Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 14, No. 2, 2022 Article ID IJIM-1550, 12 pages
DOI: http://dx.doi.org/10.30495/ijim.2022.63531.1550
Research Article

A New Approach to n-Ary Dynamical Hypersystem

A. Dehghan Nezhad *†, N. Saderi ‡

Received Date: 2021-09-29 Revised Date: 2021-10-21 Accepted Date: 2022-01-23

Abstract

The primary aim of this paper is to investigate useful generalizations of the classical concept of action of a hyperstructure on a non-empty set. The main goal is to develop the theory of dynamical system to the theory of n-ary dynamical hypersystem. We also give some principal properties of an n-ary dynamical hypersystem.

Keywords : Universal n-ary hyperalgebra; n-Ary dynamical hypersystem; Hyperstructure; Hypergroup; Action group.

1 Introduction

The main motivation for the work in this paper is the study of the theory of n-ary dynamical hypersystem. Algebraic hyperstructures were introduced by F. Marty [31] in 1934. One of the first books, dedicated especially to hypergroups is "Prolegomena of Hypergroup Theory", written by P. Corsini in 1993 [8]. Another book on "Hyperstructures and Their Representations" was published one year later [31]. A recent book on these topics is "Applications of Hyperstructure Theory", written by P. Corsini and V. Leoreanu [9], see also [14, 15].

Definitions and theorems about hyperstructure and applications that are needed along our study and can be found in the References. When good references are available, we may not include the details of all the introduction and proofs.

We use [2, 8, 9, 12, 21, 35] and summarize the general preliminary definitions of algebraic hyperstructures.

Definition 1.1. *Let H be a non-empty set. Let $\mathcal{P}^*(H)$ be the set of all non-empty subsets of H, we define the concepts of hyperoperation, semihypergroup, hypergroup, H_v-group and regular hypergroup as following:*

(i) A hyperoperation on H is defined as a map $\otimes : H \times H \longrightarrow \mathcal{P}^*(H)$. The couple $(H, \otimes)$ is called a hypergroupoid. If X and Y are non-empty subsets of H, then we denote $X \otimes Y = \bigcup_{x\in X,\, y\in Y} x \otimes y$, $a \otimes X = \{a\} \otimes X$ and $X \otimes a = X \otimes \{a\}$, where $a \in H$.

(ii) A hypergroupoid $(H, \otimes)$ is called a semihypergroup if we have $(x \otimes y) \otimes z = x \otimes (y \otimes z)$

*Corresponding author. dehghannezhad@iust.ac.ir, Tel:+98(919)2441960.
†School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran.
‡Department of Mathematics, Yazd University Yazd, Iran.

for all x, y, z of H, which means

$$\bigcup_{u \in x \otimes y} u \otimes z = \bigcup_{v \in y \otimes z} x \otimes v.$$

(iii) *We say that a semi-hypergroup $(H, \otimes)$ is a hypergroup if we have $x \otimes H = H \otimes x = H$ for all $x \in H$.*
A hypergroupoid $(H, \otimes)$ is an H_v-group, if for all $x, y, z \in H$, the following conditions hold:

(1) *$x \otimes (y \otimes z) \cap (x \otimes y) \otimes z \neq \emptyset$ (weak associativity),*

(2) *$x \otimes H = H \otimes x = H$ (reproduction axiom).*

(iv) *A hypergroupoid $(H, \otimes)$ is said to be commutative (or abelian) if $x \otimes y = y \otimes x$ for all $x, y \in H$.*

(v) *A hypergroup $(H, \otimes)$ is called regular if it has at least an identity, that is an element e of H, such that for all $x \in H, x \in e \otimes x \cap x \otimes e$ and each element has at least one inverse, that is if $x \in H$, then there exists $x' \in H$ such that $e \in x \otimes x' \cap x' \otimes x$. The set of all identities of H is denoted by $E(H)$*

(vi) *If $x \in H, i_l(x) = \{x' : e \in x' \otimes x\}$ is the set of all left inverses of x in H (resp. $i_r(x)$) and $i(x) = i_l(x) \cap i_r(x)$.*

(vi) *A regular hypergroup $(H, \otimes)$ is called reversible if for all $(x; y; a) \in H^3$:*
(1) $y \in a \otimes x$, then there exists $a' \in i(a)$ such that $x \in a' \cap y$;
(2) $y \in x \otimes a$, then there exists $a'' \in i(a)$ such that $x \in y \otimes a''$.

(vii) *Let $(H, \otimes)$ be an H_v-group and K be a nonempty subset of H. Then K is called an H_v-subgroup of H if $(K, \otimes)$ is an H_v-group.*

(iix) *Let $(H, \otimes)$ be a hypergroup, K a non-empty subset of H. We say that K is invertible to the left if the implication $y \in K \otimes x \Longrightarrow x \in K \otimes y$ valid. We say K is invertible if K is invertible to the right and to the left.*

Proposition 1.1. *If $(H, \otimes)$ is a hypergroup such that $E(H) \neq \phi$; and K is an invertible subhypergroup of it, then $E(H) \subseteq K$.*

Definition 1.2. *Let $(H_1, \cdot)$, $(H_2, *)$ be two H_v-groups. A map $f : H_1 \to H_2$ is called an H_v-homomorphism or a weak homomorphism if $f(x \cdot y) \cap f(x) * f(y) \neq \emptyset$* for all $x, y \in H_1$.
The map f is called an inclusion homomorphism if $f(x \cdot y) \subseteq f(x) * f(y)$ for all $x, y \in H_1$.
Finally, f is called a strong homomorphism if $f(x \cdot y) = f(x) * f(y)$ for all $x, y \in H_1$.
If f is onto, one to one and strong homomorphism, then it is called an isomorphism. In this case, we write $H_1 \cong H_2$. Moreover, if the domain and the range of f are the same H_v-group, then the isomorphism is called automorphism. We can easily verify that the set of all automorphisms of H, denoted by $AutH$, is a group.

We first present some basic notions and results about n-hypergroups (see [9]), which are needed in this paper.

Let H be a non-empty set and $n \in \mathbb{N}, n \geqslant 2$. Consider $\otimes_n : \underbrace{H \times H \cdots \times H}_{n-time} \longrightarrow \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ be the set of all non-empty subsets of H. Then the hyperoperation $\otimes_n$ is called an n-ary hyperoperation on H and the pair $(H, \otimes_n)$ is called an n-hypergroupoid. If B_i for $i = 1, ..., n$ are non-empty subset of H. Then we denote

$$\phi_n(B_1, \cdots, B_n) = \tag{1.1}$$

$$\bigcup \{\phi_n(b_1, \cdots, b_n); (b_1, \cdots, b_n) \in \prod_{i=1}^{n} B_i\}. \tag{1.2}$$

We shall denote the sequence $h_i, h_{i+1}, \cdots, h_j$ by h_i^j. For $j < i$, the symbol h_i^j is the empty set.

Definition 1.3. *[12]*

(i) *The n-hypergroupoid $(H, \otimes_n)$ is called an n-ary semihypergroup if for $i, j \in \{1, 2, ..., n\}$ and h_1^{2n-1}, we have*

$$\otimes_n (h_1^{i-1}, \otimes_n(h_i^{n+i-1}), h_{n+i}^{2n-1}) = \tag{1.3}$$

$$\otimes_n (h_1^{j-1}, \otimes_n(h_j^{n+j-1}), h_{n+j}^{2n-1}). \tag{1.4}$$

(ii) We say that $(H, \otimes_n)$ is an n-ary quasihypergroup if for all $h_0, h_1, \cdots, h_n \in H$ and fixed $i \in \{1, ..., n\}$ there exists $x \in H$ such that

$$h_0 \in \otimes_n(h_1^{i-1}, x, h_{n+i}^{2n-1}). \quad (1.5)$$

(iii) An n-ary hypergroup is both an n-ary semihypergroup and an n-ary quasihypergroup.

