) \longrightarrow^* (p,0_\Sigma)) \leq b_1$. Also, r is a successor of p with threshold (a_2,b_2), then there exists $\beta \in \Sigma^{\oplus}$ such that $\mu_{\mathcal{M}}((p,\beta) \longrightarrow^* (r,0_\Sigma)) \geq a_2$ and $\nu_{\mathcal{M}}((p,\beta) \longrightarrow^* (r,0_\Sigma)) \leq b_2$. So,

$$\mu_{\mathcal{M}}((q,\alpha \oplus \beta) \longrightarrow^* (r,0_\Sigma)) = \vee_{s \in Q} \mu_{\mathcal{M}}((q,\alpha \oplus \beta)) \longrightarrow^* (s,\beta)) \wedge \mu_{\mathcal{M}}((s,\beta) \longrightarrow^* (r,0_\Sigma))$$
$$\geq \mu_{\mathcal{M}}((q,\alpha \oplus \beta)) \longrightarrow^* (p,\beta)) \wedge \mu_{\mathcal{M}}((p,\beta) \longrightarrow^* (r,0_\Sigma)) \geq a_1 \wedge a_2 = a,$$

also,

$$\nu_{\mathcal{M}}((q,\alpha \oplus \beta) \longrightarrow^* (r,0_\Sigma)) = \wedge_{s \in Q} \nu_{\mathcal{M}}((q,\alpha \oplus \beta)) \longrightarrow^* (s,\beta)) \vee \nu_{\mathcal{M}}((s,\beta) \longrightarrow^* (r,0_\Sigma))$$
$$\leq \nu_{\mathcal{M}}((q,\alpha \oplus \beta)) \longrightarrow^* (p,\beta)) \vee \nu_{\mathcal{M}}((p,\beta) \longrightarrow^* (r,0_\Sigma)) \leq b_1 \vee b_2 = b.$$

Hence, the claim holds.

Definition 7. *Let $\mathcal{M} = (Q,\Sigma,A,B,C)$ be an IFMFA and $q \in Q$. We denote by $S^{(a,b)}(q)$ the set of all successor of q with threshold (a,b).*

Definition 8. *Let $\mathcal{M} = (Q,\Sigma,A,B,C)$ be an IFMFA and $T \subseteq Q$. The set of all successors of T with threshold (a,b) is defined as follows: $S^{(a,b)}(T) = \cup_{q \in T} S^{(a,b)}(q)$.*

Theorem 2. *Let $\mathcal{M} = (Q, \Sigma, A, B, C)$ be an IFMFA, and E and F be the subsets of Q. Then the following holds:*

1. If $E \subseteq F$, then $S^{(a,b)}(E) \subseteq S^{(a,b)}(F)$,
2. $E \subseteq S^{(a,b)}(E)$,
3. $S^{(a,b)}(S^{(a,b)}(E)) = S^{(a,b)}(E)$,
4. $S^{(a,b)}(E \cup F) = S^{(a,b)}(E) \cup S^{(a,b)}(F)$,
5. $S^{(a,b)}(E \cap F) \subseteq S^{(a,b)}(E) \cap S^{(a,b)}(F)$.

P*roof.* Proving 1, 4 and 5 are simple. 2. Let $q \in E$. Then $\mu_{\mathcal{M}}((q, 0_\Sigma) \longrightarrow^* (q, 0_\Sigma)) = 1 \geq a$ and $\nu_{\mathcal{M}}((q, 0_\Sigma) \longrightarrow^* (q, 0_\Sigma)) = 0 \leq b$. Then q is a successor of q with threshold (a, b). So, $q \in S^{(a,b)}(E)$. Hence, the claim holds. 3. Let $q \in S^{(a,b)}(E)$. Then there exists $p \in E$ such that q is a successor of p with threshold (a, b). On the other hand, p is a successor of p with threshold (a, b). Then $q \in S^{(a,b)}(p)$ and $p \in S^{(a,b)}(p)$. So, $q \in S^{(a,b)}(S^{(a,b)}(p))$. Now, let $q \in S^{(a,b)}(S^{(a,b)}(E))$. Then there exists $p \in S^{(a,b)}(E)$ such that q is a successor of p with threshold (a, b). Also, there exists $r \in E$ such that p is a successor of r with threshold (a, b). So, q is a successor of r with threshold (a, b) that it means $q \in S^{(a,b)}(E)$.

