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Research Article

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n-tuple Fixed Point Theorems Via α-series in C^*-algebra-valued Metric Spaces with an Application in Integral Equations

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Abstract

The purpose of this paper is to present some n-tuple fixed point and n-tuple coincidence point results in C^*-algebra-valued metric spaces using the concept of an α-series applied to a series of mappings. At the end of the paper, we give an example and an application to support our main results.

Keywords : C^*-algebra-valued; α-series; n-tuple fixed point; n-tuple coincidence point; Compatible; Weakly reciprocally continuous.

1 Introduction

Since 1922 (when Banach presented his famous result, known as the Banach fixed point theorem), the fixed point theory has piqued the interest of numerous researchers.

There is a great literature on this topic, and it is currently a very active research field. See [4, 5, 7, 8, 13, 15] for more information.

The concept of α-series was introduced by Sihag in 2014. This concept was later generalized to two-dimensional and three-dimensional fixed point theorems in generalized metric spaces (see [5, 11, 14] for more information).

The concept of a C^*-algebra valued metric space was introduced in 2014, in which, instead of the set of real numbers, the set of all positive elements of a unit C^*-algebra was used. Later, coupled fixed point theorems in C^*-algebra-valued b-metric spaces, as well as tripled fixed points and tripled coincidence points in C^*-algebra-valued metric spaces, were investigated in [1, 11], and [10], respectively.

The goal of this article is to prove some n-tuple coincidence point and fixed point theorems in C^*-algebra-valued metric spaces for a self-mapping g and a sequence $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ of n-variate mappings.

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An application and an example are also provided to support our results. In addition, we demonstrate the existence and uniqueness of a n-tuple common fixed point for a function $g : X \to X$ and the sequence of mappings $T_\zeta : X^n \to X$, where (X, A, d) represents a C^*-valued algebra metric space.

Throughout this article, $\mathbb{N}$ is the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. An unital algebra with the unit I is denoted by A and θ is the zero element. An involution on A correspond to the conjugate linear map $\kappa \mapsto \kappa^*$ on A is a mapping such that $\jmath^{**} = \jmath$ and $(\jmath\wp)^* = \wp^*\jmath^*$ for all $\jmath, \wp \in A$. The pair $(A, *)$ is called an $*$-algebra. A Banach $*$-algebra is an $*$-algebra A with the complete submultiplicative norm so that $\| \jmath^* \| = \| \jmath \|$, for all $\jmath \in A$. A C^*-algebra is a Banach $*$-algebra so that $\| \jmath^*\jmath \| = \| \jmath \|^2$ for all $\jmath \in A$. Let H be a Hilbert space and $B(H)$ be the set of all bounded linear operators on H. Then $B(H)$ is a C^*-algebra with the operator norm. Let A_{sa} be the family of all self-adjoint elements in A. An element $\jmath \in A$ is positive $(\jmath \geq \theta)$ if $\jmath \in A_{sa}$ and $\sigma(\jmath) = \{\lambda \in C \mid \lambda I - \jmath \textit{ is not invertible}\} \subseteq \mathbb{R}_+$, where $\sigma(\jmath)$ is the spectrum of $\jmath \in A$. Taking $A_+ = \{\jmath \in A : \jmath \geq \theta\}$, then $A_+ = \{\jmath^*\jmath : \jmath \in A\}$ [9]. Note that a partial order $\preceq$ on A_{sa} can be defined as follows:

$\jmath \preceq \wp$ if and only if $\wp - \jmath \succeq \theta$.

If $\jmath, \wp \in A_{sa}$ and $q \in A$, then $\jmath \leq \wp \Rightarrow q^*\jmath q \leq q^*\wp q$, and if $\jmath, \wp \in A_+$ are invertible, then $\jmath \leq \wp \Longrightarrow \theta \leq \wp^{-1} \leq \jmath^{-1}$.

2 Preliminaries

This section provides preliminaries that include definitions and results about the n-tuple fixed point for nonlinear contractive mappings defined on complete C^*-algebra-valued metric spaces.

Now, we state some basic definitions and results regarding C^*-algebra-valued metric spaces and tripled (and n-tuple) fixed point concepts.

Definition 2.1. *[7] If the function $d : X^2 \to A$ (X is a nonempty set) be such that for all $\upsilon, \varpi, \eta \in X$:*
(i) $\theta \leq d(\upsilon, \varpi)$ and $d(\upsilon, \varpi) = \theta$ iff $\upsilon = \varpi$;
(ii) $d(\upsilon, \varpi) = d(\varpi, \upsilon)$;
(iii) $d(\upsilon, \varpi) \leq d(\upsilon, \eta) + d(\eta, \varpi)$,
then (X, A, d) is called a C^*-algebra-valued metric space.

Definition 2.2. *[12] Let $F : X^3 \longrightarrow X$. We say that $(\upsilon, \iota, \kappa) \in X^3$ is a tripled fixed point of F if*

$$F(\upsilon, \iota, \kappa) = \upsilon,\ F(\iota, \upsilon, \iota) = \iota,\ F(\kappa, \iota, \upsilon) = \kappa.$$

Definition 2.3. *[12] Let (X, d) be a metric space. The function $\overline{d} : X^3 \to X$ which is given as follows*

$$\overline{d}\,[(\upsilon, \iota, \kappa), (u, v, w)] = d(\upsilon, u) + d(\iota, v) + d(\kappa, w),$$

is a metric on X^3. It will be denoted for convenience by d, too.

We provide related n-dimensional definitions (note that X is a nonempty set).