(iv) An n-ary hypergroup $(H, \otimes_n)$ is commutative if for all h_1^n of H, and any permutation σ of $\{1, 2, ..., n\}$, we have $\otimes_n(h_1^n) = \otimes_n(h_{\sigma(1)}, \cdots, h_{\sigma(n)})$.

(v) Let $(H, \otimes_n)$ be an n-ary hypergroup and K a non-empty subset of H. If K is closed under the n-ary hyperoperation $\otimes_n$, then we say that K is an n-ary semi-subhypergroup. An n-ary semi-subypergroup K is called an n-ary subhypergroup of H if for all $k_0, k_1, \cdots, k_n \in K$ and fixed $i \in \{1, 2, ..., n\}$, there exists $x \in K$ such that $k_0 \in \otimes_n(k_1^{i-1}, x, h_{i+1}^n)$.

Remark 1.1. *Every n-ary operation can be conceived as a hyperoperation whose value set is the singleton $\{\otimes_n(x_1, ..., x_n)\}$, for all $x_1, ..., x_{n_t} \in H\}$.*

Example 1.1. *Let us consider the distributive lattice $(\mathcal{P}^*(X), \cup, \cap)$ of the parts of a set X, which contains at least three elements. Define the following n-ary hyperoperation on $\mathcal{P}^*(X)$: for all $X_1, \cdots, X_n \in \mathcal{P}^*(X)$,*

$$\otimes_n (X_1, \cdots, X_n) = \quad (1.6)$$

$$\{Z \in \mathcal{P}^*(X) | X_1 \cap \cdots \cap X_n \subseteq Z \subseteq X_1 \cup \cdots \cup X_n\}. \quad (1.7)$$

Therefore $(\mathcal{P}^*(X), \otimes_n)$ is a commutative n-ary hypergroup.

2 Main Results

In this section, we define our basic object of study.

2.1 The new approach to concepts of universal n-ary hyperalgebra

Definition 2.1. *Let n be a non-negative integer and $\{\mathcal{H}_i, i = 1, ..., n\}$ be a system of (finite or infinite) non-empty sets. We define the concepts of n-ary hyperstructure (or n-HS), universal n-ary hyperoperation (or n-UHO) as follows:*

By an n-ary hyperstructure (or n-HS), we mean the pair $(\{\mathcal{H}_i; i = 1, ..., n\}, \phi_n)$, where

$$\phi_n : \prod_{i=1}^{n} \mathcal{H}_i \longrightarrow \mathcal{P}^*(\bigcup_{i=1}^{n} \mathcal{H}_i) \quad (2.8)$$

maps any n-tuple $(\mathcal{H}_1, ..., \mathcal{H}_n) \in \prod_{i=1}^n \mathcal{H}_i$ to a non-empty subsets $\phi_n(h_1, ..., h_n) \subset \bigcup_{i=1}^n \mathcal{H}_i$. That is defined universal n-ary hyperoperation (or n-UHO).

If A_i for $i = 1, ..., n$ are non-empty subset of $\mathcal{H}_n$, then we denote

$$\phi_n(A_1, ...A_n) = \quad (2.9)$$

$$\bigcup\{\phi_n(x_1, ...x_n); (x_1, ..., x_n) \in \prod_{i=1}^{n} A_i\}. \quad (2.10)$$

Remark 2.1. *In the special case let $\mathcal{H}_i = \mathcal{H}$ for all $i = 1, .., n$. We obtain an n-HO on $\mathcal{H}$ that is an operation ϕ_n from $\mathcal{H}^n$ to $\mathcal{P}^*(\mathcal{H})$. Similarly, we can identify the set $\{x\}$ with the element x. Therefore any n-HO is an n-UHO.*

Example 2.1. *Now we specialize our considerations to the classical differential ring of real functions $f \in C^\infty(\mathbb{R})$, here $J = (a, b) \subseteq \mathbb{R}$, (not excluding the case $J = \mathbb{R}$) with the usual differentiation. For any $f \in C^\infty(\mathbb{R})$, we denote by $\int f(x)dx$ the set of all primitive functions to f. For any n-ary of functions $\psi_i \in C^\infty(J)$ with $i = 1, ..., n$. Let $\Psi = (\psi_1, ..., \psi_n)$. We define an n-UHO $*_{(n,\Psi)}$ on the ring $C^\infty(J)$ by*

$$*_{(n,\Psi)} : \underbrace{C^\infty(J) \times ... \times C^\infty(J)}_{n-time} \longrightarrow P^*(C^\infty(J))$$

$$*_{(n,\Psi)}(f_1, ..., f_n) = \int(\textstyle\sum_{i=1}^n(\psi_i'(x)f_i(x))dx), \quad f_i \in C^\infty(J).$$

Evidently $(C^\infty(J), *_{n,\Psi})$ is a universal n-ary hyperalgebra (or n-UHA).

Example 2.2. *Let $\{\mathcal{V}_i)\}_{i=1}^n$ be a family of real vector spaces endowed with an n-ary hyperbracket*

$$\underbrace{[., ..., .]}_{n-time} : \mathcal{V}_1 \times ... \times \mathcal{V}_n \longrightarrow P^*(\textstyle\bigcup_{i=1}^n \mathcal{V}_i)$$

$$\underbrace{[v_1, ..., v_n]}_{n-time} = \bigcup_{i=1}^{n} Span\{v_i\}, \quad v_i \in \mathcal{V}_i.$$

Therefore, the pair $(\{\mathcal{V}_i\}_{i=1}^{n}, \underbrace{[., ..., .]}_{n-time})$*, is an n-UHA.*

Example 2.3. *A* 3*-ary Lie hyperalgebra is a vector space* $\mathcal{V}$ *over* $\mathbb{R}$*, equipped with a* 3*-ary linear hyperbracket map*

$$\underbrace{[.,.,.]}_{3-time} : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V} \subset P^*(\mathcal{V})$$

satisfying the properties;
(i) $[X, Y, Z] = -[Y, X, Z]$ *(anti-commutativity),*
(ii) $[X_1, X_2, [Y_1, Y_2, Y_3]] = [[X_1, X_2, Y_1], Y_2, Y_3] + [Y_1, [X_1, X_2, Y_2], Y_3] + [Y_1, Y_2, [X_1, X_2, Y_3]]$,
(iii) $[X, Y, Z] + [Y, Z, X] + [Z, X, Y] = 0, \forall X_i, Y_i, Z \in \mathcal{V}$ *(Jacobi identity).*
Thus, the pair $(\mathcal{V}, [.,.,.])$*, is an universal* 3*-ary Lie hyperalgebra.*

Example 2.4. *Let* M *be a differentiable n-manifold with differentiable structure* $\mathfrak{F}$*. The* $(C^\infty M)$*-module of vector fields on* M *is denoted by* $\chi(M)$*. If* X, Y *and* Z *are vector fields on* M*. It is well known that, given* $[X, Y] := XY - YX$*. he standard Jacobi identity (JI)* $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ *is automatically satisfied if the product is associative. For a Lie algebra* $\mathcal{A}$*, expressed by the Lie commutators* $[X_i, X_j] = C^k{}_{ij} X_k$ *in a certain basis* $\{X_i\}$*,* $i = 1, ..., r = dim\mathcal{A}$*. The JI implies the Jacobi condition (JC)* $\frac{1}{2}\epsilon^{j_1 j_2 j_3}_{i_1 i_2 i_3} C^{\rho}_{j_1 j_2} C^{\delta}_{\rho j_3} = 0$*.*