In the next example, we show that $S^{(a,b)}(E \cap F) \neq S^{(a,b)}(E) \cap S^{(a,b)}(F)$.

Example 1. *Let $\mathcal{M} = (Q, \Sigma, A, B, C)$ be an IFMFA. Let $Q = \{p_1, p_2, p_3, p_4, p_5\}$, and $\Sigma = \{u, v\}$. Then A is defined as follows:*

$A(p_1, <u>, p_1) = (0.4, 0.6)$, $A(p_2, <u>, p_1) = (0.8, 0.1)$,
$A(p_3, <u> \oplus <v>, p_1) = (0.9, 0.1)$, $A(p_3, <u>, p_2) = (0.1, 0.9)$,
$A(p_4, <u> \oplus <v>, p_5) = (0.2, 0.3)$, $A(p_4, <v> \oplus <v>, p_4) = (0.5, 0.5)$,
$A(p_5, <u> \oplus <v>, p_5) = (0.2, 0.3)$, $A(p_5, <v>, p_4) = (0.5, 0.5)$.

Let $E = \{p_2\}$ and $F = \{p_3\}$ and $a = 0.2, b = 0.8$. Then we have $S^{(0.2,0.8)}(E) = \{p_1\}, S^{(0.2,0.8)}(F) = \{p_1\}$. Then $S^{(0.2,0.8)}(E) \cap S^{(0.2,0.8)}(F) = \{p_1\}$, but $E \cap F = \emptyset$ so $S^{(0.2,0.8)}(\emptyset) = \emptyset$. Hence, $S^{(0.2,0.8)}(E) \cap S^{(0.2,0.8)}(F) \neq S^{(0.2,0.8)}(E \cap F)$.

Definition 9. *Let $\mathcal{M} = (Q, \Sigma, A, B, C)$ be an IFMFA. Let $p, q \in Q$ and $T \subseteq Q$. We say that $\mathcal{M}$ is a swap IFMFA with threshold (a, b) if $p \in S^{(a,b)}(T \cup \{q\})$, $p \notin S^{(a,b)}(T)$, then $q \in S^{(a,b)}(T \cup \{p\})$.*

Theorem 3. *Let $\mathcal{M} = (Q, \Sigma, A, B, C)$ be an IFMFA. Then the following are equivalent:*

1. $\mathcal{M}$ is a swap IFMFA with threshold (a, b),
2. $q \in S^{(a,b)}(p)$ if and only if $p \in S^{(a,b)}(q)$, for every $p, q \in Q$.

Proof. $1 \rightarrow 2$. Let $p, q \in Q$ and $q \in S^{(a,b)}(p)$. Also, $q \notin S^{(a,b)}(\emptyset)$. Since $\mathcal{M}$ is a swap IFMFA, then $p \in S^{(a,b)}(q)$. Hence, $q \in S^{(a,b)}(p)$ if and only if $p \in S^{(a,b)}(q)$. $2 \rightarrow 1$. Let $p, q \in Q$ and $T \subseteq Q$. Let $p \in S^{(a,b)}(T \cup \{q\})$ and $p \notin S^{(a,b)}(T)$. Then $p \in S^{(a,b)}(q)$. So, $q \in S^{(a,b)}(p)$. Hence, the claim holds.