Definition 2.4. *[3] An element $(x^1, ..., x^n) \in X^n$ is called a n-tuple fixed point of $F : X^n \to X$ if*

$$x^i = F(x^i, x^{i+1}, ..., x^n, x^1, ..., x^{i-1}),$$

where $1 \leq i \leq n$.

Definition 2.5. *[3] An element $(x^1, ..., x^n) \in X^n$ is called a n-tuple coincidence point of $F : X^n \to X$ and $g : X \to X$ if*

$$gx^i = F(x^i, x^{i+1}, ..., x^n, x^1, ..., x^{i-1}),$$

where $1 \leq i \leq n$.

Definition 2.6. *[6] The mappings $F : X^n \to X$ and $g : X \to X$ are called commutating if*

$$gF(x^1, ..., x^n) = F(gx^1, ..., gx^n),$$

for all $x^1, \ldots, x^n \in X$.

We now generalize the definitions of compatibility and weakly reciprocal continuity, as presented in [2] and [3], to include all n-variate mappings F and self-mappings of g.

Definition 2.7. *Let (X, A, d) be a C^*-algebra-valued metric space. The mappings $F : X^n \to X$ and $g : X \longrightarrow X$ are called compatible if*

$$\lim_{\zeta\to+\infty} d\Big(g(F(x^i_\zeta, x^{i+1}_\zeta, \ldots, x^n_\zeta, x^1_\zeta, \ldots, x^{i-1}_\zeta)), F(gx^i_\zeta, gx^{i+1}_\zeta, \ldots, gx^n_\zeta, gx^1_\zeta, \ldots, gx^{i-1}_\zeta)\Big) = 0,$$

where $1 \leq i \leq n$, *whenever* $\{x^i_\zeta\}, 1 \leq i \leq n$ *are sequences in* X, *so that*

$$\lim_{\zeta\to+\infty} F(x^i_\zeta, x^{i+1}_\zeta, \ldots, x^n_\zeta, x^1_\zeta, \ldots, x^{i-1}_\zeta) = \lim_{\zeta\to+\infty} gx^i_\zeta := x^i,$$

for some $x^i \in X$. *They are called weakly compatible if*

$$gx^i = F(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}),$$

implies

$$g(F(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1})) = F(gx^i, gx^{i+1}, \ldots, gx^n, gx^1, \ldots, gx^{i-1}),$$

where $1 \leq i \leq n$, *for some* $(x^1, \cdots x^n) \in X^n$.

Definition 2.8. *The mappings* $F : X^n \to X$ *and* $g : X \longrightarrow X$ *are called reciprocally continuous if*

$$\lim_{\zeta\to+\infty} g(F(x^i_\zeta, x^{i+1}_\zeta, \ldots, x^n_\zeta, x^1_\zeta, \ldots, x^{i-1}_\zeta)) = gx^i$$

and

$$\lim_{\zeta\to+\infty} F(gx^i_\zeta, gx^{i+1}_\zeta, \ldots, gx^n_\zeta, gx^1_\zeta, \ldots, gx^{i-1}_\zeta) = F(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}),$$

whenever $\{x^i_\zeta\}, 1 \leq i \leq n$, *are sequences in* X, *such that*

$$\lim_{\zeta\to+\infty} F(x^i_\zeta, x^{i+1}_\zeta, \ldots, x^n_\zeta, x^1_\zeta, \ldots, x^{i-1}_\zeta) = \lim_{n\to+\infty} gx^i_\zeta := x^i,$$

for some $x^i \in X, 1 \leq i \leq n$.

Definition 2.9. *The mappings* $F : X^n \to X$ *and* $g : X \longrightarrow X$ *are called weakly reciprocally continuous if*

$$\lim_{\zeta\to+\infty} g(F(x^i_\zeta, x^{i+1}_\zeta, \ldots, x^n_\zeta, x^1_\zeta, \ldots, x^{i-1}_\zeta)) = gx^i,$$

or

$$\lim_{\zeta\to+\infty} F(gx^i_\zeta, gx^{i+1}_\zeta, \ldots, gx^n_\zeta, gx^1_\zeta, \ldots, gx^{i-1}_\zeta) = F(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}),$$

whenever $\{x^i_\zeta\}, 1 \leq i \leq n$, *are sequences in* X *such that*

$$\lim_{\zeta\to+\infty} F(x^i_\zeta, x^{i+1}_\zeta, \ldots, x^n_\zeta, x^1_\zeta, \ldots, x^{i-1}_\zeta) = \lim_{n\to+\infty} gx^i_\zeta := x^i,$$

for some $x^i \in X$ $1 \leq i \leq n$.

Definition 2.10. *[12] The nonempty set* X *is said to be regular provided that:*

(i) $x_\zeta \preceq x$ *for all* $\zeta \geq 0$, *where* $x_\zeta \to x$ *is a non-decreasing sequence,*

(ii) $y \preceq y_\zeta$ *for all* $\zeta \geq 0$ *where* $y_\zeta \to x$ *is a non-increasing sequence.*

3 Main results

In this section, we first state Definitions 2.7-2.9 for the sequence of n-variate mappings $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ and a self-mapping g as follows:

Definition 3.1. *Let* (X, A, d) *be a* C^**-algebra-valued metric space, and let* $T_\zeta : X^n \to X$ *and* $g : X \longrightarrow X$. $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ *and* g *are called compatible if*

$$\lim_{\zeta\to+\infty} d\Big(g(T_\zeta(x^i_\zeta, x^{i+1}_\zeta, \ldots, x^n_\zeta, x^1_\zeta, \ldots, x^{i-1}_\zeta)), T_\zeta(gx^i_\zeta, gx^{i+1}_\zeta, \ldots, gx^n_\zeta, gx^1_\zeta, \ldots, gx^{i-1}_\zeta)\Big) = 0,$$

where $1 \leq i \leq n$ *and* $\{x^i_\zeta\}, 1 \leq i \leq n$ *are sequences in* X *such that*

$$\lim_{\zeta\to+\infty} T_\zeta(x^i_\zeta, x^{i+1}_\zeta, \ldots, x^n_\zeta, x^1_\zeta, \ldots, x^{i-1}_\zeta) = \lim_{n\to+\infty} gx^i_{\zeta+1} := x^i,$$

for some $x^i \in X$. *They are said to be weakly compatible if*

$$gx^i = T_\zeta(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}),$$

implies

$$g(T_\zeta(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1})) = T_\zeta(gx^i, gx^{i+1}, \ldots, gx^n, gx^1, \ldots, gx^{i-1}),$$

where $1 \leq i \leq n$.

Definition 3.2. *Let (X, A, d) be a C^*-algebra-valued metric space, and $T_\zeta : X^n \to X$ and $g : X \longrightarrow X$. $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ and g are called weakly reciprocally continuous if*

$$\lim_{\zeta\to+\infty} g(T_\zeta(x_\zeta^i, x_\zeta^{i+1}, \ldots, x_\zeta^n, x_\zeta^1, \ldots, x_\zeta^{i-1})) = gx^i,$$

whenever $\{x_\zeta^i\}, 1 \le i \le n$ are sequences in X, such that

$$\lim_{\zeta\to+\infty} T_\zeta(x_\zeta^i, x_\zeta^{i+1}, \ldots, x_\zeta^n, x_\zeta^1, \ldots, x_\zeta^{i-1}) = \lim_{n\to+\infty} gx_{\zeta+1}^i := x^i,$$

for some $x^i \in X$ and $1 \le i \le n$.

Definition 3.3. *Let $\{a_n\}$ be a sequence in a C^*-algebra with positive elements. The series $\sum_{n=1}^{+\infty} a_n$ is called an α-series, if there are $\alpha \in A$ with $\| \alpha \| < 1$ and $m_\alpha \in \mathbb{N}$ such that $\sum_{i=1}^{k} a_i \le \alpha k$ for each $k \ge m_\alpha$.*

Example 3.1. *Any convergent series with positive elements is an α-series. Also, the divergent series $\sum_{n=1}^{+\infty} \frac{1}{n}$, for each $\alpha \in A$ with $\| \alpha \| < 1$, is an α-series.*

Definition 3.4. *Let (X, A, d) be a C^*-algebra-valued metric space, $T_\zeta : X^n \to X$ and $g : X \longrightarrow X$. $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ and g satisfy the (K) property if*

$$\begin{aligned} &d(T_\zeta(x_1, \ldots, x_n), T_{\zeta'}(y_1, \ldots, y_n)) \\ &\le \beta_{\zeta,\zeta'}^*[d(gx_1, T_\zeta(x_1, \ldots, x_n)) \\ &+ d(gy_1, T_{\zeta'}(y_1, \ldots, y_n))]\beta_{\zeta,\zeta'} \\ &+ \gamma_{\zeta,\zeta'}^* d(gy_1, gx_1)\gamma_{\zeta,\zeta'} \end{aligned} \tag{3.1}$$

for all $x_i, y_i \in X$, where $1 \le i \le n$, $x_i \ne y_i$, $\| \beta_{\zeta,\zeta'} \| < 1, \| \gamma_{\zeta,\zeta'} \| < 1$ for all $\zeta, \zeta' \in \mathbb{N}_0$, $\sum_{\zeta=1}^{+\infty} (\beta_{\zeta,\zeta+1} + \gamma_{\zeta,\zeta+1})^2 (1 - \beta_{\zeta,\zeta+1}^2)^{-1}$ is an α-series, and $\lim_{r\to+\infty} \sup \beta_{r,\zeta} < 1$.

Let (X, A, d) be a C^*-algebra-valued metric space. Let g be a continuous self-mapping on X and $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ be a sequence of mappings from X^n into X, so that $T_\zeta(X^n) \subseteq g(X)$, $g(X) \subseteq X$ is complete, $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ and g are weakly reciprocally continuous, compatible, and satisfy the condition (K). If $g(X)$ is regular, then $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ and g have a n-tuple coincidence point, i.e., there is $x^i \in X, 1 \le i \le n$, so that

$$gx^i = T_\zeta(x^i, x^{i+1}, ..., x^n, x^1, ..., x^{i-1}),$$

for all $1 \le i \le n$.

Proof. Let $x_0^i \in X$ where $1 \le i \le n$. Since $T_0(X^n) \subseteq g(X)$, we can define $x_1^i \in X$, $(1 \le i \le n)$ such that

$$gx_1^i = T_0(x_0^i, x_0^{i+1}, \ldots x_0^n, x_0^1, \ldots x_0^{i-1}).$$

Again, since $T_0(X^n) \subseteq g(X)$, there exist $x_2^i \in X$, $1 \le i \le n$ such that

$$gx_2^i = T_1(x_1^i, x_1^{i+1}, \ldots x_1^n, x_1^1, \ldots x_1^{i-1}).$$

Continuing this process, we make the sequence $\{x_r^i\}$, $(1 \le i \le n)$ in order to

$$gx_{r+1}^i = T_r(x_r^i, x_r^{i+1}, \ldots x_r^n, x_r^1, \ldots x_r^{i-1}), \tag{3.2}$$