Let n be even. A n-ary bracket or skew-symmetric Lie multi-bracket is a Lie algebra valued n-ary linear skew-symmetric mapping

$$\underbrace{[., ..., .]}_{n-time} : \underbrace{\mathcal{A} \times ... \times \mathcal{A}}_{n-time} \longrightarrow \mathcal{A} \subset P^*(\mathcal{A})$$

$$(X_{i_1}, ..., X_{i_n}) \longmapsto [X_{i_1}, ..., X_{i_n}] = \omega^{\sigma}_{i_1 ... i_n} X_\sigma,$$

where the constants $\omega^{\sigma}_{i_1...i_n} X_\sigma$ *is satisfied the condition*

$\epsilon^{j_1 j_2 ... j_{2n-1}}_{i_1 i_2 ... i_{2n-1}} \omega^{\rho}_{j_1 ... i_n} \omega^{\rho}_{j_{n+1} ... j_{2n-1}} = 0$ *(the generalised Jacobi condition (GJC)).*

For $n = 2$ *it gives the ordinary (JC). Therefore, the pair* $(\mathcal{A}, \underbrace{[., ..., .]}_{n-time})$*, is a universal n-ary Lie hyperalgebra.*

Remark 2.2. *A n-UHO (1) yields a map of power-sets determined by this hyperoperation. Thus the map*

$$\Phi_n : \prod_{i=1}^{n} \mathcal{P}^*(\mathcal{X}_i) \longrightarrow \mathcal{P}^*(\bigcup_{i=1}^{n} \mathcal{X}_i) \qquad (2.11)$$

is defined by $\Phi_n(X_1, ..., X_n) = \bigcup\{\phi_n(x_1, ..., x_n); (x_1, ..., x_n) \in \prod_{i=1}^{n} X_i\}$ *and conversely an n-UHO on* $\prod_{i=1}^{n} \mathcal{P}^*(X_i)$ *yields an n-UHO on* $\prod_{i=1}^{n} \mathcal{X}_i$ *.*

Definition 2.2. *Let* $\mathcal{X}^\omega = (\{\mathcal{X}_i; i = 1, ..., n_t\}, (\phi_t)_{t\in\omega})$ *and* $\mathcal{Y}^\omega = (\{\mathcal{Y}_i; i = 1, ..., n_t\}, (\psi_t)_{t\in\omega})$ *be a pair of* n_t*-UHO of the same type* ω*. A homomorphism* $F^\omega : \mathcal{X}^\omega \longrightarrow \mathcal{Y}^\omega$ *between two* n_t*-UHO is any system of mappings* $\mathcal{F} = \{f_i : \mathcal{X}_i \longrightarrow \mathcal{Y}_i\}$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} \prod_{i=1}^{n_t} \mathcal{X}_i & \xrightarrow{\phi_t} & \mathcal{P}^*(\bigcup_{i=1}^{n_t} \mathcal{X}_i) \\ \downarrow \prod_{i=1}^{n_t} f_i & & \downarrow \vartheta^\star \\ \prod_{i=1}^{n_t} \mathcal{Y}_i & \xrightarrow{\psi_t} & \mathcal{P}^*(\bigcup_{i=1}^{n_t} \mathcal{Y}_i) \end{array}$$

where for any n_t*-tuple* $(x_1, x_2, ..., x_{n_t}) \in \prod_{i=1}^{n_t} \mathcal{X}_i$ *we have*

$$\prod_{i=1}^{n_t} f_i(x_1, x_2, ..., x_{n_t}) = (f_1(x_1), f_2(x_2), ..., f_{n_t}(x_{n_t}))$$

$$and \quad \vartheta^\star : \mathcal{P}^*(\bigcup_{i=1}^{n_t} \mathcal{X}_i) \longrightarrow \mathcal{P}^*(\bigcup_{i=1}^{n_t} \mathcal{Y}_i)$$

is the lifting of a mapping $\vartheta : \bigcup_{i=1}^{n_t} \mathcal{X}_i \longrightarrow \bigcup_{i=1}^{n_t} \mathcal{Y}_i$ *defined by the induction. For* $x \in \mathcal{X}_1$ *suppose* $\vartheta(x) = f_1(x)$*. So* $\vartheta : \bigcup_{j=1}^{i} \mathcal{X}_j \longrightarrow \bigcup_{j=1}^{i} \mathcal{Y}_j$ *is well defined and for any* $x \in \mathcal{X}_{i+1} \backslash \bigcup_{j=1}^{i} \mathcal{X}_j$ *suppose* $\vartheta(x) = f_{i+1}(x)$ *.*

By the above definition, the following lemma is easily proved.

Lemma 2.1. *Let*

$$\mathcal{X}^\omega = (\{\mathcal{X}_i; i = 1, ..., n_t\}, (\phi_t)_{t\in\omega}),$$

$$\mathcal{Y}^\omega = (\{\mathcal{Y}_i; i = 1, ..., n_t\}, (\psi_t)_{t\in\omega})$$

and $\mathcal{Z}^\omega = (\{\mathcal{Z}_i; i = 1, ..., n_t\}, (\eta_t)_{t\in\omega})$ *are three* n_t*-UHO of the same type* ω. *If* $\mathcal{F}^\omega : \mathcal{X}^\omega \longrightarrow \mathcal{Y}^\omega$ *and* $\mathcal{G}^\omega : \mathcal{Y}^\omega \longrightarrow \mathcal{Z}^\omega$ *are homomorphisms. Then we can define a homomorphism between two hyperalgebras* $\mathcal{X}^\omega$ *and* $\mathcal{Z}^\omega$ *such that*
$\mathcal{G}^\omega o \mathcal{F}^\omega = \{g_i o f_i : \mathcal{X}_i \longrightarrow \mathcal{Z}_i\}$.

In the above definition if $\mathcal{X}_i = \mathcal{X}$ and $\mathcal{Y}_i = \mathcal{Y}$ for all $i = 1, ..., n$. Then we obtain the classical hyperstructure theory is as follow:

Definition 2.3. $\mathcal{X}^\omega = (\mathcal{X}, (\phi_t)_{t\in\omega})$, $\mathcal{Y}^\omega = (\mathcal{Y}, (\psi_t)_{t\in\omega})$ *be two* n_t*-HA of the same type* ω. *A map* $f : \mathcal{X}^\omega \longrightarrow \mathcal{Y}^\omega$ *is called a homomorphism if for every* $t \in \omega$ *and all* $x_1, ..., x_{n_t} \in \mathcal{X}$:

$$f(\phi_t(x_1, ..., x_{n_t})) \subseteq \psi_t(f(x_1), ..., f(x_{n_t})) \quad (2.12)$$

f is called a dual homomorphism if:

$$f(\phi_t(x_1, ..., x_{n_t}) \supseteq \psi_t(f(x_1), ..., f(x_{n_t})) \quad (2.13)$$

f is called a weak homomorphism if:

$$f(\phi_t(x_1, ..., x_{n_t})) \cap \psi_t(f(x_1), ..., f(x_{n_t})) \neq \emptyset \quad (2.14)$$

And finally f is called a strong homomorphism if:

$$f(\phi_t(x_1, ..., x_{n_t}) = \psi_t(f(x_1), ..., f(x_{n_t})). \quad (2.15)$$

Remark 2.3. *Let f is bijection then it is called an isomorphism, a dual isomorphism, and a strong isomorphism, if both f and* f^{-1} *are homomorphisms, dual homomorphisms, and strong homomorphisms, respectively. In the case of strong isomorphism, we write* $\mathcal{X}^\omega \cong \mathcal{Y}^\omega$. *If the domain and the range of f are the same hyperalgebras, then the isomorphism is called automorphism. It is easily verified that the set of all automorphisms of* $\mathcal{X}^\omega$, *denoted by Aut* $\mathcal{X}^\omega$, *is a group.*

Corollary 2.1. *The following are equivalent for a function f between two hyperalgebras* $\mathcal{X}^\omega$ *and* $\mathcal{Y}^\omega$ *of the same type* ω.
(i) The map f is an isomorphism.
(ii) The map f is a dual isomorphism.
(iii) The map f is a strong isomorphism.