Definition 10. *Let $\mathcal{M} = (Q, \Sigma, A, B, C)$ be an IFMFA and $T \subseteq Q$. Let A_T be a intuitionistic fuzzy subset of $T \times \Sigma \times T$, B_T be a intuitionistic fuzzy subset of T, C_T be a intuitionistic fuzzy subset of T and suppose that $\mathcal{N} = (T, \Sigma, A_T, B_T, C_T)$. The IFMFA $\mathcal{N}$ is called an intuitionistic fuzzy multiset finite subautomata (IFMFSA) of $\mathcal{M}$ if:*

1. $A|_{T \times \Sigma \times T} = A_T$,

2. $S^{(a,b)}(T) \subseteq T$,
3. $B|_T = B_T$,
4. $C|_T = C_T$.

It is clear that if $\mathcal{M}$ is an intuitionistic fuzzy multiset finite subautomata of $\mathcal{N}$ with threshold (a_1, b_1) and $\mathcal{N}$ is an intuitionistic fuzzy multiset finite subautomata of $\mathcal{R}$ with threshold (a_2, b_2), then $\mathcal{M}$ is an IFMFSA of $\mathcal{R}$ with threshold (a, b), where $a = a_1 \wedge a_2$ and $b = b_1 \vee b_2$.

Theorem 4. *Let $\mathcal{M} = (Q, \Sigma, A, B, C)$ be an IFMFA. Let $\mathcal{M}_i = (Q_i, \Sigma, A_i, B_i, C_i), i \in I$ be a family of IFMFSA of $\mathcal{M}$ with threshold (a, b), where $Q_i \subseteq Q$. Then the following claim holds:*

1. $\cap_{i\in I} \mathcal{M}_i = (\cap_{i\in I} Q_i, \Sigma, A', B', C')$ is an IFMFSA of $\mathcal{M}$ with threshold (a, b), where $A' = A|_{\cap_{i\in I} Q_i \times \Sigma \times \cap_{i\in I} Q_i}$, $B' = B|_{\cap_{i\in I} Q_i}$ and $C' = C|_{\cap_{i\in I} Q_i}$.
2. $\cup_{i\in I} \mathcal{M}_i = (\cup_{i\in I} Q_i, \Sigma, A'', B'', C'')$ is an IFMFSA of $\mathcal{M}$ with threshold (a, b), where $A'' = A|_{\cup_{i\in I} Q_i \times \Sigma \times \cup_{i\in I} Q_i}$, $B'' = B|_{\cup_{i\in I} Q_i}$ and $C'' = C|_{\cup_{i\in I} Q_i}$.

Proof. I. By Theorem 2, we have $S^{(a,b)}(\cap_{i\in I} Q_i) \subseteq \cap_{i\in I} S^{(a,b)}(Q_i)$. On the other hand, $\mathcal{M}_i$ is IFMFSA of $\mathcal{M}$ with threshold (a, b), for every $i \in I$, then $S^{(a,b)}(Q_i) \subseteq Q_i$, for every $i \in I$. So, $S^{(a,b)}(\cap_{i\in I} Q_i) \subseteq \cap_{i\in I} (Q_i)$. Hence, the claim holds. II. By Theorem 2, we have $S^{(a,b)}(\cup_{i\in I} Q_i) = \cup_{i\in I} S^{(a,b)}(Q_i)$. Also, $S^{(a,b)}(Q_i) \subseteq Q_i$, for every $i \in I$. Then $S^{(a,b)}(\cup_{i\in I} Q_i) \subseteq \cup_{i\in I} Q_i$. Therefore, the claim holds.