for all $r \ge 0$. By (3.1), we get

$$\begin{aligned} &d(gx_r^1, gx_{r+1}^1) \\ &= d(T_{r-1}(x_{r-1}^1, \ldots, x_{r-1}^n), T_r(x_r^1, \ldots, x_r^n)) \\ &\le \beta_{r-1,r}^*[d(gx_{r-1}^1, T_{r-1}(x_{r-1}^1, \ldots, x_{r-1}^n)) \\ &\quad + d(gx_r^1, T_r(x_r^1, \ldots, x_r^n))]\beta_{r-1,r} \\ &\quad + \gamma_{r-1,r}^* d(gx_r^1, gx_{r-1}^1)\gamma_{r-1,r} \\ &= \beta_{r-1,r}^*[d(gx_{r-1}^1, gx_r^1) + d(gx_r^1, gx_{r+1}^1)]\beta_{r-1,r} \\ &\quad + \gamma_{r-1,r}^* d(gx_r^1, gx_{r-1}^1)\gamma_{r-1,r} \\ &= \beta_{r-1,r}^* d(gx_{r-1}^1, gx_r^1)\beta_{r-1,r} \\ &\quad + \beta_{r-1,r}^* d(gx_r^1, gx_{r+1}^1)^{\frac{1}{2}} d(gx_r^1, gx_{r+1}^1)^{\frac{1}{2}} \beta_{r-1,r} \\ &\quad + \gamma_{r-1,r}^* d(gx_r^1, gx_{r-1}^1)\gamma_{r-1,r} \\ &= (\beta_{r-1,r} + \gamma_{r-1,r})^* d(gx_{r-1}^1, gx_r^1) \\ &\quad (\beta_{r-1,r} + \gamma_{r-1,r}) \\ &\quad + | \beta_{r-1,r} d(gx_r^1, gx_{r+1}^1)^{\frac{1}{2}} |^2 \\ &= | (\beta_{r-1,r} + \gamma_{r-1,r}) d(gx_{r-1}^1, gx_r^1)^{\frac{1}{2}} |^2 \\ &\quad + | \beta_{r-1,r} d(gx_r^1, gx_{r+1}^1)^{\frac{1}{2}} |^2 . \end{aligned}$$

It follows that

$$\begin{aligned} &d(gx_r^1, gx_{r+1}^1) \\ &\le (\beta_{r-1,r} + \gamma_{r-1,r})^2 (1 - \beta_{r-1,r}^2)^{-1} \\ &\quad d(gx_r^1, gx_{r-1}^1). \end{aligned} \tag{3.3}$$

Similarly, we obtain

$$\begin{aligned}
&d(gx_r^2, gx_{r+1}^2) \\
&\ \le (\beta_{r-1,r} + \gamma_{r-1,r})^2 (1 - \beta_{r-1,r}^2)^{-1} \\
&d(gx_r^2, gx_{r-1}^2), \\
&\vdots, \\
&d(gx_r^n, gx_{r+1}^n) \\
&\ \le (\beta_{r-1,r} + \gamma_{r-1,r})^2 (1 - \beta_{r-1,r}^2)^{-1} \\
&d(gx_r^n, gx_{r-1}^n).
\end{aligned} \tag{3.4}$$

Adding (3.3)-(3.4), we have

$$\begin{aligned}
\delta_r &= \sum_{i=1}^{n} d(gx_r^i, gx_{r+1}^i) \\
&\le (\beta_{r-1,r} + \gamma_{r-1,r})^2 (1 - \beta_{r-1,r}^2)^{-1} \\
&\quad [\sum_{i=1}^{n} d(gx_r^i, gx_{r-1}^i)] \\
&= (\beta_{r-1,r} + \gamma_{r-1,r})^2 (1 - \beta_{r-1,r}^2)^{-1} \delta_{r-1} \\
&\le (\beta_{r-1,r} + \gamma_{r-1,r})^2 (1 - \beta_{r-1,r}^2)^{-1} \\
&\quad (\beta_{r-2,r-1} + \gamma_{r-2,r-1})^2 \\
&\quad (1 - \beta_{r-2,r-1}^2)^{-1} \delta_{r-2} \\
&\le \dots \\
&\le \prod_{\zeta=0}^{r-1} (\beta_{\zeta,\zeta+1} + \gamma_{\zeta,\zeta+1})^2 \\
&\quad (1 - \beta_{\zeta,\zeta+1}^2)^{-1} \delta_0.
\end{aligned}$$

Moreover, for all $s > 0$, we get

$$\begin{aligned}
&\sum_{i=1}^{n} d(gx_r^i, gx_{r+s}^i) \\
&\le \sum_{i=1}^{n} d(gx_r^i, gx_{r+1}^i) + \sum_{i=1}^{n} d(gx_{r+1}^i, gx_{r+2}^i) \\
&+ \cdots + \sum_{i=1}^{n} d(gx_{r+s-1}^i, gx_{r+s}^i) \\
&\le \prod_{\zeta=0}^{r-1} (\beta_{\zeta,\zeta+1} + \gamma_{\zeta,\zeta+1})^2 (1 - \beta_{\zeta,\zeta+1}^2)^{-1} \delta_0 \\
&\quad + \prod_{\zeta=0}^{r} (\beta_{\zeta,\zeta+1} + \gamma_{\zeta,\zeta+1})^2 (1 - \beta_{\zeta,\zeta+1}^2)^{-1} \delta_0 \\
&\quad + \dots \\
&\quad + \prod_{\zeta=0}^{r+s-2} (\beta_{\zeta,\zeta+1} + \gamma_{\zeta,\zeta+1})^2 (1 - \beta_{\zeta,\zeta+1}^2)^{-1} \delta_0 \\
&= \sum_{k=r}^{r+s-1} \prod_{\zeta=0}^{k-1} (\beta_{\zeta,\zeta+1} + \gamma_{\zeta,\zeta+1})^2 (1 - \beta_{\zeta,\zeta+1}^2)^{-1} \delta_0.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\| \sum_{i=1}^{n} d(gx_r^i, gx_{r+s}^i) \| \\
&\le \| \sum_{k=r}^{r+s-1} \prod_{\zeta=0}^{k-1} (\beta_{\zeta,\zeta+1} + \gamma_{\zeta,\zeta+1})^2 \\
&\quad (1 - \beta_{\zeta,\zeta+1}^2)^{-1} \| \delta_0.
\end{aligned}$$