Proof. (i) Right-arrow (iii) suppose that $f : \mathcal{X}^\omega \longrightarrow \mathcal{Y}^\omega$ is an isomorphism. Thus both f and f^{-1} are homomorphisms. Then, since $fof^{-1} = id$ is a strong homomorphism (actually a dual homomorphism), so for every $t \in \omega$ and all $x_1, ..., x_{n_t} \in \mathcal{X}^\omega$ we have

$$(fof^{-1})(\phi_t(x_1, ..., x_{n_t})). \quad (2.16)$$

The proof of (i) $\Longrightarrow$ (ii) is similar and the other implications are obvious. □

Remark 2.4. *In the definition of an n-UHO, if* $\mathcal{H}_i = \mathcal{H}$ *for all* $i = 1, ..., n$. *Then we obtain the classical n-ary algebraic hypersystem (hyperstructure theory) [16].*

2.2 Topology of Hyperalgebra

Recall that a topological group is a group G together with a topology on G that makes the multiplication and inversion operations continuous; where the topology on $G \times G$ is the corresponding product topology. The discrete and trivial topologies are group topologies on every group, but the question of finding interesting hyperstructure topologies has received a great deal of attention in the literature. We begin with a brief overview of this literature, to motivate our work in this paper.

Let H be a set and (H, τ) be a topological space where for any $x \in H$ there exist at least one open set $O(x)$ such that $x \in O(x)$, which is called fundamental open set.

Example 2.5. *Let H be a set,* τ *is a topology on H and* $\otimes_n$ *is a hyperoperation on H defined by* $\otimes_n(x_1, ..., x_n) = \bigcup_{i=1}^n \zeta(x_i)$ *where* $\zeta : H \rightarrow P^*(H)$ *is a function that for any* $x_i \in H$ *,* $\zeta(x_i) = O(x_i)$. *So the hypergroupoid* $(H, \otimes_n)$ *is a hypergroup.*

As defined topology n-groups, we had hoped to be able to define the topology on n-ary hypergroups as the hyperoperation be continue. But we cannot define a topology on the $P^*(H)$ with the help of the topology of the hypergroup H. So we recall first the semicontinuity and then introduce an adequate definition of topological n-ary hypergroups.

Definition 2.4. *Let $(H, \otimes_n)$ be a hypergroupoid and (H, τ) be a topological space, the Cartesian product H^n will be equipped with the product topology. The hyperoperation $*_n$ is called:*
(1) *upper semcontinuous, if for every open set $O \in \tau$, the set $O^* = \{(x_1, ..., x_n) \in H^n : \otimes_n(x_1, ...x_n) \subset O\}$ is an open in H^n;*
(2) *lower semicontinuous, if for every open set $O \in \tau$, the set $O_* = \{(x_1, ..., x_n) \in H^n : \otimes_n(x_1, ...x_n) \cap O \neq \emptyset\}$ is an open in H^n;*
(3) *Similarly, the hyperoperation $\otimes_n$ is semicontinuous if it is upper and lower semicontinuous.*

Remark 2.5. *Let $\otimes_n$ be a hyperoperation on H^n. Then the hyperoperation $\otimes_n$ is upper semicontinuous at $(x_1, ..., x_n) \in H^n$ if and only if for every open set $U \in \tau$, such that $\otimes_n(x_1, ...x_n) \subseteq U$ there exists open sets V_i , $(i = 1, ..., n)$ of $(x_1, ..., x_n)$ such that for all i ,$x_i \subseteq V_i$ implies*

$$\otimes_n(y_1, ..., y_n) \subseteq U \qquad \text{for all } y_i \in V_i.$$

Similarly, $\otimes_n$ is lower semicontinuous at $(x_1, ..., x_n) \in H^n$ if and only if for every open set $U \in \tau$ satisfying $\otimes_n(x_1, ...x_n) \cap U \neq \emptyset$, there exists open sets V_i , $(i = 1, ..., n)$ of $(x_1, ..., x_n)$ such that for all i ,$x_i \subseteq V_i$ implies

$$\otimes_n(y_1, ..., y_n) \cap U \neq \emptyset \qquad \text{for all } y_i \in V_i.$$

Proposition 2.1. *The hyperoperation $\otimes_n$ of any n-ary semihypergroup H endowed with the topology τ is upper semicontinuous.*

Proof. Let O be an open set of H. If $(x_1, ...x_n) \in O^*$ then $\bigcup_{i=1}^{n} O(x_i) \subset O$. Since $\forall x_i \in H, x_i \in O(x_i) \subseteq O$ we get $(x_1, ...x_n) \in O^*$. Conversely; let $(x_1, ...x_n) \in O^*$. It is easy to see that for any the open sets $O(x_i)$ are included in O. Therefore, it is their union and finally $(x_1, ...x_n) \in O^*$. □

Proposition 2.2. *The hyperoperation $\otimes_n$ of any topological n-ary semihypergroup (H, τ) is lower semicontinuous if $O(x) \cap O = \emptyset \Longrightarrow O(a) \cap O = \emptyset$; $\forall a \in O(x)$.*

Proof. Let O be an open set of H. Since $O(x)$ is a neighbourhood of x for any $x \in H$. To prove that O_* is open; we will prove that for any $(x_1, ...x_n) \in O_*$ there exists a neighbourhood V of $(x_1, ...x_n)$ such that $(x_1, ...x_n) \in V \subset O_*$. Let $(x_1, ...x_n) \in O_*$ and set $V = O(x_1) \times ... \times O(x_n)$. This set is an open set of H^n and then a neighbourhood of $(x_1, ...x_n)$. The condition $(\bigcup_{i=1}^{n} O(x_i)) \cap O = \emptyset$ implies that $O(x) \cap O = \emptyset$ or ... $O(x_n) \cap O = \emptyset$. For $(a_1, ..., a_n) \in O(x_1) \times ... \times O(x_n)$ and from our condition, we can deduce that $O(a_i) \cap O = \emptyset$. So $(O(a_1) \cup ... \cup O(a_n)) \cap O = \emptyset$ and $(a_1, ..., a_n) \in O_*$. Finally $(x_1, ...x_n) \in O(x_1) \times ... \times O(x_n) \subset O_*$. Thus O_* is an open set. □

Definition 2.5. *A topological n-ary hypergroup $(H, *_n)$ is a hypergroup endowed with a topology τ such that the hyperoperation is semicontinue.*

Corollary 2.2. *If $(H, \otimes_n)$ is a hypergroup and the topology τ on H is such $O(x_1, ..., x_n) \cap O = \emptyset \Rightarrow O(a_1, ...a_n) \cap O = \emptyset$; $\forall (a_1, ...a_n) \in O(x_1, ..., x_n)$ for a fundamental saturated family $\{O(x); x \in H\}$, then H is a topological hypergroup.*

Remark 2.6. *It is trivial that the hyperoperation $\otimes_n$ as defined above is commutative.*

Example 2.6. *The discrete topology on $(H, \otimes_n)$ defined by $\otimes_n(x_1, ..., x_n) = \{x_1, ..., x_n\}$ has the required properties. Therefore $(H, \otimes_n)$ is a topological hypergroup.*

Proposition 2.3. *Any open set K of a n-ary semihypergroup H endowed with the topology (τ) is a n-ary sub-semihypergroup of H.*

Proof. 1. If $x \in K$ then $x \in O(x) \cap K$ which is an open set. Consequently $O(x) \subseteq K$. Thus $ab \subseteq K$, for all $a; b \in K$.
2. By the definition of our topology, any $x \in K$ is such $x \in O(x)$ and so $\forall a \in K; x \in O(a) \cup O(x)$. Then we get $K \subseteq Ka \subseteq K \Longrightarrow K = Ka$.
The other equality can be obtained similarly. H

is a sub-semihypergroup of itself. Any other sub-semihypergroup of H will be called a proper sub-semihypergroup. □

2.3 Topological universal n-ary hyperalgebra (or topological n-UHA)

Recall first the basic terms and definitions of topological n-ary hyperalgebra.