Definition 11. *Let $\mathcal{M} = (Q, \Sigma, A, B, C)$ be an IFMFA. Then $\mathcal{M}$ is called strongly connected with threshold (a, b) if for every $p, q \in Q$, $p \in S^{(a,b)}(q)$.*

Definition 12. *Let $\mathcal{M} = (Q, \Sigma, A, B, C)$ be an IFMFA and $\mathcal{N} = (T, \Sigma, A', B', C')$ be an IFMFSA of $\mathcal{M}$ with threshold (a, b). Then we say that $\mathcal{N}$ is nontrivial if $T \neq Q$ and $T \neq \emptyset$.*

Theorem 5. *Let $\mathcal{M} = (Q, \Sigma, A, B, C)$ be an IFMFA. Then $\mathcal{M}$ is strongly connected with threshold (a, b) if and only if $\mathcal{M}$ has not nontrivial IFMSA.*

Proof. Let $\mathcal{M}$ be strongly connected with threshold (a, b) and $\mathcal{N} = (T, \Sigma, A', B', C')$ be an IFMSA of $\mathcal{M}$ with threshold (a, b), where $T \subseteq Q$ and $T \neq \emptyset$. Then there exists $q \in T$. Let $p \in Q$. Since $\mathcal{M}$ is strongly connected with threshold (a, b), then $p \in S^{(a,b)}(q)$. So, $p \in S^{(a,b)}(q) \subseteq S^{(a,b)}(T) \subseteq T$. Therefore, $Q \subseteq T$. Hence, $T = Q$ and $\mathcal{M} = \mathcal{N}$. Now, let $\mathcal{M}$ has not nontrivial IFMFSA with threshold (a, b). Let $p, q \in Q$ and let $\mathcal{N} = (S^{(a,b)}(q), \Sigma, A', B', C')$, where $A' = A|_{S^{(a,b)}(q) \times \Sigma \times S^{(a,b)}(q)}$, $B' = B|_{S^{(a,b)}(q)}$ and $C' = C|_{S^{(a,b)}(q)}$. Then $\mathcal{N}$ is an IFMFSA of $\mathcal{M}$ with threshold (a, b). Also, $q \in S^{(a,b)}(q)$, then $S^{(a,b)}(q) \neq \emptyset$. So, $S^{(a,b)}(q) = Q$. Therefore, $p \in S^{(a,b)}(q)$. Hence, $\mathcal{M}$ is strongly connected with threshold (a, b).

Proposition 1. *Let $\mathcal{M} = (Q, \Sigma, A, B, C)$ be an IFMFA and $T \subseteq Q$. Then $\mathcal{N} = (S^{(a,b)}(T), \Sigma, A_T, B_T, C_T)$ is an IFMFSA of $\mathcal{M}$ with threshold (a, b), where $A_T = A|_{S^{(a,b)}(T) \times \Sigma^{\oplus} \times S^{(a,b)}(T)}$, $B_T = B|_{S^{(a,b)}(T)}$ and $C_T = C|_{S^{(a,b)}(T)}$.*

Definition 13. *Let $\mathcal{M} = (Q, \Sigma, A, B, C)$ be an IFMFA. Let $T \subseteq Q$ and $\{\mathcal{N}_i | i \in I\}$ be the collection of all IFMFSA with threshold (a, b) of $\mathcal{M}$, where state set contains T. Let $\ll T \gg^{(a,b)} = \cap_{i\in I} \{\mathcal{N}_i | i \in I\}$. Then $\ll T \gg^{(a,b)}$ is called the intuitionistic fuzzy multiset submachine generated by T with threshold (a, b).*

Clearly, $\ll T \gg^{(a,b)}$ is the smallest IFMFSA of $\mathcal{M}$ with threshold (a,b) whose state set contains T.

Definition 14. *Let $\mathcal{M} = (Q, \Sigma, A, B, C)$ be an IFMFA and*

$$L^{(a,b)}(\mathcal{M}) = \{\alpha \in \Sigma^{\oplus} | \mu_A(p) \wedge \mu_{\mathcal{M}}((p,\alpha) \to^* (q, 0_\Sigma)) \wedge \mu_C(q) \geq a,$$

$$\nu_A(p) \vee \nu_{\mathcal{M}}((p,\alpha) \to^* (q, 0_\Sigma)) \vee \nu_C(q) \leq b, \text{forsome } p, q \in Q\}.$$

Then $L^{(a,b)}(\mathcal{M})$ is called the multiset language with threshold (a,b) recognized by IFMFA $\mathcal{M}$.