Let α and n_α be as in Definition 3.3. For all $r \ge n_\alpha$, using the fact that the non-negative numbers geometric mean is less than or equal to the arithmetic mean, it follows that

$$\begin{aligned}&\| \sum_{i=1}^{n} d(gx_r^i, gx_{r+s}^i) \| \\ &\leq \| \sum_{k=r}^{r+s-1} \\ &\left[\frac{1}{k} \sum_{\zeta=0}^{k-1} (\beta_{\zeta,\zeta+1} + \gamma_{\zeta,\zeta+1})^2 (1-\beta^2_{\zeta,\zeta+1})^{-1} \right]^k \| \\ &\delta_0 \\ &\leq \| \left(\sum_{k=r}^{r+s-1} \alpha^k \right) \| \delta_0 \\ &\leq \frac{\| \alpha^r \|}{1- \| \alpha \|} \delta_0.\end{aligned}$$

Now, letting $r \to +\infty$, we conclude that

$$\lim_{r\to+\infty} \| \sum_{i=1}^{n} d(gx_r^i, gx_{r+s}^i) \| = 0, \ 1 \leq i \leq n,$$

which implies that

$$\lim_{r\to+\infty} \| d(gx_r^i, gx_{r+s}^i) \| = 0.$$

Thus, $\{gx_r^i\}$, $(1 \leq i \leq n)$ are Cauchy sequences in X. Since $g(X)$ is complete, there are $p^i \in X$ so that

$$\lim_{r\to+\infty} \{gx_r^i\} = g(p^i) := x^i, \ (1 \leq i \leq n).$$

So, we have

$$\begin{aligned}&\lim_{r\to+\infty} g(x_{r+1}^i) \\ &= \lim_{r\to+\infty} T_r(x_r^i, x_r^{i+1}, \dots x_r^n, x_r^1, \dots x_r^{i-1}) \\ &= x^i, \ (1 \leq i \leq n).\end{aligned}$$

Since $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ and g are compatible and weakly reciprocally continuous, one has

$$\begin{aligned}&\lim_{r\to+\infty} T_r(gx_r^i, gx_r^{i+1}, \dots, gx_r^n, gx_r^1, \dots, gx_r^{i-1}) \\ &\qquad = gx^i, \ (1 \leq i \leq n).\end{aligned}$$

Now, we show that $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ have n-tuple coincidence points. According to (3.1), we obtain

$$\begin{aligned}&d(T_r(gx_r^1, \dots gx_r^n), T_\zeta(x^1, \dots x^n)) \\ &\leq \beta_{r,\zeta}^*[d(ggx_r^1, T_r(gx_r^1, \dots gx_r^n)) \\ &\quad + d(gx^1, T_\zeta(x^1, \dots x^n))]\beta_{r,\zeta} \\ &\quad + \gamma_{r,\zeta}^* d(gx^1, ggx_r^1)\gamma_{r,\zeta}.\end{aligned}$$

Taking the limit as $r \to +\infty$, we obtain $T_\zeta(x^1, \dots, x^n) = g(x^1)$ as $\beta_{r,\zeta} < 1$. Similarly, it can be proved that

$$\begin{aligned}&T_\zeta(x^2, \dots, x^n, x^1) = g(x^2), \\ &\qquad \dots \\ &T_\zeta(x^n, \dots, x^{n-1}) = g(x^n).\end{aligned}$$

Thus, $(x^1, \cdots, x^n)$ is a n-tuple coincidence point of $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ and g. □

Taking g the identity mapping in Theorem 3, we have Let (X, A, d) be a complete C^*-algebra-valued metric space. Let $\{T_\zeta\}_{\zeta\in\mathbb{N}\cup\{0\}}$ be a sequence of mappings from X^n into X, so that $\{T_\zeta\}$ and $Id : X \to X$ satisfy the (K) property. If X is regular, then $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ has a n-tuple fixed point, that is, there exists $x^i \in X^n$ $(1 \leq i \leq n)$ such that

$$x^i = T_\zeta(x^i, x^{i+1}, ..., x^n, x^1, ..., x^{i-1}),$$

for all $\zeta \in \mathbb{N}_0$.

For the existence and uniqueness of a n-tuple common fixed point, we need the next definition. For simplicity, we set $\mathbf{n} = \{1, \cdots, n\}$.