Definition 2.6. *Assume that $(\{\mathcal{X}_i; i = 1, ..., n\}, \Phi_n)$ be a universal n-ary hyperalgebra, where Φ_n is a n-ary hyperoperation on $\mathcal{X}_i (i = 1, ..., n)$, for each $\tau_1, ..., \tau_n$ is a topology on $\mathcal{X}_1, ..., \mathcal{X}_n$ and τ^* be a topology on $\mathcal{P}^*(\bigcup_{i=1}^{n} \mathcal{X}_i)$ as follow:*
The family $\mathcal{B}$ consisting of all sets $S_V = \{U \in \mathcal{P}^*(\bigcup_{i=1}^{n} \mathcal{X}_i) \mid U \subseteq V, V = \bigcup_{i=1}^{n} v_i, v_i \in \tau_i\}$ is a base for a topology on $\mathcal{P}^*(\bigcup_{i=1}^{n} \mathcal{X}_i)$.
Let $\mathcal{X} = \prod_{i=1}^{n} \mathcal{X}_i$ and $\tau = (\tau_1, ..., \tau_n)$, if the universal n-ary hyperoperation Φ_n is continuous, the triple$(\{\mathcal{X}_i; i = 1, ..., n\}, \Phi_n, \tau)$ is called a topology of universal n-ary hyperalgebra. The continuity of Φ_n means that for every $(x_1, ..., x_n) \in \mathcal{X}$ the following statement holds:

$$\forall O_{\Phi_n(x_1,...,x_n)} \in \tau^* \qquad \exists (O_{x_i} \in \tau_i)_1^n$$

$$\Phi_n(O_{x_1}, ..., O_{x_n}) \subseteq O_{x_1^n}$$

Definition 2.7. *Let for $n \geq 2$, the pair $(\mathcal{H}, \Phi_n)$ be a classical n-ary algebraic hypersystem. We say that $(\mathcal{H}, \Phi_n)$ is an n-ary group (or n-group) if and only if is an n-semigroup and an n-quasigroup.*

Definition 2.8. *Assume that $(\mathcal{H}, \Phi_n)$ be an n-groupoid, $n \geq 2$ and τ*

H isatopologyonH.Ifthen−hyperoperationΦ_n is continuous, the triple $(\mathcal{H}, \Phi_n, \tau_{\mathcal{H}})$ is called a topological n-groupoid. The continuity of Φ_n means that for every $h_1, h_2, ..., h_n \in \mathcal{H}$, (or $h_1^n \in \mathcal{H}$) the following statement holds:

$$\forall O_{\Phi_n(h_1^n)} \in \tau_{\mathcal{H}}, \qquad \exists \tag{2.17}$$

$$(O_{h_i} \in \tau)_1^n, \quad \Phi_n(O_{h_1}, ..., O_{h_n}) \subseteq O_{h_1^n} \tag{2.18}$$

(equipped with ordinary product topology $\tau_{\mathcal{H}} \times ... \times \tau_{\mathcal{H}}$).

In the sequel for n-group $(\mathcal{H}, \Phi_n)$ suppose $^{-1}$ its inverse operation, $n \geq 2$.

Definition 2.9. *Let $\mathcal{H}$ be equipped with a topology $\tau_{\mathcal{H}}$. Then we say that $(\mathcal{H}, \Phi_n, \tau_{\mathcal{H}})$ is a topological n-group if:*
1) the n-hyperoperation, Φ_n is continuous in $\tau_{\mathcal{H}}$ and
2) the $(n-1)$-hyperoperation, $^{-1}$ is continuous in $\tau_{\mathcal{H}}$.

In other words, we say that $(\mathcal{H}, \Phi_n, \tau_{\mathcal{H}})$ is a *topological n-group* if:
1)$\forall O_{\Phi_n(h_1^n)} \in \tau_{\mathcal{H}} \quad \exists (O_{h_i} \in \tau)_1^n$
$\Phi_n(O_{h_1}, ..., O_{h_n}) \subseteq O_{\Phi_n(h_1^n)}$,
2)$\forall O_{(x_1^{n-1})^{-1}} \in \tau_{\mathcal{H}} \quad \exists (O_{h_i} \in \tau_{\mathcal{H}})_1^{n-1}$
$\Phi_n(O_{h_1}, ..., O_{h_{n-1}})^{-1} \subseteq O_{(x_1^{n-1})^{-1}}$,
inspired by the definition of the topological n-group.

Proposition 2.4. *Assume that $(\{\mathcal{X}_i; i = 1, ..., n_t\}, (\Phi_{n_t})_{t\in\omega})$ be an n_t-UHA, where Φ_t is an n_t-HO on $\mathcal{X}_i (i = 1, ..., n_t)$, for each $t \in \omega$ and $\tau_1, ..., \tau_{n_t}$ is a topology on $\mathcal{X}_1, ..., \mathcal{X}_{n_t}$ and τ^{*t} be a topology on $\mathcal{P}^*(\bigcup_{i=1}^{n_t} \mathcal{X}_i)$ as follow:*
The family $\mathcal{B}_t$ consisting of all sets $S_V = \{U \in \mathcal{P}^*(\bigcup_{i=1}^{n_t} \mathcal{X}_i) \mid U \subseteq V, V = \bigcup_{i=1}^{n_t} v_i, v_i \in \tau_i\}$ is a base for a topology on $\mathcal{P}^*(\bigcup_{i=1}^{n_t} \mathcal{X}_i)$.

We define the topological n_t-UHA as follow.

Definition 2.10. *Let $\mathcal{X} = \prod_{i=1}^{n_t} \mathcal{X}_i$ and $\tau = (\tau_1, ..., \tau_{n_t})$. If the n_t-UHO, Φ_t is continuous, then the triple$(\{\mathcal{X}_i; i = 1, ..., n_t\}, (\Phi_t)_{t\in\omega}, \tau)$ is called a topological n_t-UHA . The continuity of Φ_t means that for every $(x_1, ..., x_{n_t}) \in \mathcal{X}$ the following statement holds:*

$$\forall O_{\Phi(x_1,...,x_{n_t})} \in \tau^{*t} \qquad \exists (O_{x_i} \in \tau_i)_1^{n_t} \tag{2.19}$$

$$\Phi(O_{x_1}, ..., O_{x_{n_t}}) \subseteq O_{x_1^{n_t}} \tag{2.20}$$

Example 2.7. *We define a the 3-UHO, ϕ_3 as follows;*

$$\phi_3 : (0,1) \times \mathbb{N} \times (0,1) \longrightarrow \mathcal{P}^*((0,1) \cup \mathbb{N} \cup (0,1)) \tag{2.21}$$

$$\phi_3(x, n, y) = \{\frac{xy}{2^k} | 0 \leqslant k \leqslant n\}, \forall x, y \in (0,1). \tag{2.22}$$

Therefore $\phi_3(x,n,y) \subsetneqq (0,1)$ *and for every* $m,n \in \mathbb{N}$ *and* $x,y,z \in (0,1)$*, we have,*

$$\phi_3(\phi_3(x,n,y),m,z) = \{\frac{xy}{2^k} | 0 \leqslant k \leqslant (n+m)\} \quad (2.23)$$
$$= \phi_3(x,n,\phi_3(y,m,z)). \quad (2.24)$$

The triple $(((0,1),\mathbb{N},(0,1)),\phi_3,(\tau,\tau_0,\tau))$ *is a topological* 3*-UHA, where* τ *is the standard topology on* $(0,1)$ *and* τ_0 *is the discrete topology on* $\mathbb{N}$*.*

The Cartesian product $\underbrace{\mathcal{H}\times ... \times\mathcal{H}}_{n-time} = \mathcal{H}^n$ consists of all n-tuples $(h_1,...,h_2)$, such that $h_i \in \mathcal{H}, i = 1,...,n$. The i-projection of the Cartesian product $\mathcal{H}^n$ on its i-th axis is the map $Pr_i^{(n)} : \mathcal{H}^n \longrightarrow \mathcal{H}$ such that $(h_1,...,h_2) \longmapsto h_i$.