Theorem 6. *Let $\mathcal{M} = (Q, \Sigma, A, B, C)$ be an IFMFA and $T \subseteq Q$. Let $\mathcal{M}_T = (S^{(a,b)}(T), \Sigma, A_T, B_T, C_T)$, where $A_T = A|_{S^{(a,b)}(T)\times\Sigma\times S^{(a,b)}(T)}, B_T = B|_{S^{(a,b)}(T)}$ and $C_T = C|_{S^{(a,b)}(T)}$. Then $L^{(a\prime,b\prime)}(\mathcal{M}_T) = L^{(a\prime,b\prime)}(\ll T \gg^{(a,b)})$, where $a' \geq a$ and $b' \leq b$, $a', b' \in [0,1]$ and $0 \leq a' + b' \leq 1$.*

Proof. Let $\ll T \gg^{(a,b)} = (\cap_{i\in I} Q_i, \Sigma, A', B', C')$, where $\{\mathcal{N}_i | i \in I\}$ is the collection of all IFMFSA of $\mathcal{M}$ with threshold (a,b) whose state set contains T and $\mathcal{N}_i = (Q_i, \Sigma, A_i, B_i, C_i), A' = A|_{\cap_{i\in I} Q_i \times\Sigma\times \cap_{i\in I} Q_i}, B' = B|_{\cap_{i\in I} Q_i}$ and $C' = C|_{\cap_{i\in I} Q_i}$. Now, we show that $S^{(a,b)}(T) = \cap_{i\in I} Q_i$. $(S^{(a,b)}(T), \Sigma, A_T, B_T, C_T)$ is an IFMFSA of $\mathcal{M}$ with threshold (a,b) and $T \subseteq S^{(a,b)}(T)$. So, $\cap_{i\in I} Q_i \subseteq S^{(a,b)}(T)$. Then there exists $t \in T$ and $\alpha \in \Sigma^{\oplus}$ such that $\mu_{\mathcal{M}}((t,\alpha) \longrightarrow^* (p, 0_\Sigma)) \geq 0$ and $\nu_{\mathcal{M}}((t,\alpha) \longrightarrow^* (p, 0_\Sigma)) \leq 1$. On the other hand, $t \in \cap_{i\in I} Q_i$ and since $\ll T \gg^{(a,b)}$ is an IMFSA of $\mathcal{M}$ with threshold (a,b), then $p \in \cap_{i\in I} Q_i$. So, $S^{(a,b)}(T) \subseteq \cap_{i\in I} Q_i$. Therefore, $S^{(a,b)}(T) = \cap_{i\in I} Q_i$. So, $\ll T \gg^{(a,b)} = \mathcal{M}_T$. Hence, $L^{(a\prime,b\prime)}(\mathcal{M}_T) = L^{(a\prime,b\prime)}(\ll T \gg^{(a,b)})$

Definition 15. *Let $\mathcal{M} = (Q, \Sigma, A, B, C)$ be an IFMFA. $\mathcal{M}$ is called intuitionistic single generated if there exists $q \in Q$ such that $\mathcal{M} = \ll \{q\} \gg^{(a,b)}$. In this case, q is called an intuitionistic generator of $\mathcal{M}$ and we say that $\mathcal{M}$ is generated by q.*