Definition 3.5. *For $x_i, u_i \in X$ $(1 \leq i \leq n)$, we say that $(x_i)_{i=1}^n$ is n-tuple comparable with $(u_i)_{i=1}^n$ if*

$$x_i \geq u_{\sigma(i)} \ \text{ or } \ x_i \leq u_{\sigma(i)}$$

where

$$\begin{aligned}&\sigma \in \Pi = \{\sigma^j | \quad \sigma^j : \boldsymbol{n} \to \boldsymbol{n}, \ \sigma^j(i) = k, \\ &\qquad k \overset{\text{mod } n}{\equiv} i + j, \ 0 \leq k \leq n-1\}.\end{aligned}$$

On the other hand, the elements of Π are the permutations of $\boldsymbol{n}$, which preserve the order (modulus n).
Replacing x_i and $u_{\sigma(i)}$ with gx_i and $gu_{\sigma(i)}$, respectively, one says that $(x_i)_{i=1}^n$ is n-tuple comparable with $(u_i)_{i=1}^n$ with respect to g.

Let (X, A, d) be a C^*-algebra-valued metric space. Let g be a self-mapping on X and $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ be a sequence of mappings from X^n into X. Let $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ and g be w-compatible and satisfy the condition (K). If $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ have n-tuple coincidence points comparable with respect to g, then $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ and g have a unique n-tuple common fixed point, that is, there exists a unique $(x^1,\cdots,x^n)\in X^n$ such that

$$\begin{aligned} x^i &= g(x^i) \\ &= T_\zeta(x^i,x^{i+1},...,x^n,x^1,...,x^{i-1}), \end{aligned}$$

where $1\le i\le n$.

Moreover, the common fixed point of $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ and g is of the form $(\jmath,\cdots,\jmath)$ for some $\jmath\in X$.

Proof. From Theorem 3, the set of n-tuple coincidence points is non-empty. First, we show that if $(x^1,\cdots,x^n)$ and $(u^1,\cdots,u^n)$ be n-tuple coincidence points, then $gx^i = gu^i (1\le i\le n)$. Since the set of n-tuple coincidence points is n-tuple comparable, applying condition (3.1), we get

$$\begin{aligned} &d(gx^1,gu^1) \\ &= d(T_\zeta(x^1,\dots,x^n),T_{\zeta'}(u^1,\dots,u^n)) \\ &\le \beta^*_{\zeta,\zeta'}[d(gx^1,T_\zeta(x^1,\dots,x^n) \\ &\quad + d(gu^1,T_{\zeta'}(y^1,,\dots,y^n))]\beta_{\zeta,\zeta'} \\ &\quad + \gamma^*_{\zeta,\zeta'}d(gu^1,gx^1)\gamma_{\zeta,\zeta'}, \end{aligned}$$

that is,

$$\begin{aligned} &(1-\gamma^2_{\zeta,\zeta'})d(gx^1,gu^1) \\ &\le \beta^2_{\zeta,\zeta'}[d(gx^1,T_\zeta(x^1,\dots,x^n) \\ &\quad + d(gu^1,T_{\zeta'}(y^1,,\dots,y^n))]. \end{aligned}$$

So,

$$\begin{aligned} &d(gx^1,gu^1) \\ &\le \beta^2_{\zeta,\zeta'}(1-\gamma^2_{\zeta,\zeta'})^{-1}[d(gx^1,T_\zeta(x^1,\dots,x^n) \\ &\quad + d(gu^1,T_{\zeta'}(y^1,,\dots,y^n))]. \end{aligned}$$

This induces that

$$\begin{aligned} \| d(gx^1,gu^1)\| \le \| \beta^2_{\zeta,\zeta'}\| \| (1-\gamma^2_{\zeta,\zeta'})^{-1}\| \\ \| d(gx^1,T_\zeta(x^1,\dots,x^n) \\ + d(gu^1,T_{\zeta'}(y^1,,\dots,y^n))\| . \end{aligned}$$

Therefore, since $\gamma_{\zeta,\zeta'} < 1$, it follows that $d(gx^1,gu^1)=0$, that is, $gx^1=gu^1$. Similarly, it can be proved that $gx^i = gu^j$, where $1\le i,j\le n$. So

$$gx^1=\cdots=gx^n=gu^1\cdots=gu^n.$$

Therefore, $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ and g have a unique n-tuple point of coincidence
$(gx^1,\cdots,gx^1)$. Now, let $gx^1=\jmath$. Then we have

$$\jmath = gx^1 = T_\zeta(x^1,\cdots,x^1).$$

By w-compatibility of $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ and g, we have

$$\begin{aligned} g\jmath &= ggx^1 \\ &= g(T_\zeta(x^1,\cdots,x^1)) \\ &= T_\zeta(gx^1,\cdots,gx^1) \\ &= T_\zeta(\jmath,\cdots,\jmath). \end{aligned}$$

On the other hand, $T_\zeta(gx^1,\cdots,gx^1) = gx^1$. Hence, $(g\jmath,\dots,g\jmath)$ is a n-tuple coincidence point of $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ and g. So, $g\jmath = gx^1$. Thus,

$$\jmath = g\jmath = T_\zeta(\jmath,\cdots,\jmath).$$

Therefore, $(\jmath,\cdots,\jmath)$ is the unique n-tuple common fixed point of $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ and g. □

4 Example

We give an example to support our main results.