2.4 The n-ary dynamical hypersystem

Definition 2.11. *[20] Let* H *be a group and* X *be a set. Then* H *is said to act on* X *(on the left) if there is a mapping* $\Omega : H \times X \to X$ *satisfying two conditions:*
(i) If e *is the identity element of* H*, then* $\Omega(e,x) = x, \quad \forall x \in X$ *(identity) and*
(ii) If $h_1,h_2 \in H$*, then* $\Omega(h_1,\Omega(h_2,x)) = \Omega(h_1h_2,x), \quad \forall x \in X$ *(compatibility).*
When H *is a topological group,* X *is a topological space, and* Ω *is continuous, then the action is called continuous.*

Example 2.8.

(1) Let $X = S^1 = \{z \in \mathbb{C} | |Z| = 1\}$ *and* H *be the group of nth roots of unity for some n. Then* H *acts on* S^1 *by rotations :* $e^{\frac{i2\pi}{n}}$ *acts on* $e^{i\theta}$ *by* $\Omega(e^{\frac{i2\pi}{n}}, e^{i\theta}) = e^{i(\theta+\frac{i2\pi}{n})}$*.*

(2) Take $X = \mathbb{R}^2$ *and* $H = \mathbb{Z}^2$*. For each pair of integers* $(m,n) \in \mathbb{Z}^2$*, we define* $\Omega((m,n),(x,y)) = (m+x,n+y)$*. The pair* $(\mathbb{Z}^2,\mathbb{R}^2)$ *is a continuous group action.*

This section explores the novel notion of the n-ary hyperstructure actions, which is a natural generalization of the usual notion of group actions. As a first step toward the study of the n-ary hyperstructure actions from the algebraic viewpoint.

Definition 2.12. *[17] (i) An element* $e \in H$*, where* $(H;\phi)$ *is a hyperstructure, is called an identity if for all* $x \in H$ *there holds* $x \in \phi(x,e)$ *and* $x \in \phi(e,x)$*.*
(ii) The element e of an n-ary hypergroup (H,ϕ_n) *is called a neutral (identity) element if*

$$\phi_n(\underbrace{e\times\cdots\times e}_{(i-1)-time}, x, \underbrace{e\times\cdots\times e}_{(n-i)-time})$$

includes x *for all* $x \in H$ *and all* $i \in \{1,2,\cdots,n\}$*.*

Definition 2.13. *An* n*-ary dynamical hypersystem or* n*-DHS* Λ_n *is a triple* $(X,\Lambda_n,\mathcal{A})$*, where* $\mathcal{A} = (\{H_i; i = 1,...,n\},\phi_n)$ *(time set) is an* n*-UHA, the function* ϕ_n *is a hyperoperation on* $H_i (i = 1,...,n)$*, the non-empty set* X *is the state-space (a topological space with topology* τ_X*) and* Λ_n *is a map* $\Lambda_n : \mathcal{H}\times X \to \mathcal{P}^*(X)$ *(we set* $\mathcal{H} = \prod_{i=1}^n H_i$*), that satisfying two conditions:*
(i) $\Lambda_n(E_1,...,E_n,x) = \bigcup_{e_i\in E_i}\Lambda_n(e_1,...,e_n,x) \supseteq \{x\}, \quad \forall x \in X$*, where* E_i *is the identity set for* H_i*, for all* $i = 1,...,n$*,*
(ii) If $h_1,...,h_n \in \mathcal{H}$*, then* $\forall x \in X$*;*

$$\Lambda_n(\phi_n(h_1,E_2,...,E_n),\Lambda_n(\phi_n(E_1,h_2,...,E_n), \Lambda_n(...,\Lambda_n(\phi_n(E_1,...,E_{n-1},h_n),x),...))) \in \Lambda((\phi_n)(h_1,...,h_n),x)$$

where $\Lambda_n((\phi_n)(h_1,...,h_n),x) = \{\Lambda_n(h,x) : h \in \phi_n(h_1,...,h_n)\}$ *and* E_i *is the identity set for* H_i*, for all* $i = 1,...,n$*.*

Example 2.9. *Let* $H = SPD(n)$ *be the set of* $n\times n$ *symmetric, positive definite matrices. Suppose* $X = GL(n,\mathbb{R})$*, then the act of* H *on* X *as follows;*

$$\Lambda_2 : GL(n,\mathbb{R}) \times SPD(n) \to \mathcal{P}^*(SPD(n))$$

for all $G \in GL(n,\mathbb{R})$ *and all* $s \in SPD(n)$*,* $\Lambda_2(G,S) = \{S, GSG^T, G^TSG\}$*. It is easily checked that* GSG^T, G^TSG *is in* $SPD(n)$ *if* S *is in* $SPD(n)$*. For every SPD matrix* S*, can be written as* $S = GG^T$*, for some invertible matrix* G*. Therefore the triple* $(SPD(n),\Lambda_2,GL(n,\mathbb{R}))$ *is a* 2*-DHS.*

Our study is sufficiently general to apply to finite-as well as infinite-dimensional n-DHS whose motions may evolve along a continuum (continuous-time n-DHS), discrete-time (discrete-time n-DHS). In the case of continuous-time n-DHS, we consider motions that is continuous concerning for to time (continuous n-DHS) and motions that allow discontinuities in time(discontinuous n-DHS).

Let $(X, \Lambda_n, \mathcal{A})$ be an n-DHS. Then a $(Y, \Lambda_n, \mathcal{A})$ is called a *n-ary subdynamical hypersystem* of $(X, \Lambda_n, \mathcal{A})$, when Y is a subset of X.
Furthermore $(X, \Lambda_n|\mathcal{H}', \mathcal{A}')$ is called a *n-ary dynamical subhypersystem* of $(X, \Lambda_n, \mathcal{A})$, when $H'_i \subseteq H_i$ and $\mathcal{H}' = H'_1 \times, ..., \times H'_n$.

Definition 2.14. *Let $\mathcal{A} = (\{H_i; i = 1, ..., n\}, \phi_n)$ and $\mathcal{B} = (\{H'_i; i = 1, ..., n\}, \phi'_n)$ be two n-HA where $E_i (i = 1, ..., n)$ is the identity set for H_i and $E'_i (i = 1, ..., n)$ is the identity set for H'_i. Two n-DHSs $(X, \Lambda_n, \mathcal{H})$ and $(X', \Lambda'_n, \mathcal{H}')$, are called conjugate n-DHS if there exist one to one and onto maps $\mathcal{L} : X \to X'$ and $\mathcal{T} : H \to H'$, where $\mathcal{T}_i : H_i \to H'_i$ and $\mathcal{T}^* : P^*(\bigcup_{i=1}^n H_i) \to P^*(\bigcup_{i=1}^n H'_i)$ such that $\mathcal{T}^*(h) = \mathcal{T}_i(h)$ that the following two axioms hold;*

(1) $\mathcal{T}^*(\phi_n(h_1, ..., h_n)) = \phi'_n(\mathcal{T}_1(h_1), ..., \mathcal{T}_n(h_n)),$

$$\forall h_i \in H_i$$

$$\begin{array}{ccc}
\mathcal{H} = H_1 \times ... \times H_n & \longrightarrow & \mathcal{P}^*(\bigcup_{i=1}^{n} H_i) \\
\downarrow & & \downarrow \\
\mathcal{H}' = H'_1 \times ... \times H'_n & \longrightarrow & \mathcal{P}^*(\bigcup_{i=1}^{n} H'_i)
\end{array}$$