Theorem 7. *Let $\mathcal{M} = (Q, \Sigma, A, B, C)$ be an IFMFA and $T, R \subseteq Q$. Let $\ll R \cup T \gg^{(a,b)} = (S^{(a,b)}(R \cup T), \Sigma, A', B', C')$, such that $A' = A|_{S^{(a,b)}(R\cup T)\times\Sigma\times S^{(a,b)}(R\cup T)}, B' = B|_{S^{(a,b)}(R\cup T)}, C' = C|_{S^{(a,b)}(R\cup T)}$, and $\ll R \gg^{(a,b)} \cup \ll T \gg^{(a,b)} = (S^{(a,b)}(R) \cup S^{(a,b)}(T), \Sigma, A'', B'', C'')$, such that A'=A |_S^(a, b)(R) S^(a, b)(T) S^(a, b)(R) S^(a, b)(T), $B' = B|_{S^{(a,b)}(R)\cup S^{(a,b)}(T)}$ and $C' = C|_{S^{(a,b)}(R)\cup S^{(a,b)}(T)}$. Then $L^{(a\prime,b\prime)}(\ll R \cup T \gg^{(a,b)}) = L^{(a\prime,b\prime)}(\ll R \gg^{(a,b)} \cup \ll T \gg^{(a,b)})$, where $a' \geq a, b' \leq b$ and $a', b' \in [0,1]$ and $0 \leq a' + b' \leq 1$.*

Proof. Let $\alpha \in L^{(a\prime,b\prime)}(\ll R \cup T \gg^{(a,b)})$. Then there exist $p, q \in Q$ such that

$$\mu_{A\prime}(p) \wedge \mu_{\ll R\cup T \gg^{(a,b)}}((p,\alpha) \to^* (q, 0_\Sigma)) \wedge \mu_{C\prime}(q) \geq a',$$

$$\nu_{A\prime}(p) \vee \nu_{\ll R\cup T \gg^{(a,b)}}((p,\alpha) \to^* (q, 0_\Sigma)) \vee \nu_{C\prime}(q) \leq b'.$$

By considering Theorem 2, we have $S^{(a,b)}(R \cup T) = S^{(a,b)}(R) \cup S^{(a,b)}(T)$. Now,

$$A_{S^{(a,b)}(R\cup T)} = A|_{S^{(a,b)}(R\cup T)\times\Sigma^{\oplus}\times S^{(a,b)}(R\cup T)}$$

$$= A|_{S^{(a,b)}(R)\cup S^{(a,b)}(T)\times\Sigma^{\oplus}\times S^{(a,b)}(R)\cup S^{(a,b)}(T)}$$

$$= A_{S^{(a,b)}(R) \cup S^{(a,b)}(T)},$$

similarly, $B|_{S^{(a,b)}(R \cup T)} = B|_{S^{(a,b)}(R) \cup S^{(a,b)}(T)}$ and $C|_{S^{(a,b)}(R \cup T)} = C|_{S^{(a,b)}(R) \cup S^{(a,b)}(T)}$. So, it is clear that,

$$\mu_{A''}(p) \wedge \mu_{\ll R \gg^{(a,b)} \cup \ll T \gg^{(a,b)}}((p, \alpha) \to^* (q, 0_\Sigma)) \wedge \mu_{C''}(q) \geq a',$$

$$\nu_{A''}(p) \vee \nu_{\ll R \gg^{(a,b)} \cup \ll T \gg^{(a,b)}}((p, \alpha) \to^* (q, 0_\Sigma)) \vee \nu_{C''}(q) \leq b'.$$

So, $L^{(a',b')}(\ll R \cup T \gg^{(a,b)}) \subseteq L^{(a',b')}(\ll R \gg^{(a,b)} \cup \ll T \gg^{(a,b)})$. Similarly, $L^{(a',b')}(\ll R \gg^{(a,b)} \cup \ll T \gg^{(a,b)}) \subseteq L^{(a',b')}(\ll R \cup T \gg^{(a,b)})$. Hence, the claim holds.