Example 4.1. *Let $X=[0,1]$. Define $d(x,y)=|x-y|$. Then (X,A,d) is a complete C^*-valued-algebra metric space. Fix $n\in\mathbb{N}$ and $n>1$. We define*

$$\beta_{\zeta,\zeta'}=\frac{1}{n^{2\zeta+1}} \text{ and } \gamma_{\zeta,\zeta'}=\frac{1}{n^\zeta} \text{ for all } \zeta,\zeta'\in\mathbb{N},$$

and consider the mappings $T_\zeta : X^n\to X$ and $g: X\to X$ defined by

$$T_\zeta(x_1,\cdots,x_n)=\frac{x_1+\cdots+x_n}{n^\zeta} \text{ and}$$
$$g(x)=3nx$$

for all $\zeta=1,2,\dots$ and $x_1,\cdots,x_n\in X$. We prove by mathematical induction that $\{T_\zeta\}_{\zeta\in\mathbb{N}_0}$ and g satisfy the condition (K). We know that the greatest value of the first side in (3.1) corresponds to $\zeta=1$ and $\zeta'\to\infty$. Suppose that for $\zeta=1$ and $\zeta'=k$, we have

$$\begin{aligned} &\left| \frac{x_1+\cdots+x_n}{n} - \frac{y_1+\cdots+y_n}{n^k} \right| \\ &\le \frac{1}{n^3}[\left| 3nx_1 - \frac{x_1+\cdots+x_n}{n} \right| \\ &\quad + \left| 3ny_1 - \frac{y_1+\cdots+y_n}{n^k} \right|]\frac{1}{n^3} \\ &\quad + \frac{1}{n}\left| 3n(y_1-x_1)\right| \frac{1}{n}. \end{aligned}$$

Now, for $\zeta' = k+1$, we have

$$\begin{aligned} C :=& \mid \frac{x_1+\cdots+x_n}{n} - \frac{1}{n}(\frac{y_1+\cdots+y_n}{n^k}) \mid \\ \leq& \frac{1}{n^3}[\mid 3nx_1 - \frac{x_1+\cdots+x_n}{n} \mid \\ &+ \mid 3y_1 - \frac{1}{n}(\frac{y_1+\cdots+y_n}{n^k}) \mid]\frac{1}{n^3} \\ &+ 3 \mid \frac{y_1}{n} - x_1 \mid \frac{1}{n} := D. \end{aligned}$$

That is,

$$\begin{aligned} C \leq& \frac{1}{n}\left[\mid \frac{x_1+\cdots+x_n}{n} - \frac{y_1+\cdots+y_n}{n^k} \mid\right] \\ &+ \frac{n-1}{n} \mid \frac{x_1+\cdots+x_n}{n} \mid \\ \leq& \frac{1}{n}(\frac{1}{n^3}\left[\mid 3nx_1 - \frac{x_1+\cdots+x_n}{n} \mid\right. \\ &+ \mid 3ny_1 - \frac{y_1+\cdots+y_n}{n^k} \mid \frac{1}{n^3}) \\ &+ \frac{1}{n}(3 \mid y_1 - x_1 \mid)\frac{1}{n} \\ &+ \frac{n-1}{n} \mid \frac{x_1+\cdots+x_n}{n} \mid \leq D. \end{aligned}$$

Since $d(x,y)$ is symmetric, the role of ζ and ζ' can be changed with each other and reach a similar result. Thus, the inequality (3.1) for every ζ, ζ' holds. Moreover, the series

$$\sum_{\zeta=1}^{+\infty} \left(\frac{\beta_{\zeta,\zeta+1} + \gamma_{\zeta,\zeta+1}}{1-\beta_{\zeta,\zeta+1}} \right) = \sum_{\zeta=1}^{+\infty} \frac{n^{\zeta+1}+1}{n^{2\zeta+1}-1},$$

where $n > 1$, is an α-series with $\alpha = \frac{1}{2}$. Thus, all the hypotheses of Theorem 3 hold. Here, $(0, \cdots, 0)$ is a n-tuple coincident point (it is the unique n-tuple common fixed point) of g and $\{T_\zeta\}_{\zeta \in \mathbb{N}_0}$.

5 Application

We search an $x(t) = \{x_\zeta(t)\}_{\zeta=1}^n$ (a finite sequence in the variable ζ) to be a solution of the integral equation

$$\begin{aligned} x(t) =& h(t) \\ &+ \int_\mu f_\zeta^1(t,s,x(s)) + \ldots + f_\zeta^n(t,s,x(s))ds, \end{aligned} \quad (5.5)$$

for all $t, s \in \mu$, where μ is a Lebesgue measurable set so that $m(\mu) < \infty$.

Denote by $X = L^\infty(\mu)$ the set of essentially bounded measurable functions on μ.

Suppose that

(i) $f_\zeta^1, \ldots, f_\zeta^n : \mu^2 \times \mathbb{R} \longrightarrow \mathbb{R}$ and $h \in L^\infty(\mu)$;

(ii) there is $k \in (0, \frac{1}{2})$ so that for all $x^1, x^2 \in \mathbb{R}$,

$$\begin{aligned} 0 \leq& |f_\zeta^1(t,s,x^1(s)) - f_\zeta^1(t,s,x^2(s))| \\ &\leq k(x^1 - x^2), \\ -k(x^1-x^2) \leq& |f_\zeta^2(t,s,x^1(s)) \\ &\qquad - f_\zeta^2(t,s,x^2(s))| \leq 0, \\ &\vdots \\ 0 \leq& |f_\zeta^n(t,s,x^1(s)) - f_\zeta^n(t,s,x^2(s))| \\ &\leq k(x^1 - x^2), \end{aligned} \quad (5.6)$$

for all $s, t \in \mu$ with

$$k \leq \beta_{\zeta,\zeta'}^2 = \frac{1}{n^{4\zeta+2}} \text{ and } k \leq \gamma_{\zeta,\zeta'}^2 = \frac{1}{n^{2\zeta}}.$$

If the conditions $(i) - (ii)$ hold, then (5.5) admits a unique solution in $L^\infty(\mu)$.