(2) $\mathcal{L}(\Lambda_n(h, x)) = \Lambda'_n(\mathcal{T}^n(h), \mathcal{L}(x)),$

$$\forall h \in \mathcal{H}, x \in X.$$

Proposition 2.5. *Let $(\mathcal{L}, \mathcal{T})$ be a conjugate relation between two n-DHSs $(X, \Lambda_n, \mathcal{H})$ and $(X', \Lambda'_n, \mathcal{H}')$ (or $(\{H_i; i = 1, ..., n\}, \phi_n)$ and $(\{H'_i; i = 1, ..., n\}, \phi'_n)$) and $(\mathcal{L}', \mathcal{T}')$ be a conjugate relation between two n-DHSs $(X', \Lambda'_n, \mathcal{H}')$ and $(X'', \Lambda''_n, \mathcal{H}'')$ (or $(\{H'_i; i = 1, ..., n\}, \phi'_n)$ and $(\{H''_i; i = 1, ..., n\}, \phi''_n)$). Then*

(1) the relation $(\mathcal{L}^{-1}, \mathcal{T}^{-1})$ is a conjugate relation between two n-DHSs $(X', \Lambda'_n, \mathcal{H}')$ and $(X, \Lambda_n, \mathcal{H})$ (or $(\{H'_i; i = 1, ..., n\}, \phi'_n)$ and $(\{H_i; i = 1, ..., n\}, \phi_n)$),

(2) the relation $(\mathcal{L}' o \mathcal{L}, \mathcal{T}' o \mathcal{T})$ is a conjugate relation between two n-DHSs $(X, \Lambda_n, \mathcal{H})$ and $(X'', \Lambda''_n, \mathcal{H}'')$ (or $(\{H_i; i = 1, ..., n\}, \phi_n)$ and $(\{H''_i; i = 1, ..., n\}, \phi''_n)$).

Proof. (1) If $h'_i \in H'_i$ for all $i \in 1, ...n$. Then the following sequence of equalities holds

$(\mathcal{T}^*)^{-1}(\phi'_n(h'_1, ..., h'_n)) =$
$(\mathcal{T}^*)^{-1}(\phi'_n(\mathcal{T}_1(h_1), ..., \mathcal{T}_n(h_n))) =$
$(\mathcal{T}^*)^{-1}(\mathcal{T}^*(\phi_n(h_1, ..., h_n)) = \phi_n(h_1, ..., h_n) =$
$\phi_n(\mathcal{T}_1^{-1}(\mathcal{T}_1(h_1)), ..., \mathcal{T}_n^{-1}(\mathcal{T}_n(h_n))) =$
$\phi_n(\mathcal{T}_1^{-1}(h'_1), ..., \mathcal{T}_n^{-1}(h'_n))$ where $h_i \in H_i$.

For all $h' \in \mathcal{H}'$ and $x' \in X'$, we conclude that the following sequence of equalities hold

$\mathcal{L}^{-1}(\Lambda'_n(h'), x') =$
$\mathcal{L}^{-1}(\Lambda'_n(\mathcal{T}^n((\mathcal{T}^n)^{-1}(h')), \mathcal{L}(\mathcal{L}^{-1}(x')))) =$
$\mathcal{L}^{-1}(\mathcal{L}(\Lambda_n((\mathcal{T}^n)^{-1}(h')), \mathcal{L}^{-1}(x'))) =$
$\Lambda_n((\mathcal{T}^n)^{-1}(h'), \mathcal{L}^{-1}(x')).$
(2) If $h_i \in H_i$.

Then the following sequence of equalities holds
$\mathcal{T}'^* o \mathcal{T}^*(\phi_n(h_1, ..., h_n)) =$
$\mathcal{T}'^*(\phi'_n(\mathcal{T}_1(h_1), ..., \mathcal{T}_n(h_n))) =$
$\phi''_n(\mathcal{T}'_1 o \mathcal{T}_1(h_1), ..., \mathcal{T}'_n o \mathcal{T}_n(h_n)).$

Finally, we conclude that for every $h \in \mathcal{H}$ and $x \in X$ the following sequence of equalities holds
$\Lambda''_n(\mathcal{T}'^n o \mathcal{T}^n(h), \mathcal{L}' o \mathcal{L}(x)) =$
$\mathcal{L}'(\Lambda'_n(\mathcal{T}^n(h), \mathcal{L}(x))) = \mathcal{L}'(\mathcal{L}(\Lambda_n(h, x))) =$
$(\mathcal{L}' o \mathcal{L})(\Lambda_n(h, x)).$ □

Example 2.10. *Let $\{\mathbb{N}_i)\}_{i=1}^n = \mathcal{N}$ be a family of the set of natural numbers endowed with an n-HO*

$$*_n : \mathbb{N}_1 \times ... \times \mathbb{N}_n \longrightarrow P^*(\mathbb{N}_1 \times ... \times \mathbb{N}_n)$$

$$*_n(m_1, ..., m_n) = \bigcup_{i=1}^n \{(l_1, ..., l_n) | l_1 + ... + l_n = m_1 + ... + m_n, l_1, ..., l_n \in \mathbb{N}\}, \quad m_i \in \mathbb{N}_i.$$

Therefore, the pair $(\{\mathbb{N}_i)\}_{i=1}^n, *_n)$, is an n-UHA with an identity element $\underbrace{(0, ..., 0)}_{n-time}$.

We define an n-UHO Λ_n *on the ring* $C^\infty(\underbrace{(J\times ...\times J)}_{n-time})$ *by*

$$\Lambda_n : \mathcal{N}\times C^\infty(\underbrace{(J\times ...\times J)}_{n-time}) \longrightarrow P^*(C^\infty(\underbrace{(J\times ...\times J)}_{n-time}))$$

$$\Lambda_n((m_1,...,m_n),f) = \{\bigcup_{*_n(m_1,...,m_n)} \frac{\partial^{m_1+...+m_n}f}{\partial^{l_1}x_1...\partial^{l_n}x_n}\},\quad f\in C^\infty(\underbrace{(J\times ...\times J)}_{n-time}),$$

where ∂ *denotes the partial derivative in the partial differential equation (PDE). Evidently* $(C^\infty(\underbrace{(J\times ...\times J)}_{n-time}),\Lambda_n,\mathcal{N})$ *is a discontinuous n-DHS.*

Definition 2.15. *For any* $x\in X$*, the set* $O^{\mathcal{H}}(x)=\{\Lambda(h,x); h\in\mathcal{H}\}$ *is called hyperorbit of* x*.*

Example 2.11. *Let* (X,λ_n,H) *is a dynamical system. So we can define a 2-DHS* (X,λ_n^*,H) *where* $\lambda_n^*: H\times X\to P^*(X)$ *by* $(g,x)\mapsto O_g(x)$*.*

Proposition 2.6. *Let* $(\mathcal{L},\mathcal{T})$ *be a conjugate relation between* $(\{H_i; i=1,...,n\},\phi_n)$ *and* $(\{H_i'; i=1,...,n\},\phi_n')$*. Then* $\mathcal{L}(O^{\mathcal{H}}(x))=(O^{\mathcal{H}'}(\mathcal{L}(x))$*.*

Proof. If $x'\in\mathcal{L}(O^{\mathcal{H}}(x))$, then there exists $h\in\mathcal{H}$ such that

$$x'\in\mathcal{L}(\Lambda(h,x))=\Lambda'(\phi_n(h),\mathcal{L}(x))\in O^{\mathcal{H}'}(\mathcal{L}(x)).$$

Since conjugate relation is an equivalence relation, so the first part of the proof shows that

$$\mathcal{L}^{-1}(O^{\mathcal{H}'}(\mathcal{L}(x)))\subseteq O^{\mathcal{L}}(x).$$