Theorem 8. *Let $\mathcal{M} = (Q, \Sigma, A, B, C)$ be an IFMFA and $T, R \subseteq Q$. Let $\ll R \cap T \gg^{(a,b)} = (S^{(a,b)}(R \cap T), \Sigma, A', B', C')$, such that $A' = A|_{S^{(a,b)}(R \cap T) \times \Sigma \times S^{(a,b)}(R \cap T)}$, $B' = B|_{S^{(a,b)}(R \cap T)}$, $C' = C|_{S^{(a,b)}(R \cap T)}$, and $\ll R \gg^{(a,b)} \cap \ll T \gg^{(a,b)} = (S^{(a,b)}(R) \cap S^{(a,b)}(T), \Sigma, A'', B'', C'')$, such that A'=A |_S^(a, b)(R) S^(a, b)(T) S^(a, b)(R) S^(a, b)(T), $B' = B|_{S^{(a,b)}(R) \cap S^{(a,b)}(T)}$ and $C' = C|_{S^{(a,b)}(R) \cap S^{(a,b)}(T)}$. Then $L^{(a',b')}(\ll R \cap T \gg^{(a,b)}) \subseteq L^{(a',b')}(\ll R \gg^{(a,b)} \cap \ll T \gg^{(a,b)})$, where $a' \geq a, b' \leq b$ and $a', b' \in [0,1]$ and $0 \leq a' + b' \leq 1$.*

Proof. By Theorem 2, we have $S^{(a,b)}(R \cap T) \subseteq S^{(a,b)}(R) \cap S^{(a,b)}(T)$. Let $\alpha \in L^{(a',b')}(\ll R \cap T \gg^{(a,b)})$. Then there exists $p, q \in S^{(a,b)}(R \cap T)$ such that

$$\mu_{A'}(p) \wedge \mu_{\ll R \cap T \gg^{(a,b)}}((p, \alpha) \to^* (q, 0_\Sigma)) \wedge \mu_{C'}(q) \geq a',$$
$$\nu_{A'}(p) \vee \nu_{\ll R \cap T \gg^{(a,b)}}((p, \alpha) \to^* (q, 0_\Sigma)) \vee \nu_{C'}(q) \leq b'.$$

By considering $S^{(a,b)}(R \cap T) \subseteq S^{(a,b)}(R) \cap S^{(a,b)}(T)$, we have $p, q \in S^{(a,b)}(R) \cap S^{(a,b)}(T)$. So,

$$\mu_{A''}(p) \wedge \mu_{\ll R \gg^{(a,b)} \cap \ll T \gg^{(a,b)}}((p, \alpha) \to^* (q, 0_\Sigma)) \wedge \mu_{C''}(q) \geq a',$$
$$\nu_{A''}(p) \vee \nu_{\ll R \gg^{(a,b)} \cap \ll T \gg^{(a,b)}}((p, \alpha) \to^* (q, 0_\Sigma)) \vee \nu_{C''}(q) \leq b'.$$

Hence, the claim holds.

Definition 16. *Let $\mathcal{M} = (Q, \Sigma, A, B, C)$ be an IFMFA and $\mathcal{M}_i = (Q_i, \Sigma, A_i, B_i, C_i), i = 1, 2$ be two IFMFSA of $\mathcal{M}$ with threshold (a, b). If $\mathcal{M} = \ll Q_1 \cup Q_2 \gg^{(a,b)}$, then we say that $\mathcal{M}$ is the union of $\mathcal{M}_1$ and $\mathcal{M}_2$ with threshold (a, b) and we write $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$. If $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$ and $Q_1 \cap Q_2 = \emptyset$, then we say that $\mathcal{M}$ is the direct union of $\mathcal{M}_1$ and $\mathcal{M}_2$ with threshold (a, b) and we write $\mathcal{M} = \mathcal{M}_1 \sqcup_{(a,b)} \mathcal{M}_2$.*

Suppose $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$. Then $S^{(0,1)}(Q_i) = Q_i$ in $\mathcal{M}$, $i = 1, 2$. Since $\mathcal{M}_i$ is an IFMFSA of $\mathcal{M}, i = 1, 2$. Now, $S^{(1,0)}(Q_1 \cup Q_2) = S^{(1,0)}(Q_1) \cup S^{(1,0)}(Q_2) = Q_1 \cup Q_2$.