Proof. Let $X = L^\infty(\mu)$ and $B(L^2(\mu))$ be the set of bounded linear mappings on the Hilbert space $L^2(\mu)$. We equip X with the metric $d : X \times X \to B(L^2(\mu))$ given as follows:

$$d(f,g) = M_{|f-g|},$$

where $M_{|f-g|}$ is the multiplication operator on $L^2(\mu)$. It is clear that $(X, B(L^2(\mu)), d)$ is a complete C^*-algebra-valued metric space. Define $T_\zeta : X^n \longrightarrow X$ by

$$\begin{aligned} &T_\zeta(x_1, \ldots, x_n)(t) \\ &= \int_\mu (f_\zeta^1(t,s,x^1(s)) + \ldots + f_\zeta^n(t,s,x^n(s)))ds \\ &\quad + h(t), \end{aligned}$$

for all $x^1, \ldots, x^n \in X$ and $s, t \in \mu$.

Now, we have

$$\begin{aligned} &d(T_\zeta(x_1, \ldots, x_n), T_{\zeta'}(y_1, \ldots, y_n)) \\ &= M_{|T_\zeta(x_1,\ldots,x_n) - T_{\zeta'}(y_1,\ldots,y_n)|}. \end{aligned}$$

Using (5.6), we have

$$\begin{aligned}
&\Big|T_\zeta(x_1,\dots,x_n)(t)-T_{\zeta'}(y_1,\dots,y_n)(t)\Big|\\
&=\Big|\int_\mu f^1_\zeta(t,s,x^1(s))+\dots+f^n_\zeta(t,s,x^n(s))ds\\
&-\int_\mu f^1_\zeta(t,s,y^1(s))+\dots+f^n_\zeta(t,s,y^n(s))ds\Big|\\
&=\Big|\int_\mu (f^1_\zeta(t,s,x^1(s))-f^1_\zeta(t,s,y^1(s)))\\
&+\dots+(f^n_\zeta(t,s,x^n(s))-f^n_\zeta(t,s,y^n(s)))ds\Big|\\
&\le\int_\mu\Big|(f^1_\zeta(t,s,x^1(s))-f^1_\zeta(t,s,y^1(s)))\\
&+\dots+(f^n_\zeta(t,s,x^n(s))-f^n_\zeta(t,s,y^n(s)))\Big|ds\\
&\le k(|x^1-y^1|+\dots+|x^n-y^n|)\\
&\le k(\| x^1-y^1\|_\infty+\dots+\| x^n-y^n\|_\infty),
\end{aligned}$$

for all $s,t\in\mu$.

Therefore, for every $\Lambda\in L^2(\mu)$, we have

$$\begin{aligned}
&\| T_\zeta(x_1,\dots,x_n)(t)-T_{\zeta'}(y_1,\dots,y_n)(t)\|\\
&=\| M_{|T_\zeta(x_1,\dots,x_n)(t)-T_{\zeta'}(y_1,\dots,y_n)(t)|}\|\\
&=\sup_{\|\Lambda\|=1}(M_{|T_\zeta(x_1,\dots,x_n)(t)-T_{\zeta'}(y_1,\dots,y_n)(t)|}\Lambda,\Lambda)\\
&=\sup_{\|\Lambda\|=1}\int_\mu |T_\zeta(x_1,\dots,x_n)(t)\\
&\qquad -T_{\zeta'}(y_1,\dots,y_n)(t)|\Lambda(t)\overline{\Lambda(t)}dt\\
&\le\sup_{\|\Lambda\|=1}\int_\mu|\Lambda(t)|^2dt.(k(\| x^1-y^1\|_\infty\\
&\quad+\dots+\| x^n-y^n\|_\infty))\\
&\le k(\| x^1-y^1\|_\infty+\dots+\| x^n-y^n\|_\infty)\\
&\le|\gamma_{\zeta,\zeta'}|^2(\| x^1-y^1\|_\infty+\dots+\| x^n-y^n\|_\infty)\\
&=\gamma^*_{\zeta,\zeta'}(\| x^1-y^1\|_\infty+\dots+\| x^n-y^n\|_\infty)\gamma_{\zeta,\zeta'}.
\end{aligned}$$

As a result,

$$\begin{aligned}
&d(T_\zeta(x_1,\dots,x_n),T_{\zeta'}(y_1,\dots,y_n))\\
&\le\beta^*_{\zeta,\zeta'}[d(x^1,T_\zeta(x_1,\dots,x_n)\\
&\quad+d(y_1,T_{\zeta'}(y_1,\dots,y_n)]\beta_{\zeta,\zeta'}\\
&\quad+\gamma^*_{\zeta,\zeta'}d(y^1,x^1)\gamma_{\zeta,\zeta'}.
\end{aligned}$$

All conditions in Corollary 3 are fulfilled, so there is a unique solution of (5.5) in $L^\infty(\mu)$. □

6 Conclusions

There are many generalizations of metric spaces, and numerous fixed point results have been obtained. In this paper, we used the α-series for a sequence of mappings to prove certain n-tuple fixed point and n-tuple coincidence point results in C^*-algebra-valued metric spaces. Note that the α-series is wider than the convergent series. An example and an application were given to support our results and to distinguish them from the other results. Our results extended and generalized the relevant results in [5, 7, 10, 12, 14].

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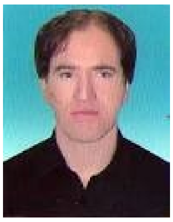

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