In the same manner, we can see that

$$\mathcal{L}^{-1}(O^{\mathcal{H}'}(\mathcal{L}(x)))\supseteq O^{\mathcal{L}}(x).$$

This finishes the proof. □

Proposition 2.7. *Let* $(\mathcal{L},\mathcal{T})$ *be a conjugate relation between* $(\{H_i; i=1,...,n\},\phi_n)$ *and* $(\{H_i'; i=1,...,n\},\phi_n')$*. If* $\overline{O^{\mathcal{H}}(x)}=X$*, then* $\overline{O^{\mathcal{H}'}(x')}=X'$*.*

Proposition 2.8. *Let* $(\mathcal{L},\mathcal{T})$ *be a conjugate relation between* $(\{H_i; i=1,...,n\},\phi_n)$ *and* $(\{H_i'; i=1,...,n\},\phi_n')$*. If* $(\{H_i; i=1,...,n\},\phi_n)$ *be a topologically transitive, then* $(\{H_i'; i=1,...,n\},\phi_n')$ *is topologically transitive.*

3 Conclusions

The study of properties of n-ary dynamical hypersystem in the context of n-ary topological hypergroups is a new research topic of n-ary hyperstructure theory. The existing research on this topic deals only with n-ary hyperstructures and for this study, the approximations in n-ary topological hyperstructures are important. In this paper, we introduce and characterize n-ary dynamical hypersystem and give some examples. Our future work on this topic will be focused on the study of some particular classes of n-ary dynamical hypersystem.

References

[1] P. Bonansinga, P. corsini, A class of hyperrings and hyperfields, *Int. J. Math. and Math. Sci.* 2 (1983) 307-312.

[2] Jan Chavalina, Commutative hypergroups in the sense of Marty, *Proceeding of the Summer School,* (1994), Horni Lipova, 19-30, Czech Republic.

[3] Jan Chvalina, S. H. Mayerova, A. D. Nezhad, General actions of hyperstructures and some applications, *An. St. Univ. Ovidius Constanta* 21 (2013) 59-82.

[4] S. D. Comer, Combinatorial aspects of relations, *Algebra Universalis*, 1993.

[5] S. D. Comer, Extension of polygroups by polygroups and their representations using colour schemes, *Lecture notes in Math., No 1004, Univattice Theory* 12 (1982) 91-103.

[6] S. D. Comer, Hyperstructures associated with character algebra and colour schemes, *New Frontiers in Hyperstructures, Hadronic Press* 32 (1996) 49-66.

[7] S. D. Comer, Polygroups derived from cogroups, *Journal of Algebra* 89 (1994) 12-23.

[8] P. Corsini, Prolegomena of hypergroup theory, *Second edition, Aviani Editor*, 1993.

[9] P. Corsini, V. Leoreanu, Applications of hyperstructures theory, *Advances in Mathematics, Kluwer Academic Publishers*, 2003.

[10] B. Davvaz, Elementary topics on weak polygroups, *Bull. Korean Math. Soc.* 40 (2003) 418-431.

[11] B. Davvaz, Isomorphism Theorems of Polygroups, *Bull. Malays. Math. Sci. Soc.* 33 (2010) 385-392.

[12] B. Davvaz, Polygroup Theory and Related Systems, *Department of Mathematics, Yazd University, Yazd, Iran.*

[13] B. Davvaz, A. Dehghan Nezhad, Chemical examples in hypergroups, *Ratio Mathematica* 14 (2003) 71-74.

[14] B. Davvaz, A. Dehghan Nezhad, A. Benvidi, Chain reactions as experimental examples of ternary algebraic hyperstructures, *Communications in Mathematical and in Computer Chemistry* 65 (2011) 491-499.

[15] B. Davvaz, A. Dehghan Nezhad, A. Benvidi, Chemical hyperalgebra: Dismutations, *Communications in Mathematical and in Computer Chemistry,* (2011).

[16] B. Davvaz, Wieslaw A. Dudek, Thomas Vougiouklis, A generalization of n-ary algebraic systems, *Communications in Algebra* 37 (2009) 1248-1263.

[17] B. Davvaz, T. Vougiouklis, n-Ary hypergroups, *Iran. J. Sci. Technol.* 30 (2006) 165-174.

[18] A. Dehghan Nezhad, B. Davvaz, Universal hyperdynamical systems, *Bull. Korean Math. Soc.* 3 (2010) 513-526.

[19] Robert L. Denaney, An Introduction to Chaotic Dynamical Systems, *Department of Mathematics, Boston University, Addison-Wesley Publishing Company,* 1989.

[20] A. Wieslaw Dudek, V. Vladimir Mukhin, On topological n-ary semigroups, *Quasigroups and related systems* 3 (1996) 73-88.

[21] M. Ghadiri, B. N. Waphare, n-ary polygroups, *Iran J. Sci. Technol. Trans. A* 33 (2009) 11-33.

[22] G. Gratzer, A representation theorem for multi-algebras, *Arch. Math.* 3 (1962) 452-456.

[23] G. E. Hansoul, A subdirect decomposition theorem for multialgebras, *Algebra universalis* 16 (1983) 274-281.

[24] H. Hoft, P. E. Howard, Representing multialgebras by algebras, the axiom of choice, and the axiom of dependent choice, *Algebra Universalis* 13 (1981) 69-77.

[25] Rabah Kellil, Ferdaous Kellil, A brief introduction to topological hypergroup, *Sci. Int. (Lahore)* 27 (2015) 3957-3959.

[26] John M. Lee, Introduction to Smooth Manifolds, *Springer, New York, New York*, 2012.

[27] V. Leoreanu-Fotea, B. Davvaz, n-hypergroups and binary relations, *European J. Combinatorics* 29 (2008) 1207-1218.

[28] V. Leoreanu-Fotea, B. Davvaz, Roughness in n-ary hypergroups, *Inform. Sci.* 178 (2008) 4114-4124.

[29] V. Leoreanu-Fotea B. Davvaz, n-hypergroups and binary relations, *European J. Combin.* 29 (2008) 1207-1218.

[30] A. Madanshekaf, A. R. Ashrafi, Generalized action of a hypergroup on a set, *Italian J.of pure and Appl. math.* 3 (1998) 127-135.

[31] F. Marty, Sur une generalization de la notion de groupe, 8^{th} *Congress Math. Scandenaves, Stockholm* 12 (1934) 45-49.

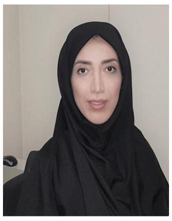
[32] J. R. Munkres, Topology, *second edition, Prentice Hall, NJ,* 2000.

[33] T. Vougiouklis, Hyperstructures and their representations, *Hadronic Press, Inc, 115, Palm Harber, USA,* 1994.

[34] T. Vougiouklis, A new class of hyperstructures, *J. Comb.nf. Syst. Sci.* 20 (1995) 229-235.

[35] J. Zhan, S. Sh. Mousavi, M. Jafarpour, On hyperaction of hypergroups, *U. P. B. Sci. Bull., Series A.* 73 (2011) 117-128.

Akbar Dehghan Nezhad is an Associate Professor the Faculty of Mathematics, Iran University of Science and Technology. He received the Ph.D. degree in Geometry (Dynamical System) from the Universite de Valenciennes et du Hainaut-Cambrsis, France, in 2006. His research interests include geometry, topology, dynamical systems, hyperstructures, foliations, metric spaces, diffeology and applications of geometry in science (physic, chemistry, art, etc.).



Navideh Saderi is a Ph.D. Candidate in pure mathematics, geometry, at Yazd University. She received the M.Sc. in pure mathematics from the Shahid Beheshti University in 2011. Her research interest is classical dynamical systems and geometry.