Definition 17. *Let $\mathcal{M} = (Q, \Sigma, A, B, C)$ be an IFMFA and $T \subseteq Q$. T is called free with threshold (a, b) if and only if for every $t \in T$, $t \notin S^{(a,b)}(T \backslash \{t\})$.*

Definition 18. *Let $\mathcal{M} = (Q, \Sigma, A, B, C)$ be an IFMFA and $T \subseteq Q$. If T is free with threshold (a, b) and $\mathcal{M} = \ll T \gg^{(a,b)}$, then T is called a basis of $\mathcal{M}$.*

Theorem 9. *Let $\mathcal{M} = (Q, \Sigma, A, B, C)$ be a swap IFMFA with threshold (a, b). Let $\{q_1, q_2, \ldots, q_n\}$ be a basis of $\mathcal{M}$ with threshold (a, b). Then $L^{(a',b')}(\mathcal{M}) = L^{(a',b')}(\ll q_1 \gg^{(a,b)} \sqcup \ll \{q_2\} \gg^{(a,b)} \sqcup \ldots \sqcup \ll q_n \gg^{(a,b)})$.*

Proof. We have $\ll q_i \gg^{(a,b)} = (S^{(a,b)}(q_i), \Sigma, A_i, B_i, C_i)$, where $A_i = A|_{S^{(a,b)}(q_i)\times\Sigma\times S^{(a,b)}(q_i)}$, $B_i = B|_{S^{(a,b)}(q_i)}$ and $C_i = C|_{S^{(a,b)}(q_i)}$. Now, let $i \neq j$. Then $S^{(a,b)}(q_i) \cap S^{(a,b)}(q_j) = \emptyset$. Also, for every $p, q \in Q$, $p \in S^{(a,b)}(q)$ if and only if $q \in S^{(a,b)}(p)$. Since $\mathcal{M} = \ll q_1, q_2, \ldots, q_n \gg^{(a,b)}$, then $\mathcal{M} = \ll q_1 \gg^{(a,b)} \sqcup \ll q_2 \gg^{(a,b)} \sqcup \ldots \sqcup \ll q_n \gg$. Hence, $L^{(a\prime,b\prime)}(\mathcal{M}) = L^{(a\prime,b\prime)}(\ll q_1 \gg^{(a,b)} \sqcup \ll \{q_2\} \gg^{(a,b)} \sqcup \ldots \sqcup \ll q_n \gg^{(a,b)})$.

Theorem 10. *Let $\mathcal{M} = (Q, \Sigma, A, B, C)$ be a swap IFMFA with threshold (a, b). Then $\mathcal{M}$ has a basis with threshold (a, b) and the cardinality of the basis is unique.*

Proof. Let $T_1 = Q$ and $p, q \in Q$. If $p \in S^{(a,b)}(q)$ and $q \in S^{(a,b)}(p)$. Consider $T_2 = T_1 \backslash \{p\}$. Then $p \in S^{(a,b)}(T_2)$. If T_2 is the smallest generator of $\mathcal{M}$ with threshold (a, b), the proof is complete, otherwise we continue in the same way until we reach T_k such that we cannot delete another state of T_k. So, T_k is the smallest generator with threshold (a, b).

4. Conclusion

In this paper, we have discussed the notion of lattice multiset finite automata and algebraic properties of intuitionistic fuzzy multiset finite automata. In fact, finite multiset automata processing has appeared frequently in various areas of mathematics, Petri nets, membrane computing, biology and biochemistry. Even, membrane computing has been connected with the theory of mealy multiset automata. The present study was an attempt to propose the notions of intuitionistic fuzzy multiset finite automata (IFMFA) and intuitionistic fuzzy multiset submachine generated by T. Subsequently, it was shown that the union and the intersection of a family of one IFMFSA are also an IFMFSA. Moreover, this research work proved that if IFMFA M has a basis, then the cardinality of the basis is unique. Multiset automata are connected with membrane computing, we will try to introduce the concept of membrane general fuzzy automata to explore the applications of deterministic intuitionistic general fuzzy multiset finite automata in the area of membrane computing.

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